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Some Pareto optimality results for nonsmooth multiobjective optimization problems with equilibrium constraints

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Abstract

In this paper, we study the nonsmooth multiobjective optimization problems with equilibrium constraints (MOMPEC). First, we extend the Guignard constraint qualification for MOMPEC, and then more constraint qualifications are developed. Also, the relationships between them are investigated. Moreover, we introduce the notion of primal Pareto stationarity and some dual Pareto stationarity concepts for a feasible point of MOMPEC. Some necessary optimality conditions are derived for any Pareto optimality solution of MOMPEC under weak assumptions. Indeed, we just need the objective functions to be locally Lipschitz. Further, we indicate our defined Pareto stationarity concepts are also sufficient conditions under the generalized convexity requirements.

Keywords: Equilibrium Constraints, Pareto Optimality, Constraint Qualifications, Upper Convexificator, Nonsmooth Optimization
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1 Introduction

In this paper, we deal with the multiobjective optimization problem with equilibrium constrains (MOMPEC) that has the form

$$\begin{aligned} \min \quad & c(z) = (c_1(z), \dots, c_r(z)) \\ s.t. \quad & z \in \Omega \end{aligned} \tag{1.1}$$

where

$$\Omega := \{z \in \mathbb{R}^n |\ a(z) \leq 0,\ b(z) = 0,\ A(z) \geq 0,\ B(z) \geq 0,\ \langle A(z), B(z) \rangle = 0\},$$

and $c_i : \mathbb{R}^n \to \mathbb{R}$, $a : \mathbb{R}^n \to \mathbb{R}^p$, $b : \mathbb{R}^n \to \mathbb{R}^q$, $A, B : \mathbb{R}^n \to \mathbb{R}^m$ are given functions. Problem (1.1) is clearly a combination of the class of mathematical program with equilibrium constraints (MPEC) and the class of multiobjective optimization problems (MOP).

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Many articles have studied the MPECs both in the smooth and nonsmooth cases. For more general knowledge and many applications of MPEC, one can see [1, 2, 7, 8, 13, 20], two books [12, 17], and references therein. Also, common multiobjective optimization problems have been investigated by various researchers. Li et al. in [11] considered a nonsmooth multiobjective optimization problem with just inequality constraints and had given a strong necessary optimality condition under the nonsmooth analogue Guignard constraint qualification. In their work, they assumed that all effective functions are locally Lipschitz. To the best of our knowledge, articles published in MOMPEC are very limited; see for example [10, 14, 19, 21].

The authors in [23] extended some constraint qualifications and stationary notions for MOMPEC under the assumption that all functions which describe MOMPEC are continuously differentiable. They proved that under appropriate constraint qualifications, the proper Pareto optimality conditions hold for the locally Pareto optimal solutions of MOMPEC. Later on, in [22], they obtained the Kuhn-Tucker type strong Pareto optimality conditions for MOMPEC under a Guignard constraint qualification and smooth assumptions. They showed that this constraint qualification is the weakest constraint qualification for the strong Pareto stationary.

In this paper, we take the nonsmooth analysis approach and try to give Pareto optimality conditions for MOMPEC. Our tool, here, is the notion of the convexificator. In 1994, Demyanov [5] introduced the notion of a convex and compact convexificator which is a generalization of the notion of upper convex and lower concave approximations. This notion was further developed in [4]. The notion of nonconvex closed convexificator was introduced by Jeyakumar and Luc [9] for extended-real-valued functions. Moreover, the notion of convexificators can be viewed as weaker versions of the idea of subdifferentials. For a function that is locally Lipschitz, most known subdifferentials are convexificators and these known subdifferentials may contain the convex hull of a convexificator; see, for example [4, 9]. The descriptions of the optimality conditions and calculus rules in terms of convexificators have been provided sharp results in optimization and applications.

Our purpose in this article is to obtain some Pareto type necessary and sufficient optimality conditions for nonsmooth MOMPEC based on the upper convexificators. To this end, following the article [16], we first present a nonsmooth version of the Guignard constraint qualification in which all of the objective functions are involved. Moreover, an extension of some other well-known constraint qualifications is offered and the relationships between them are investigated. Moreover, we introduce the nonsmooth versions of primal and dual Pareto stationary concepts for any feasible point of MOMPEC. Then, we illustrate that Pareto stationary conditions are Pareto necessary optimality conditions under weak constraint qualifications and while only the objective functions are locally Lipschitz. Since the Lagrange multipliers related to the objective functions are all positive in these conditions, then the results are stronger. Finally, we prove that under the generalized convexity assumptions, Pareto stationary conditions can also be sufficient conditions for Pareto optimality or weak Pareto optimality of MOMPEC.

The structure of the next sections of the paper is as follows. Section 2 is assigned to provide the needed notations and initial results used in the rest of the paper. In Section 3, the generalized form of Guignard CQ and the other well-known CQ are introduced and the relations between them are studied. Also, Pareto stationarity concepts are presented. Then, we focus on getting Pareto necessary and sufficient optimality conditions for MOMPEC.

2 Preliminaries

In this section, we present some notations and preliminary results from the nonsmooth analysis that will be needed in what follows see, e.g., [3, 15] for more detail.

Throughout the paper, $\langle z, y\rangle$ denote the standard inner product of vectors $z, y \in \mathbb{R}^n$. For a given subset $Q \subseteq \mathbb{R}^n$, the convex cone containing the origin generated by Q, the convex hull of Q and the closure of Q are denoted by $pos\, Q$, $co\, Q$ and $cl\, Q$ respectively. The strictly negative and negative polar cones Q^s and Q^- are defined, respectively, by

$$Q^s := \{v \in \mathbb{R}^n \,|\langle z, v\rangle < 0, \quad \forall z \in Q \setminus \{0\}\}\,.$$
$$Q^- := \{v \in \mathbb{R}^n \,|\langle z, v\rangle \leq 0, \quad \forall z \in Q\}\,.$$

It can be easily shown that if $Q^s \neq \emptyset$, then $cl(Q^s) = Q^-$. Also, note that for given closed convex cones Q_1 and Q_2 we have:

$$(Q_1 \cup Q_2)^- = Q_1^- \cap Q_2^-\,.$$

The following approximate cones for $Q \subseteq \mathbb{R}^n$ will be requested in this paper:

- The contingent cone (or the Bouligand tangent cone) of Q at $z \in cl\, Q$ is

$$\mathcal{T}(Q,z) := \{d \in \mathbb{R}^n \mid\ \exists \tau_r \downarrow 0,\ \ \exists d_r \to d \quad \text{such that} \quad z + \tau_r d_r \in Q,\ \ \forall r\}.$$

- The cone of attainable directions $\mathcal{A}(Q,z)$ and the cone of feasible directions $\mathcal{D}(Q,z)$ of Q at z are defined, respectively, by

$$\mathcal{A}(Q,z) := \Big\{ d \in \mathbb{R}^n \mid\ \exists \eta > 0 \ \text{ and } \ \alpha : \mathbb{R} \to \mathbb{R}^n \ \text{ such that } \ \alpha(\tau) \in Q,\ \ \forall \tau \in (0,\eta),$$
$$\alpha(0) = 0,\ \ \lim_{\tau \downarrow 0} \frac{\alpha(\tau) - \alpha(0)}{\tau} = d \Big\},$$

$$\mathcal{D}(Q,z) := \{d \in \mathbb{R}^n \mid\ \exists \eta > 0 \quad \text{such that} \quad z + \tau d \in Q,\ \ \forall \tau \in (0,\eta)\}.$$

Let us provide the notion of convexificator and some of its important properties.
Let $\Phi : \mathbb{R}^n \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be a given function, $z \in \text{dom}\ \Phi := \{z \in \mathbb{R}^n \,|\Phi(z) < \infty\}$ and $w \in \mathbb{R}^n$. The lower and upper directional derivatives (or Dini derivatives) of Φ at z in the direction w are defined, respectively by

$$\Phi^-(z;w) := \liminf_{\tau \downarrow 0} \frac{\Phi(z+\tau w) - \Phi(z)}{\tau},$$
$$\Phi^+(z;w) := \limsup_{\tau \downarrow 0} \frac{\Phi(z+\tau w) - \Phi(z)}{\tau}.$$

Now we remind the definitions of upper and lower convexificators from [9].

- It is said that the function Φ have an upper convexificator at $z \in \mathbb{R}^n$ if there is a closed set $\partial^*\Phi(z) \subset \mathbb{R}^n$ such that for each $u \in \mathbb{R}^n$,

$$\Phi^-(z;u) \le \sup_{\zeta \in \partial^*\Phi(z)} \langle \zeta, u \rangle.$$

- It is said that the function Φ have a lower convexificator at $z \in \mathbb{R}^n$ if there is a closed set $\partial_*\Phi(z) \subset \mathbb{R}^n$ such that for each $u \in \mathbb{R}^n$,

$$\Phi^+(z;u) \ge \inf_{\zeta \in \partial_*\Phi(z)} \langle \zeta, u \rangle.$$

  A closed set $\partial^*\Phi(z) \subset \mathbb{R}^n$ is said to be a convexificator of Φ at $z \in \mathbb{R}^n$ if it is both upper and lower convexificator for Φ at z.

The upper convexificator $\partial^*\Phi(z)$ was also called [18] the Jeyakumar-Luc subdifferential of Φ at z. It is worth mentioning that if a continuous function $\Phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ admits a locally bounded upper convexificator map at z, then it is locally Lipschitz around this point (see [9]).

Following the pattern in [6], the definitions of ∂^*-quasiconvex and ∂^*-pseudoconvex functions are given. Assume that the extended real-valued function $\Phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ has an upper convexificator at $z \in \mathbb{R}^n$, then

- The function Φ is said to be ∂^*-quasiconvex if for every $y \in \mathbb{R}^n$ one has

$$\Phi(y) \le \Phi(z) \Longrightarrow \langle \zeta, y - z \rangle \le 0, \qquad \forall \zeta \in \partial^*\Phi(z).$$

- The function Φ is said to be ∂^*-pseudoconvex at z if for every $y \in \mathbb{R}^n$ with $z \ne y$

$$\Phi(y) < \Phi(z) \Longrightarrow \langle \zeta, y - z \rangle < 0, \qquad \forall \zeta \in \partial^*\Phi(z).$$

We say that Φ is ∂^*-quasiconcave at z if $-\Phi$ is ∂^*-quasiconvex at z and similarly Φ is ∂^*-pseudoconcave at z if $-\Phi$ is ∂^*-pseudoconvex at z.

3 Constraint qualifications and Pareto optimality conditions

In this section, considering the problem (1.1), we first develop a generalized Guignard constraint qualification for it. Then, we introduce new forms of some other famous constraint qualifications for MOMPEC. Also, some nonsmooth Pareto stationary points are defined for MOMPEC based on the upper convexificators. Furthermore, we attempt to provide generalized optimality conditions for nonsmooth MOMPEC.

At first, we need to introduce some notation. Let $\bar{z} \in \Omega$ be an arbitrary feasible point. Then, we define the index sets:

$$\begin{aligned}
&\mathcal{I}_c := \{1, 2, \ldots, r\}, \quad \mathcal{J} := \{1, 2, , \ldots, q\}, \\
&\mathcal{I}_a := \{i \,| a_i(\bar{z}) = 0\}, \\
&\alpha := \alpha(\bar{z}) = \{i \,| A_i(\bar{z}) = 0, B_i(\bar{z}) > 0\}, \\
&\beta := \beta(\bar{z}) = \{i \,| A_i(\bar{z}) = 0, B_i(\bar{z}) = 0\}, \\
&\gamma := \gamma(\bar{z}) = \{i \,| A_i(\bar{z}) > 0, B_i(\bar{z}) = 0\}.
\end{aligned}$$

A point $\bar{z} \in \Omega$ is called a Pareto (weak Pareto) optimal solution of problem (1.1) if there is no another point $z \in \Omega$ such that $c(z) \leq c(\bar{z})\big(c(z) < c(\bar{z})\big)$ and $c_i(z) < c_i(\bar{z})$ for some i.

Definition 3.1. For $\bar{z} \in \Omega$, we say that the generalized Guignard constraint qualification (GG-CQ) is satisfied at $\bar{z}$ when

$$\mathcal{L}(Q, \bar{z}) \subseteq \bigcap_{k=1}^{r} clco\mathcal{T}(Q^k, \bar{z}),$$

where $Q = \bigcap\limits_{k=1}^{r} Q^k$ with $Q^k = \left\{z \in \Omega \,\middle|\, c_i(z) \leq c_i(\bar{z}),\ i \in \mathcal{I}_c,\ i \neq k \right\}$, and $\mathcal{L}(Q, \bar{z})$ is the linearizing cone to Q at $\bar{z} \in Q$ defined by

$$\begin{aligned}
\mathcal{L}(Q, \bar{z}) := \big\{d \in \mathbb{R}^n \,\big|\, & c_i^-(\bar{z}; d) \leq 0 \quad (i \in \mathcal{I}_c), \\
& a_i^-(\bar{z}; d) \leq 0 \quad (i \in \mathcal{I}_a), \\
& b_j^-(\bar{z}; d) \leq 0, \ \ (-b_j)^-(\bar{z}; d) \leq 0 \quad (j \in \mathcal{J}), \\
& A_i^-(\bar{z}; d) \leq 0, \ \ (-A_i)^-(\bar{z}; d) \leq 0 \quad (i \in \alpha), \\
& B_i^-(\bar{z}; d) \leq 0, \ \ (-B_i)^-(\bar{z}; d) \leq 0 \quad (i \in \gamma), \\
& (-A_i)^-(\bar{z}; d) \leq 0, \ \ (-B_i)^-(\bar{z}; d) \leq 0 \quad (i \in \beta)\big\}.
\end{aligned}$$

Definition 3.2. Suppose that $\bar{z} \in \Omega$, then we say that

(i). The Abadie constraint qualification (A-CQ) is satisfied at $\bar{z}$ if

$$\mathcal{L}(Q, \bar{z}) \subseteq \mathcal{T}(Q, \bar{z}).$$

(ii). The generalized Abadie constraint qualification (GA-CQ) is satisfied at $\bar{z}$ if

$$\mathcal{L}(Q, \bar{z}) \subseteq \bigcap_{k=1}^{r} \mathcal{T}(Q^k, \bar{z}).$$

(iii). The Kuhn-Tucker constraint qualification (KT-CQ) is satisfied at $\bar{z}$ if

$$\mathcal{L}(Q, \bar{z}) \subseteq cl\mathcal{A}(Q, \bar{z}).$$

(iv). The Zangwill constraint qualification (Z-CQ) is satisfied at $\bar{z}$ if

$$\mathcal{L}(Q, \bar{z}) \subseteq cl\mathcal{D}(Q, \bar{z}).$$

Definition 3.3. Let $\bar{z} \in \Omega$. We say that (i). The MOMPEC generalized Guignard constraint qualification (MOMPEC-GG-CQ) is satisfied at $\bar{z}$ if

$$\mathcal{L}_{MOMPEC}(Q,\bar{z}) \subseteq \bigcap_{k=1}^{r} clco\mathcal{T}(Q^k,\bar{z}),$$

where

$$\mathcal{L}_{MOMPEC}(Q,\bar{z}) = \mathcal{L}(Q,\bar{z}) \cap \{d \in \mathbb{R}^n |\, A_i^-(\bar{z};d) \leq 0 \vee B_i^-(\bar{z};d) \leq 0, i \in \beta\}.$$

(ii). The MOMPEC generalized Abadie constraint qualification (MOMPEC-GA-CQ) is satisfied at $\bar{z}$ if

$$\mathcal{L}_{MOMPEC}(Q,\bar{z}) \subseteq \bigcap_{k=1}^{r} \mathcal{T}(Q^k,\bar{z}),$$

(iii). The MOMPEC Abadie constraint qualification (MOMPEC-A-CQ) is satisfied at $\bar{z}$ if

$$\mathcal{L}_{MOMPEC}(Q,\bar{z}) \subseteq \mathcal{T}(Q,\bar{z}).$$

(iv). The MOMPEC Kuhn-Tucker constraint qualification (MOMPEC-KT-CQ) is satisfied at $\bar{z}$ if

$$\mathcal{L}_{MOMPEC}(Q,\bar{z}) \subseteq cl\mathcal{A}(Q,\bar{z}).$$

(v). The MOMPEC Zangwill constraint qualification (MOMPEC-Z-CQ) is satisfied at $\bar{z}$ if

$$\mathcal{L}_{MOMPEC}(Q,\bar{z}) \subseteq cl\mathcal{D}(Q,\bar{z}).$$

From the above definitions and since $cl\mathcal{D}(Q,\bar{z}) \subseteq cl\mathcal{A}(Q,\bar{z}) \subseteq \mathcal{T}(Q,\bar{z})$, the relationships between the defined CQs are summarized as follows:

$$\text{Z-CQ} \Rightarrow \text{KT-CQ} \Rightarrow \text{A-CQ} \Rightarrow \text{GA-CQ} \Rightarrow \text{GG-CQ} \Rightarrow \text{MOMPEC-GG-CQ},$$

and

$$\begin{aligned}&\text{MOMPEC-Z-CQ} \Rightarrow \text{MOMPEC-KT-CQ} \Rightarrow \text{MOMPEC-A-CQ} \Rightarrow \text{MOMPEC-GA-CQ}\\ &\text{MOMPEC-GA-CQ} \Rightarrow \text{MOMPEC-GG-CQ}.\end{aligned}$$

Now, we present the generalized Pareto stationarity notions for MOMPEC in terms of the upper convexificators.

Definition 3.4. Let $\bar{z} \in \Omega$. Then, $\bar{z}$ is called

(i) a generalized Pareto W-stationary point (GPW-stationary point) of MOMPEC if there exist vectors $\lambda = (\lambda^c, \lambda^a, \lambda^b, \lambda^A, \lambda^B) \in \mathbb{R}^{r+p+q+2m}$, $\mu = (\mu^b, \mu^A, \mu^B) \in \mathbb{R}^{q+2m}$ such that

$$\begin{aligned}0 \in cl\Bigg(&\sum_{i\in\mathcal{I}_c} \lambda_i^c co\partial^* c_i(\bar{z}) + \sum_{i\in\mathcal{I}_a} \lambda_i^a co\partial^* a_i(\bar{z})\\ &+ \sum_{j\in\mathcal{J}} \left[\lambda_j^b co\partial^* b_j(\bar{z}) + \mu_j^b co\partial^*(-b_j)(\bar{z})\right]\\ &+ \sum_{i=1}^{m} \left[\lambda_i^A co\partial^* A_i(\bar{z}) + \lambda_i^B co\partial^* B_i(\bar{z})\right]\\ &+ \sum_{i=1}^{m} \left[\mu_i^A co\partial^*(-A_i)(\bar{z}) + \mu_i^B co\partial^*(-B_i)(\bar{z})\right]\Bigg),\end{aligned} \tag{3.1}$$

$$\lambda_i^c > 0,\ i \in \mathcal{I}_c, \quad \lambda_i^a \geq 0,\ i \in \mathcal{I}_a, \quad \lambda_j^b, \mu_j^b \geq 0,\ j \in \mathcal{J}, \tag{3.2}$$

$$\lambda_i^A, \lambda_i^B, \mu_i^A, \mu_i^B \geq 0,\ i = 1, 2, \ldots, m, \tag{3.3}$$

$$\lambda_\gamma^A = \lambda_\alpha^B = \mu_\gamma^A = \mu_\alpha^B = 0. \tag{3.4}$$

(ii) a generalized Pareto A-stationary point (GPA-stationary point) of MOMPEC if $\bar{z}$ is GPW-stationary point and $\lambda_i^A = 0 \;\vee\; \lambda_i^B = 0,\; \forall i \in \beta$.

(iii) a generalized Pareto S-stationary point (GPS-stationary point) of MOMPEC if $\bar{z}$ is GPW-stationary point and $\lambda_i^A = 0 \;\wedge\; \lambda_i^B = 0,\; \forall i \in \beta$.

The following definition is the generalized form of primal stationary (Bouligand-stationarity) concept for MOMPEC.

Definition 3.5. Suppose that $\bar{z} \in \Omega$. Then, $\bar{z}$ is called a generalized Pareto B-stationary point (GPB-stationary point) if for every $k \in \mathcal{I}_c$,

$$c'_k(\bar{z}; d) \geq 0, \quad \forall\, d \in \bigcap_{k \in \mathcal{I}_c} clco\mathcal{T}(Q^k, \bar{z}).$$

Remark 3.6. The constraint qualifications and the stationary notions introduced above are already defined in differentiability mood [22, 23].

We are now ready to state the first main conclusion of this article. We will see that the assumptions made in the following Proposition are very weak. Where only the objective functions are assumed to be Lipschitz.

Proposition 3.7. Suppose that $\bar{z}$ be a Pareto optimal solution of MOMPEC and the functions $c_i, i \in \mathcal{I}_c$ are locally Lipschitz near $\bar{z}$. Also, assume that $c'_k(\bar{z}; .), k \in \mathcal{I}_c$ be concave. Then, $\bar{z}$ is a GPB-stationary point of MOMPEC.

Proof . On the contrary, let $\bar{z}$ is not a GPB-stationary point of MOMPEC. Thus, there exists an index $k_0 \in \mathcal{I}_c$ and a vector $d \in clco\mathcal{T}(Q^{k_0}, \bar{z})$ such that

$$c'_{k_0}(\bar{z}; d) < 0.$$

Since $\{d \in \mathbb{R}^n |\; c'_{k_0}(\bar{z}; d) < 0\}$ is an open set, then we can say there exists a vector $\bar{d} \in co\mathcal{T}(Q^{k_0}, \bar{z})$ such that

$$c'_{k_0}(\bar{z}; \bar{d}) < 0.$$

Therefore, there should be a finite number of vectors $d_1, d_2, \ldots, d_l \in \mathcal{T}(Q^{k_0}, \bar{z})$ and scalers $\eta_1, \eta_2, \ldots, \eta_l$ with $\eta_i > 0, i = 1, 2, \ldots, l$ and $\sum\limits_{i=1}^{l} \eta_i = 1$ such that $\bar{d} = \sum\limits_{i=1}^{l} \eta_i d_i$. Since $c'_{k_0}(\bar{z}; .)$ is concave, so

$$\sum_{i=1}^{l} \eta_i c'_{k_0}(\bar{z}; d_i) \leq c'_{k_0}(\bar{z}; \sum_{i=1}^{l} \eta_i d_i) = c'_{k_0}(\bar{z}; \bar{d}) < 0.$$

The above inequality implies that there is an index $i \in \{1, 2, \ldots, l\}$, for instance $i = 1$, such that

$$c'_{k_0}(\bar{z}; d_1) < 0.$$

Since $d_1 \in \mathcal{T}(Q^{k_0}, \bar{z})$, so there exist the sequences $t_r \downarrow 0$ and $d_r \to d_1$ such that $\bar{z} + t_r d_r \in Q^{k_0}$. Take $z_r = \bar{z} + t_r d_r$. Due to the locally Lipschitz property of c_{k_0} near $\bar{z}$, we have

$$\left| \frac{c_{k_0}(\bar{z} + t_r d_r) - c_{k_0}(\bar{z} + t_r d_1)}{t_r} \right| \leq L_{k_0} \|d_r - d_1\| \to 0 \quad (\text{when } r \to 0),$$

where L_{k_0} denotes the Lipschitz constant of c_{k_0} near $\bar{z}$. Therefore, we obtain

$$\begin{aligned}
&\lim_{r \to \infty} \frac{c_{k_0}(\bar{z} + t_r d_r) - c_{k_0}(\bar{z})}{t_r} \\
&= \lim_{r \to \infty} \frac{c_{k_0}(\bar{z} + t_r d_r) - c_{k_0}(\bar{z} + t_r d_1)}{t_r} + \lim_{r \to \infty} \frac{c_{k_0}(\bar{z} + t_r d_1) - c_{k_0}(\bar{z})}{t_r} \\
&= 0 + \lim_{r \to \infty} \frac{c_{k_0}(\bar{z} + t_r d_1) - c_{k_0}(\bar{z})}{t_r} \\
&= f'_{k_0}(\bar{z}; d_1) < 0.
\end{aligned}$$

It follows that there exists a subsequence $\{\bar{z}+t_{u_r}d_{u_r}\}$ of $\{\bar{z}+t_r d_r\}$ such that $c_{k_0}(\bar{z}+t_{u_r}d_{u_r}) < c_{k_0}(\bar{z})$ and since $\bar{z}+t_{u_r}d_{u_r} \in Q^{k_0}$, we get

$$\begin{aligned}
&c_k(\bar{z}+t_{u_r}d_{u_r}) \leq c_k(\bar{z}), \quad \forall k \in \mathcal{I}_c \setminus \{k_0\},\\
&a(\bar{z}+t_{u_r}d_{u_r}) \leq 0, \quad b(\bar{z}+t_{u_r}d_{u_r}) = 0,\\
&A(\bar{z}+t_{u_r}d_{u_r}) \geq 0, \quad B(\bar{z}+t_{u_r}d_{u_r}) \geq 0, \quad \langle A(\bar{z}+t_{u_r}d_{u_r}), B(\bar{z}+t_{u_r}d_{u_r})\rangle = 0,
\end{aligned}$$

which contradicts the assumption that $\bar{z}$ is a Pareto optimal solution of MOMPEC. Then, the proof is complete. □

Now, in the following Theorem, we provide a strong necessary condition for Pareto optimality of MOMPEC under our GG-CQ.

Theorem 3.8. Let $\bar{z}$ be a Pareto optimal solution of MOMPEC and the following conditions hold:

(i). $c_i(i \in \mathcal{I}_c)$ are locally Lipschitz near $\bar{z}$,
(ii). $c_i(i \in \mathcal{I}_c)$ and all of the constraint functions admit upper convexificators at $\bar{z}$,
(iii). $c_i(i \in \mathcal{I}_c)$ are directionally differentiable at $\bar{z}$, $c'_{i_0}(\bar{z};.)$ being linear for some $i_0 \in \mathcal{I}_c$ and $c'_k(\bar{z};.)$ being sublinear for every $k \in \mathcal{I}_c \setminus \{i_0\}$,
(iv). $a_i^-(\bar{z};.)(i \in \mathcal{I}_a)$, $(\pm b_j)^-(\bar{z};.)(j \in \mathcal{J})$, $(\pm A_i)^-(\bar{z};.)(i \in \alpha)$, $(\pm B_i)^-(\bar{z};.)(i \in \gamma)$, $(-A_i)^-(\bar{z};.)$, $(-B_i)^-(\bar{z};.)(i \in \beta)$ being sublinear.

Then, $\bar{z}$ will be a GPS-stationary point provided that GG-CQ is satisfied at $\bar{z}$.

Proof . We first claim that the following system has no solution:

$$\begin{aligned}
&c'_{i_0}(\bar{z};d) < 0,\\
&c'_k(\bar{z};d) \leq 0, \quad k \in \mathcal{I}_c \setminus \{i_0\},\\
&a_i^-(\bar{z};d) \leq 0, \quad i \in \mathcal{I}_a,\\
&b_j^-(\bar{z};d) \leq 0, \quad (-b_j)^-(\bar{z};d) \leq 0, \quad j \in \mathcal{J},\\
&A_i^-(\bar{z};d) \leq 0, \quad (-A_i)^-(\bar{z};d) \leq 0, \quad i \in \alpha,\\
&B_i^-(\bar{z};d) \leq 0, \quad (-B_i)^-(\bar{z};d) \leq 0, \quad i \in \gamma,\\
&(-A_i)^-(\bar{z};d) \leq 0, \quad (-B_i)^-(\bar{z};d) \leq 0, \quad i \in \beta.
\end{aligned}$$

Assume on the contrary that there exists a vector $d \in \mathbb{R}^n$ which is a solution of the above system. Thus, $d \in \mathcal{L}(Q,\bar{z})$, and so by the GG-CQ, we have

$$d \in \bigcap_{k \in \mathcal{I}_c} clco\mathcal{T}(Q^k, \bar{z}).$$

According to Proposition 3.7, this is a contradiction with the GPB-stationarity of $\bar{z}$. Therefore, the claim is correct.

Now, by the linearity and sublinearity assumptions and also by the Farkas theorem, there exist nonnegative multipliers $\lambda_i^c > 0$ $(i \in \mathcal{I}_c)$, $\lambda_i^a \geq 0$ $(i \in \mathcal{I}_a)$, $\lambda_j^b, \mu_j^b \geq 0$ $(j \in \mathcal{J})$, $\lambda_i^A, \mu_i^A \geq 0$ $(i \in \alpha)$, $\lambda_i^B, \mu_i^B \geq 0$ $(i \in \gamma)$, $\mu_i^A, \mu_i^B \geq 0$ $(i \in \beta)$, such that for all $d \in \mathbb{R}^n$,

$$\begin{aligned}
&\sum_{i \in \mathcal{I}_c} \lambda_i^c c'_i(\bar{z};d) + \sum_{i \in \mathcal{I}_a} \lambda_i^a a_i^-(\bar{z};d) + \sum_{j \in \mathcal{J}} \left[\lambda_j^b b_j^-(\bar{z};d) + \mu_j^b(-b_j)^-(\bar{z};d)\right]\\
&+ \sum_{i \in \alpha} \left[\lambda_i^A A_i^-(\bar{z};d) + \mu_i^A(-A_i)^-(\bar{z};d)\right] + \sum_{i \in \gamma} \left[\lambda_i^B B_i^-(\bar{z};d) + \mu_i^B(-B_i)^-(\bar{z};d)\right]\\
&+ \sum_{i \in \beta} \left[\mu_i^A(-A)_i^-(\bar{z};d) + \mu_i^B(-B_i)^-(\bar{z};d)\right] \geq 0.
\end{aligned}$$

From the definition of upper convexificators, for every $d \in \mathbb{R}^n$, we get

$$\begin{aligned}
&\sum_{i\in\mathcal{I}_c} \lambda_i^c \sup_{\xi\in\partial^* c_i(\bar{z})} \langle \xi, d\rangle + \sum_{i\in\mathcal{I}_a} \lambda_i^a \sup_{\xi\in\partial^* a_i(\bar{z})} \langle \xi, d\rangle \\
&+ \sum_{j\in\mathcal{J}} \left[\lambda_j^b \sup_{\xi\in\partial^* b_j(\bar{z})} \langle \xi, d\rangle + \mu_j^b \sup_{\xi\in\partial^*(-b_j)(\bar{z})} \langle \xi, d\rangle\right] \\
&+ \sum_{i\in\alpha} \left[\lambda_i^A \sup_{\xi\in\partial^* A_i(\bar{z})} \langle \xi, d\rangle + \mu_i^A \sup_{\xi\in\partial^*(-A_i)(\bar{z})} \langle \xi, d\rangle\right] \\
&+ \sum_{i\in\gamma} \left[\lambda_i^B \sup_{\xi\in\partial^* B_i(\bar{z})} \langle \xi, d\rangle + \mu_i^B \sup_{\xi\in\partial^*(-B_i)(\bar{z})} \langle \xi, d\rangle\right] \\
&+ \sum_{i\in\beta} \left[\mu_i^A \sup_{\xi\in\partial^*(-A_i)(\bar{z})} \langle \xi, d\rangle + \mu_i^B \sup_{\xi\in\partial^*(-B_i)(\bar{z})} \langle \xi, d\rangle\right] \geq 0.
\end{aligned}$$

Therefore, it fallows that

$$\sup_{\xi\in\Gamma(\bar{z})} \langle \xi, d\rangle \geq 0, \quad \forall d \in \mathbb{R}^n,$$

where

$$\begin{aligned}
\Gamma(\bar{z}) := \Bigg(&\sum_{i\in\mathcal{I}_c} \lambda_i^c \partial^* c_i(\bar{z}) + \sum_{i\in\mathcal{I}_a} \lambda_i^a \partial^* a_i(\bar{z}) + \sum_{j\in\mathcal{J}} \left[\lambda_j^b \partial^* b_j(\bar{z}) + \mu_j^b \partial^*(-b_j)(\bar{z})\right] \\
&+ \sum_{i\in\alpha} \left[\lambda_i^A \partial^* A_i(\bar{z}) + \mu_i^A \partial^*(-A_i)(\bar{z})\right] + \sum_{i\in\gamma} \left[\lambda_i^B \partial^* B_i(\bar{z}) + \mu_i^B \partial^*(-B_i)(\bar{z})\right] \\
&+ \sum_{i\in\beta} \left[\mu_i^A \partial^*(-A_i)(\bar{z}) + \mu_i^B \partial^*(-B_i)(\bar{z})\right]\Bigg).
\end{aligned}$$

Now, using the separation theorem for the closed convex sets $clco\Gamma(\bar{z})$ and $\{0\}$, we get

$$0 \in clco\Gamma(\bar{z}).$$

Since

$$\begin{aligned}
clco\Gamma(\bar{z}) \subseteq cl\Bigg(&\sum_{i\in\mathcal{I}_c} \lambda_i^c co\partial^* c_i(\bar{z}) + \sum_{i\in\mathcal{I}_a} \lambda_i^a co\partial^* a_i(\bar{z}) \\
&+ \sum_{j\in\mathcal{J}} \left[\lambda_j^b co\partial^* b_j(\bar{z}) + \mu_j^b co\partial^*(-b_j)(\bar{z})\right] \\
&+ \sum_{i\in\alpha} \left[\lambda_i^A co\partial^* A_i(\bar{z}) + \mu_i^A co\partial^*(-A_i)(\bar{z})\right] \\
&+ \sum_{i\in\gamma} \left[\lambda_i^B co\partial^* B_i(\bar{z}) + \mu_i^B co\partial^*(-B_i)(\bar{z})\right] \\
&+ \sum_{i\in\beta} \left[\mu_i^A co\partial^*(-A_i)(\bar{z}) + \mu_i^B co\partial^*(-B_i)(\bar{z})\right]\Bigg),
\end{aligned}$$

we have,

$$
\begin{aligned}
0 \in cl \Bigg(& \sum_{i \in \mathcal{I}_c} \lambda_i^c co\partial^* c_i(\bar{z}) + \sum_{i \in \mathcal{I}_a} \lambda_i^a co\partial^* a_i(\bar{z}) \\
& + \sum_{j \in \mathcal{J}} \left[\lambda_j^b co\partial^* b_j(\bar{z}) + \mu_j^b co\partial^* (-b_j)(\bar{z}) \right] \\
& + \sum_{i \in \alpha} \left[\lambda_i^A co\partial^* A_i(\bar{z}) + \mu_i^A co\partial^* (-A_i)(\bar{z}) \right] \\
& + \sum_{i \in \gamma} \left[\lambda_i^B co\partial^* B_i(\bar{z}) + \mu_i^B co\partial^* (-B_i)(\bar{z}) \right] \\
& + \sum_{i \in \beta} \left[\mu_i^A co\partial^* (-A_i)(\bar{z}) + \mu_i^B co\partial^* (-B_i)(\bar{z}) \right] \Bigg).
\end{aligned}
$$

Taking $\lambda_{\gamma\cup\beta}^A = \lambda_{\alpha\cup\beta}^B = \mu_\gamma^A = \mu_\alpha^B = 0$, we arrive at the conditions (3.1)-(3.4) and $\lambda_i^A = 0 \ \wedge \ \lambda_i^B = 0$, $\forall i \in \beta$. Thus $\bar{z}$ is a GPS-stationary point. $\square$

The following example illustrates that in Theorem 3.8 without the GG-CQ, a Pareto optimal solution of MOMPEC may not be a GPS-stationary point.

Example 3.9. Consider the following MOMPEC in $\mathbb{R}^2$:

$$
\begin{aligned}
& \min(z_1 + z_2, z_1 - z_2^2) \\
& s.t. \ \ z_1 \leq 0, \quad z_2^2 \geq 0, \quad z_1^2 + z_2^2 \leq 1, \\
& \qquad \langle z_1, z_2^2 \rangle = 0.
\end{aligned}
$$

Then, $\Omega = \{(z_1, z_2) \in \mathbb{R}^2 | \, z_1 = 0, z_2 \in [-1, 1] \text{ or } z_1 \in [-1, 0], z_2 = 0\}$ is the feasible set of MOMPEC. It can be easy check that $\bar{z}_1 = (-1, 0)$ and $\bar{z}_2 = (0, -1)$ are Pareto optimal solutions. For $\bar{z}_1 = (-1, 0)$, we have

$$
\mathcal{L}(Q, \bar{z}_1) = \{d \in \mathbb{R}^2 | \, d_1 = 0, d_2 \leq 0\},
$$

and

$$
\mathcal{T}(Q, \bar{z}_1) \cap \mathcal{T}(Q, \bar{z}_2) = \{(-1, 0), (0, -1)\}.
$$

So, the GG-CQ does not hold at $\bar{z}_1$. From the condition

$$
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1^c \\ \lambda_2^c \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} \lambda^a + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \lambda^A + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu^A = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

we get $\lambda_1^c = 0$ which contradicts the condition $\lambda_i^c > 0$ for all $i \in \mathcal{I}_c$. Therefore, $\bar{z}_1 = (-1, 0)$ is not a GPS-stationary point.

The next Theorem shows that under MOMPEC-GG-CQ, GPA-stationarity is another necessary optimality condition for Pareto optimality of MOMPEC.

Theorem 3.10. Let $\bar{z}$ be a Pareto optimal solution of MOMPEC. Suppose that the conditions (i)-(iv) of Theorem 3.8 are satisfied at $\bar{z}$. Moreover, assume that $A_i^-(\bar{z}; .), B_i^-(\bar{z}; .)(i \in \beta)$ are sublinear. If the MOMPEC-GG-CQ is satisfied at $\bar{z}$, then $\bar{z}$ will be a GPA-stationary point.

Proof . First, we prove that the following system is inconsistent on $\mathbb{R}^n$:

$$\begin{aligned}
&c'_{i_0}(\bar{z};d) < 0,\\
&c'_k(\bar{z};d) \le 0, \quad k \in \mathcal{I}_c \setminus \{i_0\},\\
&a_i^-(\bar{z};d) \le 0, \quad i \in \mathcal{I}_a,\\
&b_j^-(\bar{z};d) \le 0, \quad (-b_j)^-(\bar{z};d) \le 0, \quad j \in \mathcal{J},\\
&A_i^-(\bar{z};d) \le 0, \quad (-A_i)^-(\bar{z};d) \le 0, \quad i \in \alpha,\\
&B_i^-(\bar{z};d) \le 0, \quad (-B_i)^-(\bar{z};d) \le 0, \quad i \in \gamma,\\
&(-A_i)^-(\bar{z};d) \le 0, \quad (-B_i)^-(\bar{z};d) \le 0, \quad i \in \beta,\\
&A_i^-(\bar{z};d) \le 0, \quad i \in \beta.
\end{aligned}$$

On the contrary, assume that a vector such as $\hat{d} \in \mathbb{R}^n$ be a solution of the above system. Then, $\hat{d} \in \mathcal{L}_{MOMPEC}(Q, \bar{z})$, and since MOMPEC-GG-CQ is satisfied at $\bar{z}$, it follows that

$$\hat{d} \in \bigcap_{k \in \mathcal{I}_c} clco\mathcal{T}(Q^k, \bar{z}).$$

According to Proposition 3.7, this is a contradiction with the GPB-stationarity of $\bar{z}$. Therefore, the claim is correct.

Now, similar to the process of proving Theorem 3.8, it can be said that there exist nonnegative multipliers $\lambda_i^c > 0\ (i \in \mathcal{I}_c)$, $\lambda_i^a \ge 0\ (i \in \mathcal{I}_a)$, $\lambda_j^b, \mu_j^b \ge 0\ (j \in \mathcal{J})$, $\lambda_i^A, \mu_i^A \ge 0\ (i \in \alpha \cup \beta)$, $\lambda_i^B, \mu_i^B \ge 0\ (i \in \gamma)$, $\mu_i^B \ge 0\ (i \in \beta)$, such that

$$\begin{aligned}
0 \in cl \Bigg(&\sum_{i \in \mathcal{I}_c} \lambda_i^c co\partial^* c_i(\bar{z}) + \sum_{i \in \mathcal{I}_a} \lambda_i^a co\partial^* a_i(\bar{z})\\
&+ \sum_{j \in \mathcal{J}} \left[\lambda_j^b co\partial^* b_j(\bar{z}) + \mu_j^b co\partial^*(-b_j)(\bar{z})\right]\\
&+ \sum_{i \in \alpha \cup \beta} \left[\lambda_i^A co\partial^* A_i(\bar{z}) + \mu_i^A co\partial^*(-A_i)(\bar{z})\right]\\
&+ \sum_{i \in \gamma} \left[\lambda_i^B co\partial^* B_i(\bar{z}) + \mu_i^B co\partial^*(-B_i)(\bar{z})\right]\\
&+ \sum_{i \in \beta} \mu_i^B co\partial^*(-B_i)(\bar{z}) \Bigg).
\end{aligned}$$

Taking $\lambda_\gamma^A = \lambda_{\alpha\cup\beta}^B = \mu_\gamma^A = \mu_\alpha^B = 0$, we get the conditions (3.1)-(3.4) and $\lambda_i^B = 0, \forall i \in \beta$. This means $\bar{z}$ is a GPA-stationary point. □

In the two next results, we illustrate that under appropriate generalized convexity assumptions, GPS-stationarity and GPA-stationarity are each sufficient conditions for both weak Pareto optimality and Pareto optimality of MOMPEC.

Theorem 3.11. Assume that $\bar{z}$ be a GPS-stationary point of MOMPEC and consider the following index sets:

$$\alpha_\lambda^+ := \{i \in \alpha \,|\, \lambda_i^A > 0\}, \quad \gamma_\lambda^+ := \{i \in \gamma \,|\, \lambda_i^B > 0\}.$$

Also, suppose that $c_i\ (i \in \mathcal{I}_c)$ are ∂^*-pseudoconvex at $\bar{z}$ and $a_i\ (i \in \mathcal{I}_a)$, $\pm b_j\ (j \in \mathcal{J})$, $-A_i\ (i \in \alpha \cup \beta)$, $-B_i\ (i \in \gamma \cup \beta)$ are ∂^*-quasiconvex at $\bar{z}$. Then, we claim

(i). If $\alpha_\lambda^+ \cup \gamma_\lambda^+ = \emptyset$, then $\bar{z}$ is a weak Pareto optimal solution of MOMPEC.

(ii). If $c_i\ (i \in \mathcal{I}_c)$ are strictly ∂^*-pseudoconvex at $\bar{z}$, then $\bar{z}$ is a Pareto optimal solution of MOMPEC.

Proof .

(i). On the contrary, suppose that there exists $z_0 \in \Omega$ such that $c(z_0) < c(\bar{z})$. Due to the ∂^*-pseudoconvexity of $c_i\ (i \in \mathcal{I}_c)$ at $\bar{z}$,

$$\langle \xi_i^c, z_0 - \bar{z} \rangle < 0, \quad \forall \xi_i^c \in \partial^* c_i(\bar{z}),\ \forall i \in \mathcal{I}_c. \tag{3.5}$$

Now, let z be any feasible point of MOMPEC. Thus , we have $a_i(z) \le 0 = a_i(\bar{z})$, for each $i \in \mathcal{I}_a$. By the ∂^*-quasiconvexity of a_i $(i \in \mathcal{I}_a)$, it follows that,

$$\langle \xi_i^a, z - \bar{z} \rangle \le 0, \quad \forall \xi_i^a \in \partial^* a_i(\bar{z}),\ \forall i \in \mathcal{I}_a. \tag{3.6}$$

In a similar way, we can get

$$\langle \eta_j, z - \bar{z} \rangle \le 0, \quad \forall \eta_j \in \partial^* b_j(\bar{z}),\ \forall j \in \mathcal{J}, \tag{3.7}$$
$$\langle \nu_j, z - \bar{z} \rangle \le 0, \quad \forall \nu_j \in \partial^* (-b_j)(\bar{z}),\ \forall j \in \mathcal{J}, \tag{3.8}$$
$$\left\langle \xi_i^A, z - \bar{z} \right\rangle \le 0, \quad \forall \xi_i^A \in \partial^* (-A_i)(\bar{z}),\ \forall i \in \alpha \cup \beta, \tag{3.9}$$
$$\left\langle \xi_i^B, z - \bar{z} \right\rangle \le 0, \quad \forall \xi_i^B \in \partial^* (-B_i)(\bar{z}),\ \forall i \in \gamma \cup \beta. \tag{3.10}$$

From (3.5)-(3.10) and since according to assumptions $\bar{z}$ is a GPS-stationary point of MOMPEC, $\alpha_\lambda^+ \cup \gamma_\lambda^+ = \emptyset$, there exist the multipliers $\lambda_i^c > 0\,(i \in \mathcal{I}_c)$, $\lambda_i^a \ge 0\,(i \in \mathcal{I}_a)$, $\lambda_j^b \ge 0, \mu_j^b \ge 0\,(j \in \mathcal{J})$, $\mu_i^A \ge 0\,(i \in \alpha \cup \beta)$, $\mu_i^B \ge 0\,(i \in \gamma \cup \beta)$ such that

$$\begin{aligned} 0 = \Bigg\langle \sum_{i \in \mathcal{I}_c} \lambda_i^c \xi_i^c + \sum_{i \in \mathcal{I}_a} \lambda_i^a \xi_i^a + \sum_{j \in \mathcal{J}} \left[\lambda_j^b \eta_j + \mu_j^b \nu_j\right] \\ + \sum_{i=1}^{m} \left[\mu_i^A \xi_i^A + \mu_i^B \xi_i^B\right], z_0 - \bar{z} \Bigg\rangle < 0, \end{aligned} \tag{3.11}$$

for every $\xi_i^c \in \partial^* c_i(\bar{z})$, $\xi_i^a \in \partial^* a_i(\bar{z})$, $\eta_j \in \partial^* b_j(\bar{z})$, $\nu_j \in \partial^*(-b_j)(\bar{z})$, $\xi_i^A \in \partial^*(-A_i)(\bar{z})$, $\xi_i^B \in \partial^*(-B_i)(\bar{z})$. The contradiction (3.11) completes the proof of the part (i) of the Theorem.

(ii). The proof can be obtained in a similar way to the first part, hence we ignore it.

□

Theorem 3.12. Assume that $\bar{z}$ be a GPA-stationary point of MOMPEC and $\alpha_\lambda^+, \gamma_\lambda^+$ be defined as before. Also, consider the following index sets:

$$\beta_\lambda^A := \{ i \in \beta \,|\, \lambda_i^B = 0,\ \lambda_i^A > 0 \}, \quad \beta_\lambda^B := \{ i \in \beta \,|\, \lambda_i^B > 0,\ \lambda_i^A = 0 \}.$$

Moreover, suppose that c_i $(i \in \mathcal{I}_c)$ are ∂^*-pseudoconvex at $\bar{z}$ and a_i $(i \in \mathcal{I}_a)$, $\pm b_j$ $(j \in \mathcal{J})$, $-A_i$ $(i \in \alpha \cup \beta)$, $-B_i$ $(i \in \gamma \cup \beta)$ are ∂^*-quasiconvex at $\bar{z}$. Then the following claims hold:

(i). If $\alpha_\lambda^+ \cup \gamma_\lambda^+ \cup \beta_\lambda^A \cup \beta_\lambda^B = \emptyset$, then $\bar{z}$ is a weak Pareto optimal solution of MOMPEC.

(ii). If c_i $(i \in \mathcal{I}_c)$ are strictly ∂^*-pseudoconvex at $\bar{z}$, then $\bar{z}$ is a Pareto optimal solution of MOMPEC.

Proof . The proof follows the process of proving Theorem 3.11 and thus, we omit it. □

4 Conclusion

Since the Clarke and Michel–Penot subdifferentials of a locally Lipschitz function are upper convexificators, the results of this work are valid with the convexificators being replaced by these subdifferentials. On the other hand, a locally Lipschitz function may have a convexificator strictly smaller than the above subdifferentials. Thus there are situations that the above constraint qualification holds with some convexificators, and simultaneously fails to be satisfied by none of these subdifferentials.

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