Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 2037–2041
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.17721.1958

# New results for the best proximity pair in cone Riesz space

Ali Asghar Sarvari[a], Hamid Mazaheri Tehrani[b,*], Hamid Reza Khademzadeh[c]

[a]Computer Geometry and Dynamical Systems Laboratory, Yazd University, Yazd, Iran

[b]Faculty of Mathematics, Yazd University, Yazd, Iran

[c]Department of Mathematics, Technical and Vocational University (TVU), Tehran, Iran

(Communicated by Saman Babaie-Kafaki)

**Abstract**

In this paper, the best proximity pair problem is considered with a cone metric. the conditions for the existence and uniqueness of the best proximity pair problem is discussed by using interesting relationships in Riesz spaces. This problem is studied for $T$-absolutely direct sets. Also, given the conditions considered for this problem, it is shown for the cone cyclic contraction maps, the best proximity pair problem is uniquely solvable.

Keywords: The best proximity pair, Cone metric Riesz space, Order complete, Order convergence
2020 MSC: 41A65;, 41A42

## 1 Introduction

The problem of the best proximity pair is one of the significant issues that has called a lot of attention in recent years. In all relevant papers, the research done on metric space $(E, d)$ has made use of metric function $d : E \times E \to \mathbb{R}$. As examples, Eldred and Veeramani [5] discussed the best proximity pair problem for cyclic contraction maps on uniformly convex Banach spaces. This problem was examined for relatively nonexpansive maps [13] and pointwise contraction maps in [2]. The best approximation problem in Banach lattices is connected to monotonicity in [4, 6, 7, 9, 10, 11]. Afterwards, we will review some basic definitions in Riesz space $E$. If $E$ is a partially ordered vector space, then $E$ is called a Riesz space (or a vector lattice space) if $x \vee y = \sup\{x, y\}$, and $x \wedge y = \inf\{x, y\}$, both exist in $E$, for any $x, y \in E$. For any vector $x$ in Riesz space $E$, define $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$. The set $E^+ = \{x \in E : x \geq 0\}$ is called the positive cone of $E$. Riesz space $E$ is called Dedekind complete whenever every nonempty bounded above subset has a supremum (or equivalently, whenever every nonempty bounded below subset has an infimum). Also $E$ is said Archimedean if $x = 0$ holds whenever, $0 \leq nx \leq y \in E^+$ for all $n \in \mathbb{N}$. More details about Riesz spaces could be find in [1, 3, 12, 14].

## 2 Preliminaries

**Definition 2.1.** The mapping $d : E \times E \to E^+$ is said to be a cone metric on $E$ if it satisfies:

*Corresponding author
*Email addresses:* Sarvari_math@yahoo.com (Ali Asghar Sarvari), hmazaheri@yazduni.ac.ir (Hamid Mazaheri Tehrani), hrkhademzadeh@gmail.com (Hamid Reza Khademzadeh)

*Received:* April 2019 *Accepted:* May 2020

(a) $d(x,y) = 0$ if and only if $x = y$.
(b) $d(x,y) = d(y,x)$ for all $x, y \in E$.
(c) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in E$.

In this way, we recognize $(E, d)$ as a cone metric Riesz space. We define $d(x,y) = |x - y| \in E^+$ for any $x, y \in E$, so $(E, |\cdot|)$ is a cone metric Riesz space. Recall that $f_n \downarrow f$ it means the sequence $\{f_n\} \subseteq E$ is decreasing and $f = \inf f_n$ in $E$.

In the continuation of the article, it is supposed that $A$ and $B$ are two nonempty subsets of Riesz space $E$, and $T : A \to B$ is an arbitrary map, and $Dist(A,B) = \bigwedge |A - B| = \inf\{|x - y| : x \in A, y \in B\}$ exists in the set $|A - B|$. It means, there exist $a \in A$ and $b \in B$ such that $Dist(A,B) = |a - b|$. For instance, if $E$ is Dedekind complete and $|A - B|$ is order closed, then there exist $a \in A$ and $b \in B$ such that $Dist(A,B) = |A - B|$ ($|A - B|$ is a bounded below subset of $E$). Let $x \in A$. If $|x - TX| = Dist(A,B)$, we say $(x, TX)$ is a cone best proximity point for $T$. We show the set of all such points by $P^c_T(A,B)$, i.e., $P^c_T(A,B) = \{x \in A : |x - Tx| = Dist(A,B)\}$.

**Definition 2.2.** Let $(E, |\cdot|)$ be a cone metric Riesz space and $u, u_n \in E$ $(n = 1, 2, \ldots)$.

(a) The sequence $\{u_n\}$ is order convergence to $u$ if there exists a sequence $f_n \downarrow 0$ such that $|u - u_n| \leq f_n$ holds for any $n \in \mathbb{N}$. (In symbols $u_n \xrightarrow{o} u$). Also the subset $A \subseteq E$ is order closed whenever for the all sequences $\{x_n\} \subseteq A$ such that $x_n \xrightarrow{o} x$, imply $x \in A$.
(b) The sequence $\{u_n\}$ is order Cauchy if there exists a sequence $f_n \downarrow 0$ which $|u_n - u_m| \leq f_n$ for all $n \geq m \geq 1$. Clearly, order convergence sequences are order Cauchy.
(c) The cone metric Riesz space $(E, |\cdot|)$ is order complete if any order Cauchy sequence is order convergence.
(d) The mapping $T : A \cup B \to A \cup B$ is a cone cyclic contraction map if $T$ is cyclic ($T(A) \subseteq B$ and $T(B) \subseteq A$) and also $|Tx - Ty| \leq k|x - y| + (1 - k)Dist(A,B)$ for some $k \in (0,1)$ and any $(x,y) \in A \times B$.

It should be mentioned that the order convergence in a Riesz space $E$ does not necessarily correspond to a topology on $E$.

**Remark 2.3.** Let $C(K)$ be the set of all real continuous functions on $K$ by ordering $f_1 \leq f_2$ if $f_1(x) \leq f_2(x)$ for any $x \in K$. We know $f_1 \vee f_2 = \frac{1}{2}(f_1 + f_2) + \frac{1}{2}|f_1 - f_2| \in C(K)$ and $f_1 \wedge f_2 = \frac{1}{2}(f_1 + f_2) - \frac{1}{2}|f_1 - f_2| \in C(K)$ for any $f_1, f_2 \in C(K)$. Therefore $C(K)$ is a Riesz space. Also, if $K$ is a Hausdorff topological space, compact and extremally disconnected (i.e., the closure of any open set is open) then $C(K)$ is order complete and Dedekind complete [8].

**Definition 2.4.** Let $(E, |\cdot|)$ be a cone metric Riesz space.

(a) A sequence $\{x_n\} \subseteq A$ is said to be a cone $T$-minimizing sequence in $A$ whenever $|x_n - Tx_n| \xrightarrow{o} Dist(A,B)$.
(b) The subset $A \subseteq E$ is a $T$-absolutely direct set if for any $x, y \in A$, there exists $z \in A$ such that

$$|z - Tx| \leq |x - Tx| \wedge |y - Tx| \quad \text{and} \quad |z - Ty| \leq |x - Ty| \wedge |y - Ty|$$

**Example 2.5.** Suppose $A \subseteq E$ is a sublattice, it means $x \vee y$ and $x \wedge y$ both exist in $A$ for any $x, y \in A$, and also $A \geq B$ (or $B \geq A$). Then $A$ is a $T$-absolutely direct set.
The notation $A \geq B$ means that $a \geq b$ for any $a \in A$ and $b \in B$.

Cone best proximity pair problem is $T$-solvable ($T$-uniquely solvable) if $P^c_T(A,B) \neq \emptyset$ ($card\ P^c_T(A,B) = 1$).

## 3 Main results

In this part, we aim to provide conditions to investigate the existence and uniqueness of cone best proximity pair problem.

**Theorem 3.1.** Let $(E, |\cdot|)$ be a cone metric Riesz space and $A \subseteq E$ be a convex $T$-absolutely direct set. Then $card\ P^c_T(A,B) \leq 1$

**Proof .** Suppose there exist $x, y \in A$ such that $|x - Tx| = |y - Ty| = Dist(A, B)$. Given $A$ is a $T$-absolutely direct set, there exists $z \in A$ such that

$$|z - Tx| \leq |x - Tx| \wedge |y - Tx| \quad \text{and} \quad |z - Ty| \leq |x - Ty| \wedge |y - Ty|.$$

Thus $|z - Tx| = |x - Tx|$ and $|z - Ty| = |y - Ty|$. Since $A$ is convex, we have $\frac{x+z}{2} \in A$. Therefore,

$$|x - Tx| \leq \left| \frac{x+z}{2} - Tx \right| \leq \frac{|x - Tx| + |z - Tx|}{2} = |x - Tx|.$$

Notic that every Riesz space $E$ has the following property

$$|f + g| + |f - g| = 2(|f| \vee |g|), \qquad (\forall f, g \in E) \tag{3.1}$$

which leads to

$$|x - z| = 2(|x - Tx| \vee |z - Tx|) - |x + z - 2Tx| = 0,$$

that is $x = z$. in the same way, we obtain $y = z$, and as a result, $card\ P_T^c(A, B) \leq 1$. □

**Example 3.2.** Suppose $E = \mathbb{R}^2$ with coordinatewise ordering (i.e., $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$). Put $A = \{(1, x) : x \in \mathbb{R}\}$, $B = \{(2, x) : x \in \mathbb{R}\}$ and $T : A \to B$ defined by

$$T(1, x) = \begin{cases} (2, 1), & x \in \mathbb{Q} \\ (2, \sqrt{2}), & x \notin \mathbb{Q} \end{cases}$$

W can see that $Dist(A, B) = (1, 0)$ and $A$ is a convex set, but $A$ is not a $T$-absolutely direct set. It is easy to see that $P_T^c(A, B) = \{(1, 1), (1, \sqrt{2})\}$.

**Theorem 3.3.** Let $(E, |\cdot|)$ be order complete, $A \subseteq E$ be an order closed and a convex $T$-absolutely direct set. Then any cone $T$-minimizing sequence in $A$, is order convergence.

**Proof .** Suppose the sequence $\{y_n\} \subseteq A$ is a cone $T$-minimizing sequence in $A$. Then $|y_n - Ty_n| \xrightarrow{o} Dist(A, B)$, so there exists a sequence $f_n \downarrow 0$ such that $0 \leq |y_n - Ty_n| - Dist(A, B) \leq f_n$. We prove the sequence $\{y_n\}$ is order Cauchy, it means there exists a sequence $g_n \downarrow 0$ such that $|y_{n+k} - y_n| \leq g_n$ $(n, k \in \mathbb{N})$. Since $A$ is a $T$-absolutely direct set, there exists $x_n \in A$ which

$$|x_n - Ty_n| \leq |y_n - Ty_n| \wedge |y_{n+k} - Ty_n|$$

and

$$|x_n - Ty_{n+k}| \leq |y_n - Ty_{n+k}| \wedge |y_{n+k} - Ty_{n+k}|.$$

$A$ is convex so $\frac{x_n + y_n}{2} \in A$, Therefore

$$Dist(A, B) \leq \left| \frac{x_n + y_n}{2} - Ty_n \right| \leq \frac{|x_n - Ty_n| + |y_n - Ty_n|}{2} \leq |y_n - Ty_n| \xrightarrow{o} Dist(A, B).$$

Hence

$$0 \leq \left| \frac{x_n + y_n}{2} - Ty_n \right| - Dist(A, B) \leq |y_n - Ty_n| - Dist(A, B) \leq f_n \downarrow 0,$$

so

$$\left| \frac{x_n + y_n}{2} - Ty_n \right| \xrightarrow{o} Dist(A, B). \tag{3.2}$$

By (3.1) and (3.2), we have

$$|y_n - x_n| = 2(|x_n - Ty_n| \vee |y_n - Ty_n|) - |x_n + y_n - 2Ty_n| = 2|y_n - Ty_n| - |x_n + y_n - 2Ty_n|.$$

Thus $|y_n - x_n| \xrightarrow{o} 0$. (It is easy to see that if $a_n \xrightarrow{o} a$ and $b_n \xrightarrow{o} b$ then $\alpha a_n + \beta b_n \xrightarrow{o} \alpha a + \beta b$ for each $\alpha, \beta \in \mathbb{R}$).

Similarly, it can be concluded that $|y_{n+k} - x_n| \xrightarrow{o} 0$. Therefore there exist the sequences $p_n \downarrow 0$ and $q_n \downarrow 0$ such that $|y_n - x_n| \leq p_n$ and $|y_{n+k} - x_n| \leq q_n$, and as a result $|y_{n+k} - y_n| \leq p_n + q_n \downarrow 0$. □ It is necessary to mention that $f : A \to B$ is a $\sigma$-order continuse map if $f(x_n) \xrightarrow{o} f(x)$ for all sequences $\{x_n\} \subseteq A$ such that $x_n \xrightarrow{o} x$.

**Corollary 3.4.** Let $(E, |\cdot|)$ be order complete and $T : A \to B$ be a $\sigma$-order continuous map. Let $A$ be order closed, and a convex $T$-absolutely direct set. If $A$ has a cone $T$-minimizing sequence then cone best proximity pair problem is $T$-uniquely solvable.

**Proof .** Suppose that $\{x_n\} \subseteq A$ is a cone $T$-minimizing sequence, by theorem 3.3, $x_n \xrightarrow{o} x$ for some $x \in A$. Therefore $Tx_n \xrightarrow{o} Tx$, so $|x_n - Tx_n| \xrightarrow{o} |x - Tx|$. Since order limits are uniquely determined, so $|x - Tx| = Dist(A, B)$. Also by theorem 3.1, $card\ P^c_T(A, B) \le 1$. Thus $card\ P^c_T(A, B) = 1$. □

**Theorem 3.5.** Let $(E, |\cdot|)$ be an Archimedean cone metric Riesz space and $T : A \cup B \to A \cup B$ be a cone cyclic contraction map. If $x_0 \in A$ and $x_{n+1} = Tx_n = T^{n+1}x_0$ $(n = 0, 1, 2, \ldots)$, then the sequence $\{x_{2n}\} \subseteq A$ is a cone $T$-minimizing sequence in $A$.

**Proof .** We can see that $|x_{n+1} - x_n| = |Tx_n - Tx_{n-1}| \le k|x_n - x_{n-1}| + (1 - k)Dist(A, B) \le k^2|x_{n-1} - x_{n-2}| + (1 - k^2)Dist(A, B)$. By induction, we obtain

$$|x_{n+1} - x_n| \le k^n|x_1 - x_0| + (1 - k^n)Dist(A, B).$$

Since $E$ has Archimedean property, we have $|x_{n+1} - x_n| - Dist(A, B) \le k^n u \downarrow 0$ which $u = |x_1 - x_0| - Dist(A, B) \in E^+$. Thus $|Tx_n - x_n| \xrightarrow{o} Dist(A, B)$. As a result, $\{x_{2n}\} \subseteq A$ is a cone $T$-minimizing sequence in $A$. □

**Theorem 3.6.** Let $(E, |\cdot|)$ be Archimedean and order complete. Let $A \subseteq E$ be an order closed sublattice and $A \ge B$ (or $B \ge A$). If $T : A \cup B \to A \cup B$ is a cone cyclic contraction map then cone best proximity pair problem is uniquely solvable.

**Proof .** Assume $A \ge B$ (the proof is similar to the other), and $x - Tx = y - Ty = Dist(A, B)$ for some $x, y \in A$. Since $A$ is a sublattice, $x \wedge y \in A$, so $0 \le x - Tx \le x \wedge y - Tx$ and $0 \le y - Ty \le x \wedge y - Ty$. Thus $x \le x \wedge y$ and $y \le x \wedge y$, it means $x = y$ and $card\ P^c_T(A, B) \le 1$.

Suppose $x_0 \in A$ and define $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$. By theorem 3.5, the sequence $\{x_{2n}\} \subseteq A$ is a cone $T$-minimizing sequence in $A$. So based on theorem 3.3, there exists $x \in A$ that $x_{2n} \xrightarrow{o} x$. In other words, there exist the sequences $f_n \downarrow 0$ and $g_n \downarrow 0$ such that $|x_{2n} - x| \le f_n$ and $|x_{2n} - Tx_{2n}| - Dist(A, B) \le g_n$. Now, $0 \le |x - Tx_{2n}| - Dist(A, B) \le |x - x_{2n}| + |x_{2n} - Tx_{2n}| - Dist(A, B) \le f_n + g_n \downarrow 0$. Also $0 \le |x_{2n+2} - Tx| - Dist(A, B) = |Tx_{2n+1} - Tx| - Dist(A, B) \le |x_{2n+1} - x| - Dist(A, B) = |Tx_{2n} - x| - Dist(A, B) \le f_n + g_n \downarrow 0$.

Finally, $0 \le |x - Tx| - Dist(A, B) \le |x - x_{2n}| + |x_{2n} - Tx| - Dist(A, B) \le 2f_n + g_n \downarrow 0$. Therefore $|x - Tx| = Dist(A, B)$, that complete the proof. □

## References

[1] D. Aliprantis and O. Burkinshaw, *Positive operators*, Springer, Dordrecht, 2006.

[2] J. Anuradha and P. Veeramani, *Proximal pointwise contraction*, Topology Appl. **156** (2009), no. 18, 2942–2948.

[3] G. Birkhoff, *Lattice theory*, American Mathematical Society, Providence, R.I., 1979.

[4] Shu Tao Chen, Xin He, and H. Hudzik, *Monotonicity and best approximation in Banach lattices*, Acta Math. Sin. (Engl. Ser.) **25** (2009), no. 5, 785–794.

[5] A. A. Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl. **323** (2006), no. 2, 1001–1006.

[6] P. Foralewski, H. Hudzik, W. Kowalewski, and M. Wisł a, *Monotonicity properties of Banach lattices and their applications: A survey*, Ordered structures and applications, Trends Math., Birkhäuser/Springer, Cham, 2016, pp. 203–232.

[7] H. Hudzik and W. Kurc, *Monotonicity properties of Musielak-Orlicz spaces and dominated best approximation in Banach lattices*, J. Approx. Theory **95** (1998), no. 3, 353–368.

[8] A. F. Kalton and J. Nigel, *Topics in Banach space theory*, Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006.

[9] H. R. Khademzadeh and H. Mazaheri, *Monotonicity and the dominated farthest points problem in Banach lattice*, Abstr. Appl. Anal. (2014), Art. ID 616989, 7.

[10] N. S. Kukushkin, *Increasing selections from increasing multifunctions*, Order **30** (2013), no. 2, 541–555.

[11] W. Kurc, *Strictly and uniformly monotone Musielak-Orlicz spaces and applications to best approximation*, J. Approx. Theory **69** (1992), no. 2, 173–187.

[12] P. Meyer-Nieberg, *Banach lattices.*, Springer-Verlag, Berlin, 1991.

[13] V. Sankar Raj and P. Veeramani, *Best proximity pair theorems for relatively nonexpansive mappings*, Appl. Gen. Topol. **10** (2009), no. 1, 21–28.

[14] A. C. Zaanen, *Introduction to operator theory in Riesz spaces*, Springer-Verlag, Berlin, 1997.