Int. J. Nonlinear Anal. Appl.
Volume 12, Special Issue, Winter and Spring 2021, 2339-2352
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.6269

New results on fourth-order Hankel determinants for convex functions related to the sine function

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(Communicated by Ehsan Kozegar)

Abstract

In this paper, we give an upper bound for the fourth Hankel determinant $H_4(1)$ for a new class $\mathcal{S}_{\mathcal{C}}^{\#}$ associated with the sine function.

Keywords: analytic functions, univalent functions, fourth-order Hankel determinant, subordination, convex function, sine function
2010 MSC: 30C45, 47D09

1. Introduction

Assume the class of the functions f be analytic say A, in the open unitd disk $\mathcal{V} = \{z \in \mathbb{C} : |z| < 1\}$, represented by:

$$f(z) = z + a_2 z^2 + a_3 z^3 + ..., \ z \in \mathcal{V} \tag{1.1}$$

Assume $\mathcal{C}$ be subclass of $\mathcal{A}$ consist of univalent function, let $\mathcal{F}$ be a class of analytic functions L normalized by:

$$\mathrm{L}(z) = 1 + v_1 z + v_2 z^2 + v_3 z^3 + ..., \tag{1.2}$$

and satisfy the inequality below

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Received: October 2021 *Accepted:* December 2021

$$Re(\mathrm{L}(z)) > 0,\ z \in \mathcal{V}.$$

Now, let the two analytic function in $\mathcal{V}$ say f and g.

Thus, the function g be subordinate to the function f, and represented by $f(z) \prec g(z),\ z \in \mathcal{V}$. if the Schwarz function $\mathcal{W}(z)$ exists, with $|\mathcal{W}(z)| < 1$ and $\mathcal{W}(0) = 0$ [see more [21]], $g(z) = f(\mathcal{W}(z)),\ z \in \mathcal{V}$.

Cho et al. [9] introduced the following function class $\mathcal{S}_s^*$, in 2018:

$$\mathcal{S}_s^* = \left\{ \frac{zf'(z)}{f(z)} \prec 1 + \sin z,\ f \in \mathcal{A},\ z \in \mathcal{V} \right\},$$

The quantity $\frac{zf'(z)}{f(z)}$ lies in an eight-shaped region in the right-half plane.

We introduced the new class.

Definition 1.1. *Assume the function $f \in \mathcal{A}$ which given by (1.1) is said to be convex function class $\mathcal{S}_{\mathcal{C}}^{\#}$:*

$$\mathcal{S}_{\mathcal{C}}^{\#} = \left\{ 1 + \frac{zf''(z)}{f'(z)} \prec 1 + \sin z,\ f \in \mathcal{A},\ z \in \mathcal{V} \right\} \tag{1.3}$$

implies that the quantity $1 + \frac{zf''(z)}{f'(z)}$ lies in an eight-shaped region in the right- half plane.

In 1976 Noonan and Thomas [23] stated the $\mathfrak{q}^{th}$ Hankel determinant for $\mathfrak{q} \geq 1$ and $n \geq 1$ of functions f as follows:

$$H_{\mathfrak{q}}(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+\mathfrak{q}-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+\mathfrak{q}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+\mathfrak{q}-1} & a_{n+\mathfrak{q}} & \cdots & a_{n+2\mathfrak{q}-2} \end{vmatrix},\ a_1 = 1.$$

In particular, we have

For $\mathfrak{q} = 2,\ n = 1$ and $a_1 = 1,\ H_2(1) = a_3 - a_2^2$ is the well-known Fekete-Szegö functional. The second Hankel $H_2(2)$ defined as $H_2(2) = a_2a_4 - a_3^2$ for $\mathfrak{q} = 2,\ n = 2$ was studied for the classes of bi-starlike and bi-convex functions (see [1, 2, 4, 5, 10, 12, 15, 16, 25, 31]). The third Hankel determinant is given as:

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix},\ \mathfrak{q}_1 = 3,\ n = 1,$$

$H_3(1)$ whose elements are various classes of analytic functions, it is worth mentioning that [6, 7, 8, 12, 13, 17, 18, 20, 27, 28, 29, 30, 32, 34]. For instance, Murugusundaramoorthy and Bulboacă [22] defined the subclass of analytic functions $M\mathcal{L}_c^a(\lambda, \phi)$ and have upper bounds for the Fekete-Szegö functional, addition and the Hankel determinant of order two for $f \in M\mathcal{L}_c^a(\lambda, \phi)$. Islam et. al [11] introduced the $\mathfrak{q}-$ analog of starlike function linked with a trigonometric sine function and debate some geometric properties, where the well-known problems of Fekete-Szegö, the sufficient and necessary condition, the distortion and growth bound, closure theorem and convolution results with partial sums for this class. Zaprawa et al. [33] introduced the bound of the third Hankel determinant for the univalent starlike functions. Arif et al. [3] study the fourth Hankel determinant $H_4(1)$. Khan et al. [14] introduced and discussed many classes of functions with bounded turning which are

connected to the sine functions and obtained upper bounds for the third-and fourth –order Hankel determinants related to such classes.

Now, since $f \in \mathcal{C},\ a_1 = 1$ *thus*

$$\begin{aligned} H_4(1) = &\{(a_2a_4 - a_3^2)a_3 - (a_4 - a_2a_3)a_4 + (a_3 - a_2^2)a_5\}a_7 - \{(a_2a_5 - a_3a_4)a_3 - (a_5 - a_2a_4)a_4 \\ &+(a_3 - a_2^2)a_6\} a_6 + \{(a_3a_5 - a_4^2)a_3 - (a_5 - a_2a_4)a_5 + a_4 - a_2a_3\}a_6 + \left\{(a_3a_5 - a_4^2)a_3 \right. \\ &\left. -(a_5 - a_2a_4)a_5 + (a_4 - a_2a_3)a_6\right\} a_5 - \{(a_3a_5 - a_4^2)a_4 - (a_2a_5 - a_3a_4)a_5 + (a_4 - a_2a_3)a_6\}a_4. \end{aligned} \tag{1.4}$$

The first original result is the definition of the new concept on fourth Hankel determinant.

Definition 1.2. *The fourth Hankel determinant of a function* f *of the form* (1.1) *is defined by:*

$$H_4(1) = \begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix} = \delta_1 a_7 + \delta_2 a_6 + \delta_3 a_5 + \delta_4 a_4,$$

such that

$$\begin{aligned} |\delta_1| &= |a_2a_4 - a_3^2||a_3| + |a_4 - a_2a_3||a_4| + |a_3 - a_2^2||a_5|. \\ |\delta_2| &= |a_2a_5 - a_3a_4||a_3| + |a_5 - a_2a_4||a_4| + |a_3 - a_2^2||a_6|. \\ |\delta_3| &= |a_3a_5 - a_4^2||a_3| + |a_5 - a_2a_4||a_5| + |a_4 - a_2a_3||a_6|. \\ |\delta_4| &= |a_3a_5 - a_4^2||a_4| + |a_2a_5 - a_3a_4||a_5| + |a_4 - a_2a_3||a_6|. \end{aligned}$$

2. Preliminaries

To give argument to our main results, following lemmas are needed.

Lemma 2.1. *[19] If* $\mathrm{L}(z) \in \mathcal{A}$, *then exists some* x, z *with* $|z| \le 1,\ |x| \le 1$, *where*
$2v_2 = v_1^2 + x(4 - v_1^2)$.
$4v_3 = v_1^3 + 2v_1x(4 - v_1^2) - (4 - v_1^2)v_1x^2 + 2(4 - v_1^2)(1 - |x|)z$.

Lemma 2.2. *[26] Assume* $\mathrm{L}(z) \in \mathcal{A}$. *Then*
$|v_1^4 + v_2^2 + 2v_1v_3 - 3v_1^2v_2 - v_4| \le 2$,
$|v_1^5 + 3v_1v_2^2 + 3v_1^2v_3 - 4v_1^3v_2 - 2v_1v_4 - 2v_2v_3 + v_5| \le 2$,
$|v_1^6 + 6v_1^2v_2^2 + 4v_1^3v_3 + 2v_1v_5 + 2v_2v_4 + v_3^2 - v_2^3 - 5v_1^4v_2 - 3v_1^2v_4 - 6v_1v_2v_3 - v_6| \le 2$,
$|v_n| \le 2,\ n = 1, 2, 3, \dots$.

Lemma 2.3. *[24] Assume* $\mathrm{L}(z) \in \mathcal{A}$, *then we have*
$\left|v_2 - \frac{v_1^2}{2}\right| \le 2 - \frac{|v_1|^2}{2}$,
$|v_{n+k} - \rho v_n v_k| \le 2,\ 0 \le \rho \le 1$,
$|v_{n+2k} - \rho v_n v_k^2| \le 2(1 + 2\rho)$.

3. Main Results

We give now the state and prove our theorems, represent the main results of this paper.

Theorem 3.1. *Assume the function* $f \in \mathcal{S}_{\mathcal{C}}^{\#}$ *and represented by* (1.1), *then*

$$|a_2| \leq \frac{1}{2},\ |a_3| \leq \frac{1}{6},\ |a_4| \leq 0.0886,\ |a_5| \leq \frac{3}{40},\ |a_6| \leq \frac{191}{4320},\ |a_7| \leq 0.2346$$

Proof . Because $f \in \mathcal{S}_{\mathcal{C}}^{\#}$, according to subordination relation, therefore, there exists a Schwarz function $\mathcal{W}(z)$ with $|\mathcal{W}(z)| < 1$ and $\mathcal{W}(0) = 0$, satisfying $\frac{zf''(z)}{f'(z)} = 1 + \sin(\mathcal{W}(z))$, such that,

$$\begin{aligned}1 + \frac{zf''(z)}{f'(z)} &= 1 + 2a_2 z + (6a_3 - 4a_2^2)z^2 + (12a_4 - 18a_2a_3 + 8a_2^3)z^3 + \big(20a_5 - 18a_3^2 - 32a_2a_4 + 48a_2^2a_3 \\ &\quad -16a_2^4\big)\, a^4 + (30a_6 - 2a_3^2 - 50a_2a_5 + 80a_2^2a_4 - 50a_3a_4 + 90a_2a_3^2 - 120a_2^3 + 32a_2^5)a^5 \\ &\quad + \big(42a_7 - 72a_2a_6 - 30a_3a_5 + 120a_2^2a_5 - 48a_4^2 + 192a_2a_3a_4 - 192a_2^3a_4 - 180a_2^2a_3^2 \\ &\quad +240a_2^4a_3 - 64a_2^6\big)\, a^6 + ... \end{aligned} \tag{3.1}$$

Assume that
$\mathbb{p}(z) = \frac{1+\mathcal{W}(z)}{1-\mathcal{W}(z)} = 1 + v_1 z + v_2 z^2 + v_3 z^3 + ...,$
it is clear to see that $\mathbb{p}(z) \in \mathcal{F}$ and
$\mathcal{W}(z) = \frac{\mathbb{p}(z)-1}{1+\mathbb{p}(z)} = \frac{v_1z+v_2z^2+v_3z^3+...}{2+1+v_1z+v_2z^2+v_3z^3+...}.$
On the other side,

$$\begin{aligned}1 + \sin(\mathcal{W}(z)) &= 1 + \frac{1}{2}v_1 z + \left(\frac{v_2}{2} - \frac{v_1^2}{4}\right) z^2 + \left(\frac{5v_1^3}{48} + \frac{v_4 - v_1v_3}{2} + \frac{5v_1^2v_2}{16} - \frac{v_2^2}{4} - \frac{v_1^4}{32}\right) z^4 \\ &\quad + \left(\frac{v_5 - v_1v_4 - v_2v_3}{2} + \frac{5v_1^2v_3 + v_1v_2^2}{16} - \frac{v_1^3v_2}{8} + \frac{v_1^5}{3840}\right) z^5 + \left(\frac{v_6 - v_1v_5 - v_2v_4}{2}\right. \\ &\quad \left. + \frac{5v_1v_2v_3}{8} + \frac{5v_2^3}{48} - \frac{v_3^2}{4} + \frac{5v_1^6}{512} + \frac{v_1^4v_2}{768} - \frac{3v_1^2v_2^2}{16} + \frac{5v_1^2v_4}{16} - \frac{v_1^3v_3}{8}\right) z^6 + ... \end{aligned} \tag{3.2}$$

Comparing the coefficients of $z, z^2, ..., z^6$ between equations (3.1) and (3.2), we have

$$\begin{aligned} 2a_2 &= \frac{1}{2}v_1, \\ 6a_3 &= \frac{v_2}{2} - \frac{v_1^2}{4} + 4\left(\frac{v_1}{4}\right)^2, \\ a_4 &= \frac{5v_1^3}{576} + \frac{v_3}{24} + \frac{v_1^2}{24} - \frac{v_1v_2}{96}, \\ 20a_5 &= \frac{a_4 - a_1a_3}{2} + \frac{5a_1^2a_2}{16} - \frac{a_2^2}{4} - \frac{a_1^4}{32} + 18\left(\frac{a_2}{12}\right)^2 + 32\left[\frac{v_1}{4}\left(\frac{v_3}{24} - \frac{v_1v_2}{96} - \frac{v_1^3}{576}\right)\right] - 48\left[\left(\frac{v_1}{4}\right)^2 \frac{v_2}{12}\right] \\ &\quad + 16\left(\frac{v_1}{4}\right)^4, \end{aligned}$$

$$30a_6 = \frac{v_5 - v_1v_4 - v_2v_3}{2} + \frac{5v_1^2v_3 + v_1v_2^2}{16} - \frac{v_1^3v_2}{8} + \frac{v_1^5}{3840} + 2\left(\frac{v_2}{12}\right)^2 + 50\left[\frac{v_1}{4}\left(\frac{v_4}{40} - \frac{v_1v_3}{120} + \frac{v_1}{1152}\right.\right.$$
$$\left.\left. - \frac{v_1^2v_2}{960} - \frac{v_2^2}{160}\right)\right] - 80\left[\left(\frac{v_1}{4}\right)^2\left(\frac{v_3}{24} - \frac{v_1v_2}{96} - \frac{v_1^3}{576}\right)\right] + 50\left[\frac{v_2}{12}\left(\frac{v_3}{24} - \frac{v_1v_2}{96} - \frac{v_1^3}{576}\right)\right]$$
$$- 90\left[\frac{v_1}{4}\left(\frac{v_2}{12}\right)^2\right] + 120\left[\left(\frac{v_1}{4}\right)^3\frac{v_2}{12}\right] - 32\left[\left(\frac{v_1}{4}\right)^5\right].$$
$$42a_7 = \frac{v_6 - v_1v_5 - v_2v_4}{2} + \frac{5v_1v_2v_3}{8} + \frac{5v_2^3}{48} - \frac{v_3^2}{4} + \frac{5v_1^6}{512} + \frac{v_1^4v_2}{768} - \frac{3v_1^2v_2^2}{16} + \frac{5v_1^2v_4}{16} - \frac{v_1^3v_3}{8} + 72\left[\frac{v_1}{4}\left(\frac{v_5}{60}\right.\right.$$
$$\left.\left. - \frac{v_1v_4}{160} - \frac{47v_2v_3}{4320} - \frac{49v_1v_2^2}{17280} + \frac{109v_1^3v_2}{51840} + \frac{38v_1^5}{57600} - \frac{37v_2^2}{17280}\right)\right] + 30\left[\frac{v_2}{12}\left(\frac{v_4}{40} - \frac{v_1v_3}{120} + \frac{v_1^4}{1152} - \frac{v_1^2v_2}{960}\right.\right.$$
$$\left.\left. - \frac{v_2^2}{160}\right)\right] - 120\left[\left(\frac{v_1}{4}\right)^2\left(\frac{v_4}{40} - \frac{v_1v_3}{120} + \frac{v_1^4}{1152} - \frac{v_1^2v_2}{960} - \frac{v_2^2}{160}\right)\right] + 48\left[\frac{v_3}{24} - \frac{v_1v_2}{96} - \frac{v_1^3}{576}\right]^2$$
$$- 19\left[\frac{v_1v_2}{48}\left(\frac{v_3}{24} - \frac{v_1v_2}{96} - \frac{v_1^3}{576}\right)\right] + 192\left[\left(\frac{v_1}{4}\right)^3\left(\frac{v_3}{24} - \frac{v_1v_2}{96} - \frac{v_1^3}{576}\right)\right] + 64\left(\frac{v_1}{4}\right)^6$$
$$+ 180\left[\left(\frac{v_1}{4}\right)^2\left(\frac{v_2}{12}\right)^2\right] - 240\left[\left(\frac{v_1}{4}\right)^4\left(\frac{v_2}{12}\right)\right],$$

and we obtain

$$a_2 = \frac{1}{4}v_1$$
$$a_3 = \frac{1}{12}v_2$$
$$a_4 = \frac{5v_1^3}{576} + = \frac{v_3}{24} + = \frac{v_1^2}{24} - \frac{v_1v_2}{96}$$
$$a_5 = \frac{v_4}{40} - \frac{v_1v_3}{120} + \frac{v_1^4}{1152} - \frac{v_1^2v_2}{960} - \frac{v_2^2}{32} \tag{3.3}$$

$$a_6 = \frac{-v_1v_4}{160} - \frac{47v_2v_3}{4320} + \frac{38v_1^5}{57600} - \frac{49v_2^2v_1}{17280} + \frac{109v_1^3v_2}{51840} - \frac{37v_2^2}{17280} + \frac{v_5}{60}, \tag{3.4}$$

$$a_7 = \frac{v_1^2v_4}{3360} + \frac{1777v_1v_2v_3}{241920} + \frac{10379v_1^6}{14515200} - \frac{5071v_1^2v_2^2}{967680} - \frac{1314029v_1^4v_2}{1255274496} - \frac{v_1v_5}{210} - \frac{7v_2v_4}{672} + \frac{17v_2^3}{8064} + \frac{v_6}{84}$$
$$- \frac{v_3^2}{504} - \frac{111v_1v_2^2}{120960} + \frac{v_1^4}{24192} - \frac{5v_1^3v_3}{3024} \tag{3.5}$$

Applying Lemma 2.2, we obtain

$$|a_2| \le \frac{1}{2}$$
$$|a_3| \le \frac{1}{6}$$
$$|a_4| \le \left|\frac{v_3}{24} - \frac{v_1v_2}{96} - \frac{v_1^3}{576}\right| = \left|\frac{1}{24}\left[v_3 - \frac{v_1v_2}{3}\right] + \frac{v_1}{288}\left[v_2 - \frac{v_1^2}{2}\right]\right|$$

Assume $v_1 = v,\ v \in [0,2]$, by applying Lemma 2.3, we have

$$|a_4| \leq \left| \frac{1}{24}\left[v_3 - \frac{v_1 v_2}{3}\right] + \frac{v_1}{288}\left[v_2 - \frac{v_1^2}{2}\right] \right| \leq \frac{1}{12} + \frac{v\left(2-\frac{v^2}{2}\right)}{288},$$

as well, assume

$$\mu(v) = \frac{1}{12} + \frac{v\left(2-\frac{v^2}{2}\right)}{288},$$

clearly, to obtain

$$\mu'(v) = \frac{1}{144} - \frac{v^2}{192},$$

take $\mu'(v) = 0$, we have $v = \frac{2}{\sqrt{3}}$, and so, $\mu(v)$ has a maximum value satisfied at $v = \frac{2}{\sqrt{3}}$, as well which is

$$|a_4| \leq \mu\left(\frac{2}{\sqrt{3}}\right) = \frac{1}{2} + \frac{1}{108\sqrt{3}} = 0.0886.$$

$$|a_5| \leq \left|\frac{v_4}{40} - \frac{v_1 v_3}{120} + \frac{v_1^4}{1152} - \frac{v_1^2 v_2}{960} - \frac{v_2^2}{160}\right| = \left|\frac{1}{40}\left[v_4 - \frac{v_1 v_3}{3}\right] - \frac{v_1^2}{576}\left[v_2 - \frac{v_1^2}{2}\right] - \frac{v_2}{160}\left[v_2 - -\frac{v_1^2}{2}\right] - \frac{7v_1^2 v_2}{576}\right| \tag{3.6}$$

Assume that $v_1 = v,\ v \in [0,2]$, by Lemma 2.3, to get

$$|a_5| \leq \frac{1}{20} + \frac{5v^2\left(2-\frac{v^2}{2}\right)}{576} + \frac{1}{80}\left(2 - \frac{v^2}{2}\right) + \frac{7v^2}{288}.$$

We have

$$\mu(v) = \frac{1}{20} + \frac{5v^2\left(2-\frac{v^2}{2}\right)}{576} + \frac{1}{80}\left(2 - \frac{v^2}{2}\right) + \frac{7v^2}{288},$$

we obtain

$$\mu'(v) = \frac{51}{720}v - \frac{5v^3}{288}.$$

Thus, $\mu(v)$ at a point $v = 0$ has a maximum value, therefore,

$$|a_5| \leq \mu(0) = \frac{3}{40},$$

$$|a_6| = \left| -\frac{v_1 v_4}{160} - \frac{47 v_2 v_3}{4320} + \frac{38 v_1^5}{57600} - \frac{49 v_1 v_2^2}{51840} - \frac{37 v_2^2}{17280} \right|$$

$$= \left|\left[v_5 - \frac{9 v_1 v_4}{10}\right] + \frac{47}{4320}[v_5 - v_2 v_3] + \frac{38 v_1^3}{28800}\left[v_2 - \frac{v_1^2}{2}\right] - \frac{49 v_1 v_2}{51840}\left[v_2 - \frac{v_1^2}{2}\right]\right|.$$

Assume $v_1 = v,\ v \in [0,2]$, by applying Lemma 2.3, to obtain

$$|a_6| \leq \frac{1}{72} + \frac{47}{2160} + \frac{38v^3\left[2-\frac{v^2}{2}\right]}{28800} - \frac{98v\left[2-\frac{v^2}{2}\right]}{51840} + \frac{72}{17280}\left[2-\frac{v^2}{2}\right] + \frac{648c^3}{129600}.$$

Hence

$$|a_6| \leq \frac{1}{72} + \frac{47}{2160} + \frac{v^3\left[2-\frac{v^2}{2}\right]}{800} - \frac{49v\left[2-\frac{v^2}{2}\right]}{25920} + \frac{37\left[2-\frac{v^2}{2}\right]}{8640} + \frac{324v^3}{64800},$$

mean

$$|a_6| \leq \frac{1}{72} + \frac{47}{2160} + \frac{v^3\left[2-\frac{v^2}{2}\right]}{800} - \frac{49v\left[2-\frac{v^2}{2}\right]}{25920} + \frac{37\left[2-\frac{v^2}{2}\right]}{8640} + \frac{v^3}{200}.$$

Assume

$$\mu(v) = \frac{1}{72} + \frac{47}{2160} + \frac{v^3\left[2-\frac{v^2}{2}\right]}{800} - \frac{49v\left[2-\frac{v^2}{2}\right]}{25920} + \frac{37\left[2-\frac{v^2}{2}\right]}{8640} + \frac{v^3}{200}.$$

we get

$$\mu'(v) = \frac{190443v^2}{7516800} - \frac{87v^4}{9280} - \frac{1073v}{125280} + \frac{1207}{375840}.$$

Therefore, $v=0$ be the root of function $\mu'(v)=0$ and $\mu''(0)<0$, it is clear that the function $\mu(v)$ has a maximum value at $v=0$,

$$|a_6| \leq \mu(0) = \frac{191}{4320}.$$

$$\begin{aligned}|a_7| = &\left|\frac{v_1^2 v_4}{3360} + \frac{1777v_1v_2v_3}{241920} + \frac{10379v_1^6}{14515200} - \frac{5071v_1^2v_2^2}{967680} - \frac{1314029v_1^4v_2}{1255274496} - \frac{v_1v_5}{210} - \frac{7v_2v_4}{672} + \frac{17v_2^3}{8064} + \frac{v_6}{84} - \frac{v_3^2}{540}\right.\\ &\left.- \frac{111v_1v_2^2}{120960} + \frac{v_1^4}{24192} - \frac{5v_1^3v_3}{3024}\right| = \left|\frac{v_1^6}{14515200} - \frac{5071v_1^2v_2^2}{967680} - \frac{v_1v_5}{210} + \frac{v_1^2[v_4-v_2^2]}{3360} + \frac{1777[v_3-v_1v_2]}{241920}\right.\\ &+ \frac{5v_1^3[v_3-v_1v_2]}{3024} - \frac{1314029v_1^4\left[v_2-\frac{v_1^2}{2}\right]}{1255274496} + \frac{17v_2^2\left[v_2-\frac{v_1^2}{2}\right]}{8064} + \frac{\left[v_6-\frac{5}{8}v_2v_4\right]}{84} - \frac{7v_2[v_4-v_2^2]}{672}\\ &\left.+ \frac{v_1^2\left[v_2-\frac{v_1^2}{2}\right]}{24192} - \frac{111v_2\left[v_2-\frac{v_1^2}{2}\right]}{120960}\right|. \end{aligned} \tag{3.7}$$

Assume $v_1 = v,\ v \in [0,2]$, by using Lemma 2.3 we obtain

$$\begin{aligned}|a_7| \leq &\frac{v^6}{14515200} + \frac{5071v^2}{241920} + \frac{v}{105} + \frac{v^2}{1680} + \frac{1777v}{60480} + \frac{5v^3}{1512} + \frac{1314029v^4\left[2-\frac{v^2}{2}\right]}{1255274496} + \frac{17\left[2-\frac{v}{2}\right]}{2016} - \frac{1}{168}\\ &+ \frac{1}{24} + \frac{v^2\left[v_2-\frac{v^2}{2}\right]}{24192} + \frac{111\left[v_2-\frac{v^2}{2}\right]}{60480}\end{aligned}$$

Putting

$$\mu(v) = \frac{v^6}{14515200} + \frac{5071v^2}{241920} + \frac{v}{105} + \frac{v^2}{1680} + \frac{1777v}{60480} + \frac{5v^3}{1512} + \frac{1314029v^4\left[2-\frac{v^2}{2}\right]}{1255274496} + \frac{17\left[2-\frac{v}{2}\right]}{2016} - \frac{1}{168} + \frac{1}{24} + \frac{v^2\left[v_2-\frac{v^2}{2}\right]}{24192} + \frac{111\left[v_2-\frac{v^2}{2}\right]}{60480}$$

We have

$$\mu'(v) \geq 0.$$

Therefore, $\mu(v)$ has a maximum value at $v=2$, and we obtain

$$|a_7| \leq \mu(2) = 0.2346. \tag{3.8}$$

Therefore, we get the result. □

Theorem 3.2. *If f is a function represented by the form* (1.1) *and $f \in \mathcal{S}_{\mathcal{C}}^{\#}$, then, we have*

$$|a_3 - a_2^2| \leq \frac{1}{6} \tag{3.9}$$

Proof . By using the equation (3.5), we obtain

$$|a_3 - a_2^2| = \left|\frac{v_2}{12} - \frac{v_1^2}{16}\right|.$$

Thus, by using Lemma 2.1, to obtain

$$|a_3 - a_2^2| = \left|\frac{x(4-v_1^2)}{24} - \frac{v_1^2}{32}\right|.$$

Assume that $v_1 = v,\ v \in [0,2],\ x = \mathfrak{t}$ and $\mathfrak{t} \in [0,1]$. Then by applying the triangle inequality, to get

$$|a_3 - a_2^2| = \frac{\mathfrak{t}(4-v_1^2)}{24} - \frac{v_1^2}{32}.$$

Assume

$$\mu(v,\mathfrak{t}) = \frac{\mathfrak{t}(4-v_1^2)}{24} - \frac{v_1^2}{32},$$

Thus for all $v \in (0,2)$ and $\mathfrak{t} \in (0,1)$, we obtain

$$\frac{\partial\mu}{\partial\mathfrak{t}} = \frac{4-v^2}{24} > 0,$$

therefore $\mu(v,\mathfrak{t})$ is an increasing function on the interval [0,1] about $\mathfrak{t}$.
Thus, $\mu(v,\mathfrak{t})$ have the maximum value at $\mathfrak{t}=1$, hence

$$\max\mu(v,\mathfrak{t}) = \mu(v,1) = \frac{4-v^2}{24} + \frac{v^2}{32} = \frac{4(4-v^2)+3v^2}{96} = \frac{16-v^2}{96}.$$

Define $\pi(v) = \frac{16-v^2}{96}$,
Clearly that $\pi(v)$ has a maximum value at $v = 0$, thus

$$|a_3 - a_2^2| \leq \pi(v) = \frac{1}{6}.$$

The proof is complete. □

Theorem 3.3. *If f is a function represented by the form* (1.1) *and $f \in \mathcal{S}_{\mathcal{C}}^{\#}$, then, we obtain*

$$|a_2a_3 - a_4| \leq \frac{1}{12} \tag{3.10}$$

Proof . By equation (3.5), we obtain

$$|a_2a_3 - a_4| = \left|\frac{v_1v_2}{48} + \frac{v_1v_2}{96} - \frac{v_3}{24} + \frac{v_1^3}{576}\right| = \left|\frac{3v_1v_2}{96} - \frac{v_3}{24} + \frac{v_1^3}{576}\right|.$$

By using Lemma 2.1, we obtain

$$|a_2a_3 - a_4| = \left|\frac{7v_1^3}{576} + \frac{(4-v_1^2)x^2v_1}{384} - \frac{(4-v_1^2)(1-|x|^2)z}{48}\right|.$$

Assume that $v_1 = v,\ v \in [0,2],\ x = \mathfrak{t}$ and $\mathfrak{t} \in [0,1]$. Then by applying the triangle inequality, to get

$$|a_2a_3 - a_4| \leq \frac{7v_1^3}{576} + \frac{(4-v_1^2)\mathfrak{t}^2v_1}{384} - \frac{(4-v_1^2)(1-\mathfrak{t}^2)}{48}.$$

Suppose

$$\mu(v,\mathfrak{t}) = \frac{7v_1^3}{576} + \frac{(4-v_1^2)\mathfrak{t}^2v_1}{384} - \frac{(4-v_1^2)(1-\mathfrak{t}^2)}{48}.$$

Thus, for all $v \in (0,2)$ and $\mathfrak{t} \in (0,1)$, to obtain

$$\frac{\partial\mu}{\partial\mathfrak{t}} = \frac{(4-v^2)(v-8)\mathfrak{t}}{192} < 0,$$

that is, $\mu(v,\mathfrak{t})$ is an decreasing function on the interval [0,1] about $\mathfrak{t}$.
Hence, the function $\mu(v,\mathfrak{t})$ have a maximum value at $\mathfrak{t} = 0$, which is

$$\max\mu(v,\mathfrak{t}) = \mu(v,0) = \frac{7v^3}{576} + \frac{(4-v^2)}{48}.$$

Now, we define

$$\pi(v) = \frac{7v^3}{576} + \frac{(4-v^2)}{48},$$

the function $\pi(v)$ has a maximum value at $v = 0$, also, we have

$$|a_2a_3 - a_4| \leq \pi(0) = \frac{1}{12}.$$

Hence, the proof is complete. □

Theorem 3.4. *If f is a function represented by the form* (1.1) *and $f \in \mathcal{S}_{\mathcal{C}}^{\#}$, then, we obtain*

$$|a_2a_4 - a_3^2| \leq \frac{1}{36}. \tag{3.11}$$

Proof . By equation (3.5) and $f \in \mathcal{S}_{\mathcal{C}}^{\#}$, to obtain

$$|a_2a_4 - a_3^2| = \left|\frac{v_1v_3}{96} - \frac{v_1^2v_2}{384} - \frac{v_1^4}{2304} - \frac{v_2^2}{144}\right|.$$

By Lemma 2.1, we have

$$|a_2a_4 - a_3^2| = \left|-\frac{5v_1^4}{4608} - \frac{x^2v_1^2(4-v_1^2)}{384} - \frac{x^2(4-v_1^2)^2}{576} + \frac{v_1(4-v_1^2)(1-|x|^2)z}{192}\right|.$$

Assume that $v_1 = v,\ v \in [0,2],\ x = \mathfrak{t}$ and $\mathfrak{t} \in [0,1]$. Then by applying the triangle inequality, to get

$$|a_2a_4 - a_3^2| \leq \frac{\mathfrak{t}^2v^2(4-v^2)}{384} + \frac{(1-\mathfrak{t}^2)(4-v^2)vz}{192} + \frac{\mathfrak{t}^2(4-v^2)^2}{576} + \frac{5v^4}{4608}.$$

Put

$$\mu(v,\mathfrak{t}) = \frac{\mathfrak{t}^2v^2(4-v^2)}{384} + \frac{(1-\mathfrak{t}^2)(4-v^2)vz}{192} + \frac{\mathfrak{t}^2(4-v^2)^2}{576} + \frac{5v^4}{4608}.$$

Therefore, for all $v \in (0,2)$ and $\mathfrak{t} \in (0,1)$, to obtain

$$\frac{\partial \mu}{\partial \mathfrak{t}} = \frac{(4-v^2)(v^2-6v+8)\mathfrak{t}}{576} > 0,$$

which implies that the function $\mu(v,\mathfrak{t})$ increases on the interval [0,1] about $\mathfrak{t}$. That is, has a maximum value at $\mathfrak{t} = 1$, we have

$$\max \mu(v,\mathfrak{t}) = \mu(v,1) = \frac{v^2(4-v^2)}{384} + \frac{(4-v^2)^2}{576} + \frac{5v^4}{4608}.$$

Setting

$$\pi(v) = \frac{v^2(4-v^2)}{384} + \frac{(4-v^2)^2}{576} + \frac{5v^4}{4608},$$

thus, we obtain

$$\pi'(v) = \frac{2v(4-v^2)+v^2(-2v)}{384} + \frac{2(4-v^2)(-2v)}{576} + \frac{20v^3}{4608} = \frac{v(4-v^2)}{196} - \frac{v^3}{196} - \frac{v(4-v^2)}{144} + \frac{5v^3}{1152}.$$

If $\pi'(v) = 0$, thus the root is $v = 0$, furthermore, since $\pi''(0) = \frac{-1}{576}$.
Also the function $\pi(v)$ have the maximum value at $v = 0$, we obtain

$$|a_2a_4 - a_3^2| \leq \pi(0) = \frac{1}{36}.$$

Hence, the proof is complete. □

Theorem 3.5. *If f is a function represented by the form* (1.1) *and $f \in \mathcal{S}_{\mathcal{C}}^{\#}$, then we obtain*

$$|a_2a_5 - a_3a_4| \leq \frac{19}{360}.$$

Proof . Assume $f \in \mathcal{S}_{\mathcal{C}}^{\#}$, then by using (3.5), to obtain

$$\begin{aligned}
|a_2a_5 - a_3a_4| &= \left|\frac{v_1}{4}\left(\frac{v_4}{40} - \frac{v_1v_3}{120} + \frac{v_1^4}{1152} - \frac{v_1^2v_2}{960} - \frac{v_2^2}{160}\right) - \left[\frac{v_2}{12}\left(\frac{v_3}{24} - \frac{v_1v_2}{96} - \frac{v_1^3}{576}\right)\right]\right| \\
&= \left|\frac{v_1v_4}{160} - \frac{v_1^2v_3}{480} + \frac{v_1^5}{4608} - \frac{v_1^3v_2}{3840} - \frac{v_1v_2^2}{640} - \frac{v_2v_3}{288} + \frac{v_1v_2^2}{1152} + \frac{v_1^3v_2}{6912}\right| \\
&= \left|\frac{v_1^5}{4608} + \frac{v_1v_4}{160} - \frac{v_1v_2^2}{1440} - \frac{v_1^2v_3}{480} - \frac{v_1^3v_2}{34560} - \frac{v_2v_3}{288}\right| \\
&= \left|-\frac{v_1^3\left[v_2 - \frac{v_1^2}{2}\right]}{34560} - \frac{v_3\left[v_2 - \frac{v_1^2}{2}\right]}{288} + \frac{v_1[v_4 - v_1v_3]}{240} + \frac{v_1^5}{1152} + \frac{v_1\left[v_4 - \frac{1}{4}v_2^2\right]}{4320}\right|.
\end{aligned}$$

Assume that $v_1 = v,\ v \in [0,2]$, by applying Lemma 2.3, to get

$$|a_5| \leq \frac{1}{20} + \frac{5v^2\left(2 - \frac{v^2}{2}\right)}{576} + \frac{\left(2 - \frac{v^2}{2}\right)}{80} + \frac{7v^2}{288}.$$

Assume

$$\mu(v) = \frac{v^3\left[2 - \frac{v^2}{2}\right]}{34560} + \frac{\left[2 - \frac{v^2}{2}\right]}{144} + \frac{v}{80} + \frac{v^5}{1152}.$$

Thus, for all $v \in (0,2)$, to obtain

$$\mu'(v) = \frac{v^2}{5760} + \frac{59}{13824} - \frac{v}{144} + \frac{1}{80} > 0.$$

Hence, the function $\mu(v)$ increases on [0,2] about v.
Clearly, the maximum value of $\mu(v)$ at $v = 2$, also we have

$$|a_2a_5 - a_3a_4| \leq \mu(2) = \frac{19}{360}.$$

Hence, the proof is complete. □

Theorem 3.6. *If f is a function represented by the form* (1.1) *and $f \in \mathcal{S}_{\mathcal{C}}^{\#}$, then we obtain*

$$|a_5 - a_2a_4| \leq \frac{21}{320}. \tag{3.12}$$

Proof . Suppose $f \in \mathcal{S}_{\mathcal{C}}^{\#}$ and by form (3.5), to get

$$
\begin{aligned}
|a_5 - a_2a_4| &= \left|\frac{v_4}{40} - \frac{v_1v_3}{120} + \frac{v_1^4}{1152} - \frac{v_1^2v_2}{960} - \frac{v_2^2}{160} - \left[\frac{v_1}{4}\left(\frac{v_3}{24} - \frac{v_1v_2}{96} - \frac{v_1^3}{576}\right)\right]\right| \\
&= \left|\frac{v_4}{40} - \frac{3v_1v_3}{160} + \frac{v_1^4}{768} + \frac{v_1^2v_2}{640} - \frac{v_2^2}{160}\right| \\
&= \left|\frac{v_1^4 + v_2^2 + 2v_1v_3 - 3v_1^3v_2 - v_4}{160} - \frac{5v_1^2\left[v_2 - \frac{v_1^2}{2}\right]}{640} - \frac{3\left[v_4 - \frac{2}{3}v_1v_3\right]}{160}\right|.
\end{aligned}
$$

Now, by using Lemma 2.3, we obtain

$$|a_5 - a_2a_4| \le \frac{1}{20} + \frac{25v^2\left[2 - \frac{v^2}{2}\right]}{640}.$$

Assume

$$\mu(v) = \frac{1}{20} + \frac{25v^2\left[2 - \frac{v^2}{2}\right]}{640},$$

therefore, we have

$$\mu'(v) = \frac{5v}{32} - \frac{5v^3}{64}.$$

Assume $\mu'(v) = 0$ to get $v = \sqrt{2}$ or $v = 0$ and $\mu'(\sqrt{2}) < 0$, clearly that the maximum value of $\mu(v)$ at $v = \sqrt{2}$, also we get

$$|a_5 - a_2a_4| \le \mu(\sqrt{2}) = \frac{21}{320}.$$

Hence, the proof is complete. □

Theorem 3.7. *If f is a function represented by the form* (1.1) *and $f \in \mathcal{S}_{\mathcal{C}}^{\#}$, then we obtain*

$$|a_5a_3 - a_4^2| \le \frac{1961}{25920}. \tag{3.13}$$

Proof . Suppose $f \in \mathcal{S}_{\mathcal{C}}^{\#}$and by the form (3.5), we get

$$
\begin{aligned}
|a_5a_3 - a_4^2| &= \left|\frac{v_2v_4}{480} + \frac{v_1v_2v_3}{5760} + \frac{v_1^4v_2}{27648} - \frac{v_1^2v_2^2}{5120} - \frac{v_2^3}{1920} - \frac{v_3^2}{576} + \frac{v_1^3v_3}{6912} - \frac{v_1^6}{331776}\right| \\
&= \left|\frac{v_2\left[v_4 - \frac{v_1v_3}{9}\right]}{480} - \frac{v_3\left[v_3 - \frac{v_1v_2}{4}\right]}{576} - \frac{v_2^2\left[v_2 - \frac{v_1^2}{2}\right]}{1920} - \frac{v_1^2v_2\left[v_2 - \frac{v_1^2}{2}\right]}{2880} + \frac{v_1^3\left[v_3 - \frac{31v_1v_2}{32}\right]}{6912}\right. \\
&\quad \left. - \frac{25v_1^4v_2}{13824} - \frac{v_1^6}{331776}\right|
\end{aligned}
$$

Now, by using Lemma 2.3, we get

$$|a_5a_3 - a_4^2| \leq \frac{1}{120} + \frac{1}{144} + \frac{\left[2-\frac{v^2}{2}\right]}{480} + \frac{v^2\left[2-\frac{v^2}{2}\right]}{1440} + \frac{v^3}{3456} + \frac{25v^4}{6912} + \frac{v^6}{331776}.$$

Assume

$$\mu(v) = \frac{1}{120} + \frac{1}{144} + \frac{\left[2-\frac{v^2}{2}\right]}{480} + \frac{v^2\left[2-\frac{v^2}{2}\right]}{1440} + \frac{v^3}{3456} + \frac{25v^4}{6912} + \frac{v^6}{331776}.$$

Therefore, for all $v \in (0,2)$, we have $\mu'(v) > 0$, it means that the maximum value of $\mu(v)$ at $\mathfrak{t} = 2$, also we have

$$|a_5a_3 - a_4^2| \leq \mu(2) = \frac{1961}{25920}.$$

Hence, the proof is complete. □

Theorem 3.8. *If f is a function represented by the form* (1.1) *and $f \in \mathcal{S}_{\mathcal{C}}^{\#}$, then we obtain*

$$|H_4(1)| \leq 0.0103468. \tag{3.14}$$

Proof . Since

$$\begin{aligned} H_4(1) = &\{(a_2a_4 - a_3^2)a_3 - (a_4 - a_2a_3)a_4 + (a_3 - a_2^2)a_5\}a_7 - \{(a_2a_5 - a_3a_4)a_3 - (a_5 - a_2a_4)a_4 \\ &+ (a_3 - a_2^2)a_6\}a_6 - \{(a_3a_5 - a_4^2)a_3 - (a_5 - a_2a_4)a_5 + a_4 - a_2a_3\}a_6 + \{(a_3a_5 - a_4^2)a_3 \\ &- (a_5 - a_2a_4)a_5 + (a_4 - a_2a_3)a_6\}a_5 - \{(a_3a_5 - a_4^2)a_4 - (a_2a_5 - a_3a_4)a_5 + (a_4 - a_2a_3)a_6\}a_4. \end{aligned}$$

Thus, by using the triangle inequality and by Definition 1.2, we obtain

$$\begin{aligned} H_4(1) = &|a_2a_4 - a_3^2||a_3||a_7| + |a_4 - a_2a_3||a_4||a_7| + |a_3 - a_2^2||a_5||a_7| + |a_2a_5 - a_3a_4||a_3||a_6| \\ &+ |a_5 - a_2a_4||a_4||a_6| + |a_3 - a_2^2||a_6|^2 + |a_3a_5 - a_4^2||a_3||a_5| + |a_5 - a_2a_4||a_5|^2 + |a_4 - a_2a_3||a_6||a_5| \\ &+ |a_3a_5 - a_4^2||a_4|^2 + |a_2a_5 - a_3a_4||a_5||a_4| + |a_4 - a_2a_3||a_6||a_4|. \end{aligned}$$

Now, replace the equation (3.1) and (3.9), (3.10), (3.11), (3.12), (3.13) into (3.14), we readily to get the result. □

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