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Analysis of SIRC model for influenza A with Caputo-Fabrizio derivative

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Abstract

In this manuscript, we study the fractional-order SIRC epidemiological model for influenza A in the human population in the Caputo-Fabrizio sense. The existence and uniqueness of the solution of the proposed problem are established using fixed point theory. The local stability of both disease-free equilibrium and endemic equilibrium points is investigated. Using the three-step fractional Adams-Bashforth scheme, an iterative solution of our system is generated. In the numerical simulation, many plots are given for different values of the fractional-order to check the stability of equilibrium points. Also, the effect of varying some parameters of the model was presented. Furthermore, we compared our numerical solutions with those using Caputo fractional derivative model via graphical representations. The obtained results show the efficiency and accuracy of our approach.

Keywords: Caputo-Fabrizio derivative, SIRC model, stability, Fixed point theory, Three-step fractional Adams-Bashforth scheme
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1 Introduction

Influenza also called flu is a contagious viral infection, caused by three types of viruses: A, B, and C [36]. Influenza A viruses are the most dangerous, they can cause severe disease and even death. They have also been linked to pandemics and outbreaks. The viruses have the potential to modify the antigenic characteristics of their surface which makes it easier to escape the immunity generated by the previous infection. Influenza A virus's subtypes are determined by two proteins found on its surface: hemagglutinin (HA) and neuraminidase (NA). There are currently 18 subtypes of hemagglutinin and 11 subtypes of neuraminidase [38]. As reported by the Centers for Disease Control and Prevention (CDC), Only 131 subtypes of influenza A have been discovered in nature, although there are theoretically 198 distinct subtype combinations; A(H1N1), A(H2N2), and A(H3N2) are the three viruses known to have spread widely in humans [32, 38]. The seasonal influenza A(H1N1) viruses that appear every year are related to the pandemic

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2009 H1N1 virus, called the "A(H1N1)pdm09 virus". The World Health Organization (WHO) estimates that 3 to 5 million cases of severe sickness and 290 000 to 650 000 respiratory deaths among adult patients are attributable to influenza in the world. However, according to a new study led by Chow, et al.[3], Highlighted the number and impact of people hospitalized from flu for nonrespiratory complications which can be just as severe as respiratory ones. Sepsis and acute kidney injury were the most common acute diagnoses, which means that the above statistics are likely to rise. According to WHO, although that the influenza vaccine is ineffective against the COVID-19 virus, being vaccinated each year to avoid influenza infection is highly advised, because it is the most important strategy for preventing influenza and its severe consequences. In light of the foregoing, we have chosen to focus on influenza A/H1N1 viruses.

Mathematical models have become an essential tool in the study of the transmission and control of infectious diseases. It leads to more effective strategies for reducing disease spread. Several epidemic models have been proposed to analyze and study the dynamics of A(H1N1) influenza viruses spread, among them, we cite as examples SIR [14], SEIR [13, 40], SVEIR [21], and SIRC [4]. Because of changes in the antigenic characteristics of the flu viruses, the SIRC model is the most suitable for such a disease. For biological details, see [4] and references therein.

In 2006, enhancing SIR the classical model of flu, Casagrandi et al.[4] introduced the class C of cross-immune individuals, providing the system below

$$\begin{cases} \frac{dS}{dt}(t) = \mu\left(1 - S(t)\right) - \beta S(t)I(t) + \gamma C(t), \\ \frac{dI}{dt}(t) = \beta S(t)I(t) + \sigma\beta C(t)I(t) - \left(\mu + \theta\right) I(t), \\ \frac{dR}{dt}(t) = \left(1 - \sigma\right)\beta C(t)I(t) + \theta I(t) - \left(\mu + \delta\right) R(t), \\ \frac{dC}{dt}(t) = \delta R(t) - \beta C(t)I(t) - \left(\gamma + \mu\right) C(t). \end{cases} \tag{1.1}$$

with initial conditions

$$\left(S\left(0\right), I(0),\ R\left(0\right),\ C\left(0\right)\right)^T \in \mathbb{R}^4_+.$$

Where the population $N(t)$ is divided into four subclasses: $S\left(t\right)$, $I\left(t\right)$, $R\left(t\right)$, and $C\left(t\right)$ which denote the proportions of susceptible, infected, removed, and cross-immune individuals, respectively at time t.

The parameter μ represents the death rate in each compartment and also the rate of the newborn in the population. β is the contact rate of the disease, γ^{-1} is the cross-immune period, θ^{-1} is the infectious period, δ^{-1} is the total immune period and σ is the fraction of exposed cross-immune individuals who are recruited in a unit time into the infective compartment. As epidemiological systems model real situations, all parameters and state variables $S\left(t\right), I\left(t\right), R\left(t\right)$, $C\left(t\right)$ are nonnegative for all time $t \geq 0$ [4, 23].

After some works on the local stability of both disease-free equilibrium and endemic equilibrium points of the model (1.1), Li and Guo [23] have given a full study concerning their global stability. Also, its dynamical behavior has been investigated numerically [17]. In [24] Li et al. studied the SIRC model with optimal control theory. Zhang et al. [44] suggested a stochastic SIRC model for influenza A with stochastic perturbation.

It has been shown that mathematical models involving fractional-order derivatives are more realistic than those involving integer order one, in fact, it is so close to clinical data, whether in biology, bioengineering, physics, or electrochemistry, among other areas [16, 27, 28, 29, 30, 34] . Before 2015, Riemann-Liouville and Caputo [7, 20, 31, 39] are the most fractional-order derivatives considered in studies of the behavior of dynamic systems. Nevertheless, due to their singular kernels, they are unable to model natural situations in some cases. El-Shahed and Alsaedi [11] investigated the fractional-order SIRC model in Caputo sense. They gave an analysis for asymptotic stability of equilibrium points and the solutions were obtained by nonstandard finite difference methods. In [18] and [19] the authors considered the same fractional system in [11]. They presented two numerical approximations, fractional Chebyshev finite difference and Chebyshev spectral method followed by a comparison of their numerical solutions with those using the fourth-order Runge-Kutta method.

Recently, in 2015 and 2016 respectively two novel fractional derivatives without singular kernels are presented, Caputo-Fabrizio (CF) [5] and Antagana-Baleanu (ABC) [2]. The first one is with an exponential law while the second is based on the Mittag-Leffler function, which are more suitable for modeling real-world problems. These two operators have been extensively used to describe epidemiological models [10, 15, 33, 37, 40, 41, 42, 43]. Afterward, the role and properties of (CF) operator's have been discussed in [6, 26].

Motivated by previous studies, biological interpretation of model (1.1), hereditary properties of fractional calculus, and Caputo-Fabrizio characteristic's derivative the aim in this article is to establish the fractional SIRC model within the framework of Caputo-Fabrizio derivative given by the following system

$$\begin{cases} \xi^{\alpha-1}\ {}^{CF}D_t^{\alpha} S = \mu\left(1-S\right) - \beta SI + \gamma C, \\ \xi^{\alpha-1}\ {}^{CF}D_t^{\alpha} I = \beta SI + \sigma\beta CI - \left(\mu+\theta\right) I, \\ \xi^{\alpha-1}\ {}^{CF}D_t^{\alpha} R = \left(1-\sigma\right)\beta CI + \theta I - \left(\mu+\delta\right) R, \\ \xi^{\alpha-1}\ {}^{CF}D_t^{\alpha} C = \delta R - \beta CI - \left(\gamma+\mu\right) C, \end{cases} \tag{1.2}$$

with initial conditions

$$\left(S\left(0\right), I(0),\ R\left(0\right),\ C\left(0\right)\right)^T \in \mathbb{R}_+^4. \tag{1.3}$$

${}^{CF}D^{\alpha}$ is the Caputo-Fabrizio derivative, with respect to time t. According to [12], the parameter ξ was introduced to ensure that the right and left-hand sides of the model (1.2) have the same dimension.

We check the existence and uniqueness of the solution using the fixed point theory. We prove the stability of the disease-free equilibrium through the characteristic equation [22] different from the usual one, while we give more and new conditions on the stability of the endemic equilibrium point than presented in [11]. To find approximate solutions of the proposed model (1.2) subject to the initial conditions (1.3), we use the three-step fractional Adams-Bashforth method for the Caputo-Fabrizio fractional derivative [35] for different values of α. Moreover, each time α changes, we show the influence of changing parameters β and θ on the number of individuals in each compartment and we give a comparative analysis with the Caputo derivative.

This manuscript is organized as follows. In section 2, we recall some basic definitions concerning fractional Caputo-Fabrizio operators. Section 3 is devoted to our main results, firstly we examine the existence and uniqueness of the solution, then we determine the equilibrium points and investigate their stability. Finally, suggested numerical methods, simulations, and results discussions are presented.

2 Preliminaries

The new fractional derivative with non-singular kernel introduced by Caputo and Fabrizio is given by the following definition.

Definition 2.1. [5] Let $q \in H^1\left(a,b\right)$, $b > a$, $\alpha \in (0,1)$ and $a \in [-\infty, t)$. Then the fractional derivative of order α in Caputo-Fabrizio sense is defined as

$${}^{CF}D_t^{\alpha} q\left(t\right) = \frac{M\left(\alpha\right)}{1-\alpha}\int_a^t q'(s)\exp\left[-\alpha\frac{t-s}{1-\alpha}\right] ds, \tag{2.1}$$

with its corresponding fractional integral

$${}^{CF}I_t^{\alpha} f\left(t\right) = \frac{1-\alpha}{M\left(\alpha\right)} f\left(t\right) + \frac{\alpha}{M\left(\alpha\right)}\int_a^t f\left(s\right) ds,\ t \geq 0. \tag{2.2}$$

Where the normalization function $M\left(\alpha\right)$ satisfies $M\left(0\right) = M\left(1\right) = 1$.

Note that $H^1\left(a,b\right) = \left\{f :\ f \in L^2\left(a,b\right) \text{ and } f' \in L^2\left(a,b\right)\right\}$ where $L^2\left(a,b\right)$ is the space of square integrable functions on the interval $\left(a,b\right)$.

Later Losada and Nieto presented a modified version of this new fractional derivative with its corresponding integral.

Definition 2.2. [25] Let $q \in H^1\left(0,b\right)$, $b > 0$ and $\alpha \in (0,1)$.Then the fractional derivative of order α in Caputo-Fabrizio sense is given by

$${}^{CF}D_t^{\alpha} q\left(t\right) = \frac{\left(2-\alpha\right) M\left(\alpha\right)}{2\left(1-\alpha\right)}\int_0^t q\ '(s)\exp\left[-\alpha\frac{t-s}{1-\alpha}\right] ds,\ t \geq 0. \tag{2.3}$$

Definition 2.3. [25] The fractional integral of order $\alpha \in (0,1)$ in Caputo-Fabrizio sense is defined as

$${}^{CF}I_t^{\alpha} q\left(t\right) = \frac{2\left(1-\alpha\right)}{\left(2-\alpha\right) M\left(\alpha\right)} q\left(t\right) + \frac{2\alpha}{\left(2-\alpha\right) M\left(\alpha\right)}\int_0^t q\left(s\right) ds,\ t \geq 0. \tag{2.4}$$

Remark 2.4. It was noted in [25], that Definition 2.3 imposes

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}+\frac{2\alpha}{(2-\alpha)M(\alpha)}=1, \tag{2.5}$$

thus,

$$M(\alpha)=\frac{2}{2-\alpha}\ ,\ 0<\alpha<1. \tag{2.6}$$

In view of inequality (2.4), Losada and Nieto proposed the following fractional derivative definition.

Definition 2.5. [25] Let $0<\alpha<1$. The fractional derivative of order α in Caputo-Fabrizio sense is given by

$${}^{CF}D_t^{\alpha}q(t)=\frac{1}{1-\alpha}\int_0^t q'(s)\exp\left[-\alpha\frac{t-s}{1-\alpha}\right]ds,\ t\geq 0, \tag{2.7}$$

and its corresponding fractional integral is defined as

$${}^{CF}I_t^{\alpha}q(t)=(1-\alpha)f(t)+\alpha\int_0^t q(s)\,ds,\ t\geq 0. \tag{2.8}$$

3 Main results

3.1 Existence and uniqueness

Using fixed point theory, the present subsection is devoted to establishing the existence and uniqueness of the solution of the fractional model (1.2) subject to the initial conditions (1.3).

In what follows we consider the norm $\|x\|:=\sup\{|x(t)|:t\in[0,T]\}$.

As epidemiological models present real world data, we assume that $S,\ I,\ R$, and C are nonnegative and bounded functions. i.e. $\exists\ \eta_1\geq 0\ ,\ \eta_2\geq 0\ ,\ \eta_3\ \geq 0\ ,\ \eta_4\geq 0$ such that

$\|S(t)\|\leq\eta_1,\ \|I(t)\|\leq\eta_2,\ \|R(t)\|\leq\eta_3$ and $\|C(t)\|\leq\eta_4$.

In view of formula (2.2) and taking $a=0$, we get

$$\begin{aligned}
S(t)-S(0)&=\frac{1-\alpha}{M(\alpha)}\xi^{1-\alpha}(\mu(1-S)-\beta SI+\gamma C)+\frac{\alpha}{M(\alpha)}\int_0^t\xi^{1-\alpha}(\mu(1-S)-\beta SI+\gamma C)ds,\\
I(t)-I(0)&=\frac{1-\alpha}{M(\alpha)}\xi^{1-\alpha}(\beta SI+\sigma\beta CI-(\mu+\theta)I)+\frac{\alpha}{M(\alpha)}\int_0^t\xi^{1-\alpha}(\beta SI+\sigma\beta CI-(\mu+\theta)I)ds,\\
R(t)-R(0)&=\frac{1-\alpha}{M(\alpha)}\xi^{1-\alpha}((1-\sigma)\beta CI+\theta I-(\mu+\delta)R)+\frac{\alpha}{M(\alpha)}\int_0^t\xi^{1-\alpha}((1-\sigma)\beta CI+\theta I-(\mu+\delta)R)ds,\\
C(t)-C(0)&=\frac{1-\alpha}{M(\alpha)}\xi^{1-\alpha}(\delta R-\beta CI-(\gamma+\mu)C)+\frac{\alpha}{M(\alpha)}\int_0^t\xi^{1-\alpha}(\delta R-\beta CI-(\gamma+\mu)C)ds.
\end{aligned} \tag{3.1}$$

For more simplicity, we consider the following kernels

$$\begin{aligned}
N_1(t,S)&=\xi^{1-\alpha}(\mu(1-S)-\beta SI+\gamma C),\\
N_2(t,I)&=\xi^{1-\alpha}(\beta SI+\sigma\beta CI-(\mu+\theta)I),\\
N_3(t,R)&=\xi^{1-\alpha}((1-\sigma)\beta CI+\theta I-(\mu+\delta)R),\\
N_4(t,C)&=\xi^{1-\alpha}(\delta R-\beta CI-(\gamma+\mu)C),
\end{aligned} \tag{3.2}$$

Theorem 3.1. If the following hypotheses hold

$H_1)$ $\xi^{1-\alpha}(\mu+\beta\eta_2)<1$,

$H_2)$ $\xi^{1-\alpha}(\beta\eta_1+\sigma\beta\eta_4+\mu+\theta)<1$,

$H_3)$ $\xi^{1-\alpha}(\mu+\delta)<1$,

$H_4)$ $\xi^{1-\alpha}(\beta\eta_1+\mu+\gamma)<1$,

then the kernels $N_1,\ N_2,\ N_3$, and N_4 satisfy the Lipschitz condition and are contraction mappings.

Proof . Consider the kernel N_1. For anyS_1 and S_2, we have

$$\begin{aligned}\|N_1(t,S_1)-N_1(t,S_2)\| &= \xi^{1-\alpha}\|\mu(S_1(t)-S_2(t))+\beta I(S_1(t)-S_2(t))\| \\ &\leq \xi^{1-\alpha}(\mu\|S_1(t)-S_2(t)\|+\beta\eta_2\|S_1(t)-S_2(t)\|) \\ &\leq \xi^{1-\alpha}(\mu+\beta\eta_2)\|S_1(t)-S_2(t)\|, \end{aligned} \tag{3.3}$$

by hypothesis H_1, we find that the kernel N_1 satisfies the Lipschitz condition.

By similar reasoning for the kernels N_2, N_3, and N_4, we obtain

$$\begin{aligned}\|N_2(t,I_1)-N_2(t,I_2)\| &= \xi^{1-\alpha}\|\beta S(I_1-I_2)+\sigma\beta C(I_1-I_2)+(\mu+\theta)(I_1-I_2)\| \\ &\leq \xi^{1-\alpha}(\beta\eta_1+\sigma\beta\eta_4+\mu+\theta)\|I_1(t)-I_2(t)\|, \end{aligned} \tag{3.4}$$

$$\begin{aligned}\|N_3(t,R_1)-N_3(t,R_2)\| &= \xi^{1-\alpha}\|(\mu+\delta)(R_1(t)-R_2(t))\| \\ &\leq \xi^{1-\alpha}(\mu+\delta)\|R_1(t)-R_2(t)\|, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\|N_4(t,C_1)-N_4(t,C_2)\| &= \xi^{1-\alpha}\|\beta\eta_1(C_1(t)-C_2(t))+(\mu+\gamma)(C_1(t)-C_2(t))\| \\ &\leq \xi^{1-\alpha}(\beta\eta_1+\mu+\gamma)\|C_1(t)-C_2(t)\|. \end{aligned} \tag{3.6}$$

Taking into account hypotheses H_2, H_3, and H_4 we deduce the results.
□

Theorem 3.2. If the following conditions hold

$E_1)$ $\xi^{1-\alpha}(\mu+\beta\eta_2)\left[\frac{1-\alpha}{M(\alpha)}+\frac{\alpha}{M(\alpha)}T\right]<1$,

$E_2)$ $\xi^{1-\alpha}(\beta\eta_1+\sigma\beta\eta_4+\mu+\theta)\left[\frac{1-\alpha}{M(\alpha)}+\frac{\alpha}{M(\alpha)}T\right]<1$,

$E_3)$ $\xi^{1-\alpha}(\mu+\delta)\left[\frac{1-\alpha}{M(\alpha)}+\frac{\alpha}{M(\alpha)}T\right]<1$,

$E_4)$ $\xi^{1-\alpha}(\beta\eta_1+\mu+\gamma)\left[\frac{1-\alpha}{M(\alpha)}+\frac{\alpha}{M(\alpha)}T\right]<1$,

then the system (1.2) with initial conditions (1.3) has a unique solution.

Proof . Step1

From formula (3.1), we construct the following system of sequences:

$$S_n(t)=\xi^{1-\alpha}\left(\frac{1-\alpha}{M(\alpha)}N_1(t,S_{n-1})+\frac{\alpha}{M(\alpha)}\int_0^t N_1(s,S_{n-1})\,ds\right),$$

$$I_n(t)=\xi^{1-\alpha}\left(\frac{1-\alpha}{M(\alpha)}N_2(t,I_{n-1})+\frac{\alpha}{M(\alpha)}\int_0^t N_2(t,I_{n-1})\,ds\right),$$

$$R_n(t)=\xi^{1-\alpha}\left(\frac{1-\alpha}{M(\alpha)}N_3(t,R_{n-1})+\frac{\alpha}{M(\alpha)}\int_0^t N_3(t,R_{n-1})\,ds\right),$$

$$C_n(t) = \xi^{1-\alpha}\left(\frac{1-\alpha}{M(\alpha)}N_4(t, C_{n-1}) + \frac{\alpha}{M(\alpha)}\int_0^t N_4(t, C_{n-1})\,ds\right), \tag{3.7}$$

where the first terms are: $S_0(t) = S(0)$, $I_0(t) = I(0)$, $R_0(t) = R(0)$, and $C_0(t) = C(0)$.

Consider the first sequence. Since N_1 satisfies the Lipschitz condition, we obtain

$$\begin{aligned}
\|S_n(t) - S_{n-1}(t)\| &= \xi^{1-\alpha}\left\|\frac{1-\alpha}{M(\alpha)}N_1(t, S_{n-1}) + \frac{\alpha}{M(\alpha)}\int_0^t N_1(s, S_{n-1})\,ds\right. \\
&\quad \left. -\frac{1-\alpha}{M(\alpha)}N_1(t, S_{n-2}) - \frac{\alpha}{M(\alpha)}\int_0^t N_1(s, S_{n-2})\,ds\right\| \\
&\leq \xi^{1-\alpha}\frac{1-\alpha}{M(\alpha)}\|(N_1(t, S_{n-1}) - N_1(t, S_{n-2}))\| \\
&\quad +\xi^{1-\alpha}\frac{\alpha}{M(\alpha)}\int_0^t \|(N_1(s, S_{n-1}) - N_1(s, S_{n-2}))\|\,ds \\
&\leq \xi^{1-\alpha}\frac{1-\alpha}{M(\alpha)}\ (\mu + \beta\eta_2)\|S_{n-1}(t) - S_{n-2}(t)\| \\
&\quad +\xi^{1-\alpha}\frac{\alpha}{M(\alpha)}\ (\mu + \beta\eta_2)\,T\|S_{n-1}(t) - S_{n-2}(t)\|,
\end{aligned} \tag{3.8}$$

hence

$$\|S_n(t) - S_{n-1}(t)\| \leq \xi^{1-\alpha}(\mu + \beta\eta_2)\left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha}{M(\alpha)}T\right)\|S_{n-1}(t) - S_{n-2}(t)\|, \tag{3.9}$$

$$\|S_n(t) - S_{n-1}(t)\| \leq \left[\xi^{1-\alpha}(\mu + \beta\eta_2)\left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha}{M(\alpha)}T\right)\right]^2\|S_{n-2}(t) - S_{n-3}(t)\|, \tag{3.10}$$

Consequently

$$\|S_n(t) - S_{n-1}(t)\| \leq \left[\xi^{1-\alpha}(\mu + \beta\eta_2)\left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha}{M(\alpha)}T\right)\right]^n\|S_0(t)\|. \tag{3.11}$$

Similarly, we get

$$\|I_n(t) - I_{n-1}(t)\| \leq \left[\xi^{1-\alpha}(\beta\eta_1 + \sigma\beta\eta_4 + \mu + \theta)\left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha}{M(\alpha)}T\right)\right]^n\|I_0(t)\|, \tag{3.12}$$

$$\|R_n(t) - R_{n-1}(t)\| \leq \left[\ \xi^{1-\alpha}(\mu + \delta)\left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha}{M(\alpha)}T\right)\right]^n\|R_0(t)\|. \tag{3.13}$$

and

$$\|C_n(t) - C_{n-1}(t)\| \leq \left[\xi^{1-\alpha}(\beta\eta_1 + \mu + \gamma)\left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha}{M(\alpha)}T\right)\right]^n\|C_0(t)\|. \tag{3.14}$$

Inequalities (3.11), (3.12), (3.13), and (3.14) prove the existence and smoothness of the solutions of system (3.7).

Step2

We have to show that the system (3.7) converges to the solution of model (1.2) subject to initial conditions (1.3). Define $\Delta S_n(t)$, $\Delta I_n(t)$, $\Delta R_n(t)$, and $\Delta C_n(t)$ by

$$\begin{aligned}
S(t) &= S_n(t) - \Delta S_n(t), \\
I(t) &= I_n(t) - \Delta I_n(t), \\
R(t) &= R_n(t) - \Delta R_n(t), \\
C(t) &= C_n(t) - \Delta C_n(t).
\end{aligned} \tag{3.15}$$

Then

$$\|\Delta S_n(t)\| \leq \xi^{1-\alpha}\frac{1-\alpha}{M(\alpha)}\|(N_1(t, S) - N_1(t, S_{n-1}))\| + \xi^{1-\alpha}\frac{\alpha}{M(\alpha)}\int_0^t \|(N_1(s, S) - N_1(s, S_{n-1}))\|\,ds,$$

$$\|\Delta I_n(t)\| \leq \xi^{1-\alpha}\frac{1-\alpha}{M(\alpha)}\|(N_2(t,I)-N_2(t,I_{n-1}))\| + \xi^{1-\alpha}\frac{\alpha}{M(\alpha)}\int_0^t \|(N_2(s,I)-N_2(s,I_{n-1}))\|\,ds,$$

$$\|\Delta R_n(t)\| \leq \xi^{1-\alpha}\frac{1-\alpha}{M(\alpha)}\|(N_3(t,R)-N_3(t,R_{n-1}))\| + \xi^{1-\alpha}\frac{\alpha}{M(\alpha)}\int_0^t \|(N_3(s,R)-N_3(s,R_{n-1}))\|\,ds,$$

$$\|\Delta C_n(t)\| \leq \xi^{1-\alpha}\frac{1-\alpha}{M(\alpha)}\|(N_4(t,C)-N_4(t,C_{n-1}))\| + \xi^{1-\alpha}\frac{\alpha}{M(\alpha)}\int_0^t \|(N_4(s,C)-N_4(s,C_{n-1}))\|\,ds. \tag{3.16}$$

Because $S(t)$, $I(t)$, $R(t)$, and $C(t)$ are bounded functions, and due to Theorem 3.1, one can write

$$\|\Delta S_n(t)\| \leq \left[\xi^{1-\alpha}(\mu+\beta\eta_2)\left(\frac{1-\alpha}{M(\alpha)}+\frac{\alpha}{M(\alpha)}T\right)\right]^n \eta_1,$$

$$\|\Delta I_n(t)\| \leq \left[\xi^{1-\alpha}(\mu+\beta\eta_2)\left(\frac{1-\alpha}{M(\alpha)}+\frac{\alpha}{M(\alpha)}T\right)\right]^n \eta_2,$$

$$\|\Delta R_n(t)\| \leq \left[\xi^{1-\alpha}(\mu+\beta\eta_2)\left(\frac{1-\alpha}{M(\alpha)}+\frac{\alpha}{M(\alpha)}T\right)\right]^n \eta_3,$$

$$\|\Delta C_n(t)\| \leq \left[\xi^{1-\alpha}(\mu+\beta\eta_2)\left(\frac{1-\alpha}{M(\alpha)}+\frac{\alpha}{M(\alpha)}T\right)\right]^n \eta_4, \tag{3.17}$$

it is obvious that from hypotheses E_1, E_2, E_3, E_4, we have $\|\Delta S_n(t)\| \to 0$, $\|\Delta I_n(t)\| \to 0$, $\|\Delta R_n(t)\| \to 0$, and $\|\Delta C_n(t)\|$ when $n \to \infty$. Thus the existence of the solution of model (1.2)-(1.3) is proved.

Step3

Finally, we check the uniqueness of the solution.

Suppose that $(S(t), I(t), R(t), C(t))$ and $\left(\widetilde{S}(t), \widetilde{I}(t), \widetilde{R}(t), \widetilde{C}(t)\right)$ are two different solutions of model (1.2)-(1.3).

Using Theorem 3.1 for another time, we get

$$\left\|S(t)-\widetilde{S}(t)\right\| \leq \xi^{1-\alpha}(\mu+\beta\eta_2)\left(\frac{1-\alpha}{M(\alpha)}+\frac{\alpha}{M(\alpha)}T\right)\left\|S(t)-\widetilde{S}(t)\right\|,$$

$$\left\|I(t)-\widetilde{I}(t)\right\| \leq \xi^{1-\alpha}(\beta\eta_1+\sigma\beta\eta_4+\mu+\theta)\left(\frac{1-\alpha}{M(\alpha)}+\frac{\alpha}{M(\alpha)}T\right)\left\|I(t)-\widetilde{I}(t)\right\|,$$

$$\left\|R(t)-\widetilde{R}(t)\right\| \leq \xi^{1-\alpha}(\mu+\delta)\left(\frac{1-\alpha}{M(\alpha)}+\frac{\alpha}{M(\alpha)}T\right)\left\|R(t)-\widetilde{R}(t)\right\|,$$

$$\left\|C(t)-\widetilde{C}(t)\right\| \leq \xi^{1-\alpha}(\beta\eta_1+\mu+\gamma)\left(\frac{1-\alpha}{M(\alpha)}+\frac{\alpha}{M(\alpha)}T\right)\left\|C(t)-\widetilde{C}(t)\right\|. \tag{3.18}$$

Taking into account hypotheses E_1, E_2, E_3, E_4, it follows that $\left\|S(t)-\widetilde{S}(t)\right\|=0$, $\left\|I(t)-\widetilde{I}(t)\right\|=0$, $\left\|R(t)-\widetilde{R}(t)\right\|=0$, and $\left\|C(t)-\widetilde{C}(t)\right\|=0$. Therefore $(S(t), I(t), R(t), C(t)) = \left(\widetilde{S}(t), \widetilde{I}(t), \widetilde{R}(t), \widetilde{C}(t)\right)$. □

3.2 Stability of equilibrium points

In order to find the equilibrium points, we put ${}^{CF}D_t^\alpha S=0$, ${}^{CF}D_t^\alpha I=0$, ${}^{CF}D_t^\alpha R=0$ and ${}^{CF}D_t^\alpha C=0$.

Thus, we find that the disease-free equilibrium point is $E_0=(1,0,0,0)$ and the endemic equilibrium point is $E_1=(S_1,I_1,R_1,C_1)$ which will be provided and discussed later.

For studying the stability of system (1.2), we need mainly to have the basic reproduction number R_0, which is the spectral radius of the next-generation matrix FV^{-1} [9]. Therefore, we should rewrite system (1.2) as follows

$${}^{CF}D_t^\alpha X(t) = \mathcal{F}(X(t)) - \vartheta(X(t)), \text{ with } X(t) = (S(t), C(t), I(t), R(t))^T, \tag{3.19}$$

such that

$$\mathcal{F}(X(t)) = \xi^{1-\alpha} \begin{pmatrix} 0 \\ 0 \\ \beta S(t) I(t) + \sigma\beta C(t) I(t) \\ \beta C(t) I(t) \end{pmatrix}, \tag{3.20}$$

and

$$\vartheta(X(t)) = \xi^{1-\alpha} \begin{pmatrix} -\mu(1 - S(t)) + \beta S(t) I(t) - \gamma C(t) \\ -\delta R(t) + \beta C(t) I(t) + (\gamma + \mu) C(t) \\ (\mu + \theta) I(t) \\ \sigma\beta C(t) I(t) - \theta I(t) + (\mu + \delta) R(t) \end{pmatrix}. \tag{3.21}$$

where

$$F = \xi^{1-\alpha} \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix}, \; V = \xi^{1-\alpha} \begin{pmatrix} \mu + \theta & 0 \\ -\theta & \mu + \delta \end{pmatrix}, \tag{3.22}$$

consequently

$$FV^{-1} = \begin{pmatrix} \frac{\beta}{\mu+\theta} & \frac{-\beta}{(\mu+\theta)(\mu+\delta)} \\ 0 & 0 \end{pmatrix}, \tag{3.23}$$

thus

$$R_0 = \frac{\beta}{\mu + \theta}. \tag{3.24}$$

3.2.1 The stability of the disease-free equilibrium

Definition 3.3. [22] Let ${}^{CF}D_t^{\alpha} X(t) = AX(t)$ a fractional-order linear system, where $X(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $0 < \alpha < 1$.

Then, its corresponding characteristic equation is

$$\det[\lambda(I - (1 - \alpha) A) - \alpha A] = 0. \tag{3.25}$$

Lemma 3.4. [22] if $(I - (1 - \alpha) A)$ is invertible, then the system ${}^{CF}D_t^{\alpha} X(t) = AX(t)$ is asymptotically stable if and only if the real parts of the roots of its characteristic equation are negative.

Due to the non-linearity of our model (1.2), we must first linearize it. The Jacobian matrix of system (1.2) evaluated at the disease-free equilibrium point E_0 is given by

$$J(E_0) = \xi^{1-\alpha} \begin{pmatrix} -\mu & -\beta & 0 & \gamma \\ 0 & \beta - \mu - \theta & 0 & 0 \\ 0 & \theta & -\mu - \delta & 0 \\ 0 & 0 & \delta & -\mu - \gamma \end{pmatrix}. \tag{3.26}$$

The characteristic equation as defined in [22] of the linearized system of the model (1.2) at E_0 is

$$\det[\lambda(I - (1 - \alpha) J(E_0)) - \alpha J(E_0)] = 0. \tag{3.27}$$

Theorem 3.5. The disease-free equilibrium point E_0 of the model (1.2) is locally asymptotically stable if and only if the real parts of the roots of the characteristic equation (3.27) are negative.

Proof . Let $G = \lambda(I - (1 - \alpha) J(E_0)) - \alpha J(E_0)$, then

$$\begin{aligned} \det G = {} & \xi^{1-\alpha} (\lambda[1 + (1 - \alpha)\mu] + \alpha\mu) \times (\lambda[1 - (1 - \alpha)(\beta - \mu - \theta)] - \alpha(\beta - \mu - \theta)) \\ & \times (\lambda[1 + (1 - \alpha)(\mu + \delta)] + \alpha(\mu + \delta)) \times (\lambda[1 + (1 - \alpha)(\mu + \gamma)] + \alpha(\mu + \gamma)). \end{aligned} \tag{3.28}$$

Hence, the roots of this character equation are

$$\begin{aligned}\lambda_1 &= \frac{-\alpha\mu}{1+(1-\alpha)\mu}, \lambda_2 = \frac{\alpha(\beta-\mu-\theta)}{1-(1-\alpha)(\beta-\mu-\theta)}, \lambda_3 = \frac{-\alpha(\mu+\delta)}{1+(1-\alpha)(\mu+\delta)},\\ \lambda_4 &= \frac{-\alpha(\mu+\gamma)}{1+(1-\alpha)(\mu+\gamma)}.\end{aligned} \tag{3.29}$$

As $\alpha \in (0,1)$ and $R_0 < 1$, we can easily show that all eigenvalues are negative. □

3.2.2 The stability of the endemic equilibrium point

If $R_0 > 1$, the system (1.2) admits a unique positive endemic point $E_1 = (S_1, I_1, R_1, C_1)$, such that

$$\begin{aligned}S_1 &= \frac{\mu+\theta}{\beta} - \sigma\left(\frac{\delta\theta I_1}{(\mu+\delta\sigma)\beta I_1 + (\mu+\gamma)(\mu+\delta)}\right),\\ R_1 &= \frac{\theta I_1(\beta I_1+\mu+\gamma)}{(\mu+\delta\sigma)\beta I_1 + (\mu+\gamma)(\mu+\delta)},\\ C_1 &= \frac{\sigma\theta I_1}{(\mu+\delta\sigma)\beta I_1 + (\mu+\gamma)(\mu+\delta)},\end{aligned}$$

and $I_1$is the positive root of $P(I) = D_1 I^2 + D_2 I + D_3(1-R_0)$, where

$$\begin{aligned}D_1 &= \beta\mu(\theta+\mu+\delta\sigma),\\ D_2 &= \beta\mu(-\beta\mu+\gamma(\delta+\theta+\mu)+(\theta+\mu)(2\mu+\delta)+\delta(-\beta+\mu)\sigma),\\ D_3 &= \mu(\mu+\gamma)(\mu+\delta)(\theta+\mu).\end{aligned}$$

Lemma 3.6. [22] The fractional-order linear system ${}^{CF}D_t^{\alpha}X(t) = AX(t)$, where $X(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $0 < \alpha < 1$, is asymptotically stable if eigenvalues $\lambda(A)$ of matrix A satisfy one of the following conditions

1) $\|\lambda(A)\| \geq \frac{1}{1-\alpha}$, $\lambda(A) \neq \frac{1}{1-\alpha}$,
2) $Re(\lambda(A)) > \frac{1}{1-\alpha}$,
3) $Re(\lambda(A)) < 0$,
4) $|Im(\lambda(A))| > \frac{1}{2(1-\alpha)}$.

Theorem 3.7. The endemic equilibrium point E_1 of the model (1.2)is asymptotically stable if and only if the eigenvalues of jacobian matrix $\lambda(J(E_1))$ satisfy one of the following conditions

1) $\|\lambda(J(E_1))\| \geq \frac{1}{1-\alpha}$, $\lambda(J(E_1)) \neq \frac{1}{1-\alpha}$,
2) $Re(\lambda(J(E_1))) > \frac{1}{1-\alpha}$,
3) $Re(\lambda(J(E_1))) < 0$,
4) $|Im(\lambda(J(E_1)))| > \frac{1}{2(1-\alpha)}$.

Proof . The proof follows directly from Lemma 3.6. □

The Jacobian matrix of system (1.2) evaluated at the endemic equilibrium point E_1 is given by

$$J(E_1) = \xi^{1-\alpha}\begin{pmatrix} -\mu-\beta I_1 & -\beta S_1 & 0 & \gamma \\ \beta I_1 & \beta S_1 + \sigma\beta C_1 - (\mu+\theta) & 0 & \sigma\beta I_1 \\ 0 & (1-\sigma)\beta C_1 + \theta & -(\mu+\delta) & (1-\sigma)\beta I_1 \\ 0 & -\beta C_1 & \delta & -\beta I_1 - (\mu+\gamma) \end{pmatrix}. \tag{3.30}$$

The characteristic equation of the linearised system of the model (1.2) at E_1 is

$$(\lambda+\mu)\left(\lambda^3+b_1\lambda^2+b_2\lambda+b_3\right)=0, \tag{3.31}$$

where

$$\begin{aligned} b_1 &= \gamma+\delta+2\mu+2\beta I_1, \\ b_2 &= (\mu+\gamma)(\mu+\delta)+\beta I_1(\gamma+\delta+\theta+3\mu+\sigma\delta+\beta I_1), \\ b_3 &= \beta I_1\left[\frac{\omega((\mu+\delta)(\mu+\theta)+\gamma(\delta+\theta+\mu)-\delta\theta\sigma)}{\omega+\beta(\delta\sigma+\mu)I_1}\right. \\ &\quad \left.+\frac{\beta(\delta\sigma+\theta+\mu)I_1(2\omega+\beta(\delta\sigma+\mu)I_1)}{\omega+\beta(\delta\sigma+\mu)I_1}\right], \end{aligned}$$

and $\omega=(\mu+\gamma)(\mu+\gamma)$.

Let $Q(\lambda)=\lambda^3+b_1\lambda^2+b_2\lambda+b_3$, then the discriminant of $Q(\lambda)$ is given by

$$\Delta(Q)=18b_1b_2b_3+(b_1b_2)^2-4b_3b_1^3-4b_2^3-27b_3^3. \tag{3.32}$$

Proposition 3.8. We assume that the endemic equilibrium point $E_1=(S_1,I_1,R_1,C_1)$exists in $\mathbb{R}^4_+$.

1) If $\Delta(Q)>0,\ a_1>0,\ a_3>0,\ a_1a_2>a_3$, then all eigenvalues are real negative, so E_1 is locally asymptotically stable.

2) If $\Delta(Q)>0,\ a_1>0,\ a_2>0,\ a_3<0$, then there is one real positive eigenvalue that we denote $\lambda_1$and two negative eigenvalues, so the system is locally asymptotically stable if $\lambda_1>\frac{1}{1-\alpha}$.

3) If $\Delta(Q)>0,\ a_1<0,\ a_2>0,\ a_3<0$, then all the eigenvalues $\lambda_i,\ i=1,2,3$ are nonnegative, so the system is locally asymptotically stable if $\lambda_i>\frac{1}{1-\alpha},\ i=1,2,3$.

4) If $\Delta(Q)<0$, there is one real root $\lambda_1$and two complex roots $\lambda_2=\overline{\lambda_3}$, and the sign of $\lambda_1$depends on the sign of a_3, (λ_1 and a_3 have opposite signs), so we have two cases:

i) If $a_3>0$, then $\lambda_1<0$, and the system is stable if $Re(\lambda_2)=Re(\lambda_3)>\frac{1}{1-\alpha}$.

ii) If $a_3<0$, then $\lambda_1>0$, and the system is stable if $\lambda_1,\ Re(\lambda_2),\ Re(\lambda_3)>\frac{1}{1-\alpha}$.

Proof . The proof is deduced from Theorem 3.7 and Routh-Hurwitz criterion [1] □

3.3 Numerical method, simulations, and results discussions

3.3.1 Description of the numerical method

In general, exact solutions of fractional epidemiological models are not always available. Hence an iterative solution of our SIRC model (1.2) is proposed. In this work, we have chosen to adopt the three-step Adams-Bashforth technique [35].

Consider the first equation of system (1.2)

$$\xi^{\alpha-1}\ {}^{CF}D_t^{\alpha}S=N_1(t,S(t)),\ \text{with initial condition } S(0). \tag{3.33}$$

Using the fractional integral given by (2.2), we get

$$S(t)=\xi^{1-\alpha}\left[S(0)+\frac{(1-\alpha)}{M(\alpha)}N_1(t,S(t))+\frac{\alpha}{M(\alpha)}\int_0^t N_1(\tau,S(\tau))\,d\tau\right]. \tag{3.34}$$

Next, by dividing the time interval $[0,T]$ into n sub-intervals of duration h, where $t_{k+1}=t_k+h,\ k=0,1,2,...,n-1$, and substituting t in fomula (3.34) by t_k and t_{k+1},we find

$$S(t_k)=\xi^{1-\alpha}\left[S(0)+\frac{(1-\alpha)}{M(\alpha)}N_1(t_{k-1},S(t_{k-1}))+\frac{\alpha}{M(\alpha)}\int_0^{t_k} N_1(t,S(t))\,dt\right], \tag{3.35}$$

and

$$S\left(t_{k+1}\right)=\xi^{1-\alpha}\left[S\left(0\right)+\frac{\left(1-\alpha\right)}{M\left(\alpha\right)}N_1\left(t_k,S\left(t_k\right)\right)+\frac{\alpha}{M\left(\alpha\right)}\int_0^{t_{k+1}}N_1\left(t,S\left(t\right)\right)dt\right]. \tag{3.36}$$

Then, subtracting equation (3.35) from equation (3.36) , yields

$$S\left(t_{k+1}\right)-S\left(t_k\right)=\xi^{1-\alpha}\left[\frac{\left(1-\alpha\right)}{M\left(\alpha\right)}\left(N_1\left(t_k,S\left(t_k\right)\right)-N_1\left(t_{k-1},S\left(t_{k-1}\right)\right)\right)+\frac{\alpha}{M\left(\alpha\right)}\int_{t_k}^{t_{k+1}}N_1\left(t,S\left(t\right)\right)dt\right]. \tag{3.37}$$

Now, taking into acount that generally, one can get $\xi=1$ and approximating the integral $\int_{t_k}^{t_{k+1}}N_1\left(t,S\left(t\right)\right)dt$ according to [35], we obtain

$$\begin{aligned}S\left(t_{k+1}\right)=S\left(t_k\right)&+\frac{1}{M\left(\alpha\right)}\left[\left(1-\alpha\right)+\frac{23h\alpha}{12}\right]N_1\left(t_k,S\left(t_k\right)\right)-\frac{1}{M\left(\alpha\right)}\left[\left(1-\alpha\right)+\frac{4h\alpha}{3}\right]N_1\left(t_{k-1},S\left(t_{k-1}\right)\right)\\&+\frac{5h\alpha}{12M\left(\alpha\right)}N_1\left(t_{k-2},S\left(t_{k-2}\right)\right).\end{aligned} \tag{3.38}$$

Therefore, the recursive formula for SIRC model (1.2) according to three-step fractional Adams-Bashforth scheme is as follow

$$\begin{cases}S\left(t_{k+1}\right)=S\left(t_k\right)+\frac{1}{M(\alpha)}\left[\left(1-\alpha\right)+\frac{23h\alpha}{12}\right]N_1\left(t_k,S\left(t_k\right)\right)\\\quad-\frac{1}{M(\alpha)}\left[\left(1-\alpha\right)+\frac{4h\alpha}{3}\right]N_1\left(t_{k-1},S\left(t_{k-1}\right)\right)\\\quad+\frac{5h\alpha}{12M(\alpha)}N_1\left(t_{k-2},S\left(t_{k-2}\right)\right),\\I\left(t_{k+1}\right)=I\left(t_k\right)+\frac{1}{M(\alpha)}\left[\left(1-\alpha\right)+\frac{23h\alpha}{12}\right]N_2\left(t_k,I\left(t_k\right)\right)\\\quad-\frac{1}{M(\alpha)}\left[\left(1-\alpha\right)+\frac{4h\alpha}{3}\right]N_2\left(t_{k-1},I\left(t_{k-1}\right)\right)\\\quad+\frac{5h\alpha}{12M(\alpha)}N_2\left(t_{k-2},I\left(t_{k-2}\right)\right),\\R\left(t_{k+1}\right)=R\left(t_k\right)+\frac{1}{M(\alpha)}\left[\left(1-\alpha\right)+\frac{23h\alpha}{12}\right]N_3\left(t_k,R\left(t_k\right)\right)\\\quad-\frac{1}{M(\alpha)}\left[\left(1-\alpha\right)+\frac{4h\alpha}{3}\right]N_3\left(t_{k-1},R\left(t_{k-1}\right)\right)\\\quad+\frac{5h\alpha}{12M(\alpha)}N_3\left(t_{k-2},R\left(t_{k-2}\right)\right),\\C\left(t_{k+1}\right)=C\left(t_k\right)+\frac{1}{M(\alpha)}\left[\left(1-\alpha\right)+\frac{23h\alpha}{12}\right]N_4\left(t_k,C\left(t_k\right)\right)\\\quad-\frac{1}{M(\alpha)}\left[\left(1-\alpha\right)+\frac{4h\alpha}{3}\right]N_4\left(t_{k-1},C\left(t_{k-1}\right)\right)\\\quad+\frac{5h\alpha}{12M(\alpha)}N_4\left(t_{k-2},C\left(t_{k-2}\right)\right).\end{cases} \tag{3.39}$$

3.3.2 Numerical simulations and results discussions

Some numerical simulations are provided to illustrate our theoretical findings. Graphs are plotted using our Matlab code of the three-step fractional Adams-Bashforth method.

Simulation 1

First, we simulate system (1.2) subject to the initial conditions $\left(S\left(0\right),I(0),\ R\left(0\right),\ C\left(0\right)\right)^T=(0.3,0.5,0,0)$ with the following parameter values: $\mu=0.3,\ \beta=0.4,\ \gamma=0.2,\ \sigma=0.05,\ \theta=0.5,\ \delta=0.5$ [23]. In this case, we find that $R_0=0.5<1$.

Hence, system (1.2) has a unique disease-free equilibrium $E_0\left(1,0,0,0\right)$ which is locally asymptotically stable from Theorem 3.5 Where eigenvalues:
$\lambda_1=\frac{-0.3\alpha}{1.3-0.3\alpha},\lambda_2=\frac{-0.4\alpha}{1.4-0.4\alpha},\lambda_3=\frac{-0.8\alpha}{1.8-0.8\alpha},\lambda_4=\frac{-0.5\alpha}{1.5-0.5\alpha}$ are negative for all values of α ($\alpha=0.98,0.90,0.80,0.70$). In this case, influenza will die out.

Figure.1-5 show the behavior of approximate solutions $S\left(t\right),\ I\left(t\right),\ R\left(t\right)$, and $C\left(t\right)$ for various values of α. According to these plots, system (1.2) converges fastly to its steady-state when the value of α increases.

Figure.6-13 show the stability of approximate solutions of fractional SIRC model within the framework of the two operators Caputo-Fabrizio and Caputo, compared to classical one (integer-order) for various values of α. As demonstrated graphically, all solutions of the model in the sense of Caputo-Fabrizio converge to the disease-free

steady state faster than solutions of the model in the sense of Caputo. When $\alpha \to 1$ their curves get closer. Solutions curves produced by Caputo were plotted using fde12, which is an implementation of the predictor-corrector method of Adams-Bashforth-Moulton [8].

One can observe from **Figure.14-17** that the number of susceptible individuals decreases when β (contact rate of flu) increases, while the number of infected, recovered and cross-immune individuals rises. The same behavior was observed for all values of α.

Figure.18-21 depicted the influence of changing the parameter θ (θ^{-1} is the infectious period) on the variables states $S(t)$, $I(t)$, $R(t)$, and $C(t)$. When θ increases, the number of infected people declines to a lower level. On the other hand, the number of susceptible, recovered, and cross-immune individuals grows.

Consequently, to stop the spread of influenza, we have to reduce the parameter β and increase θ.

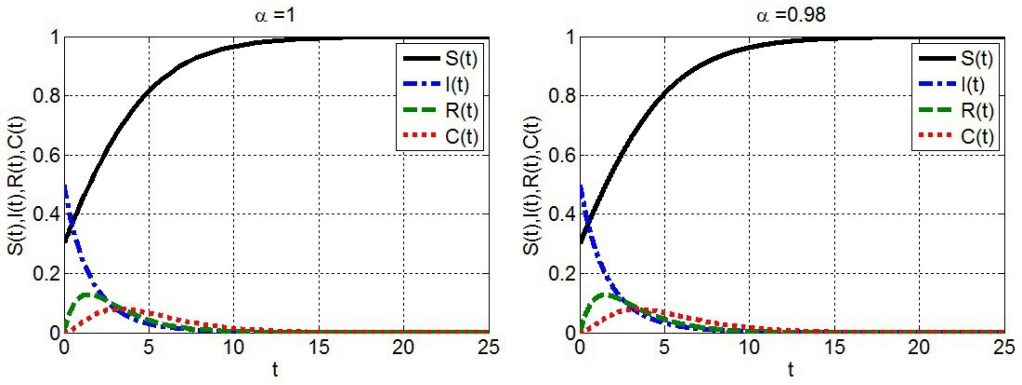

Figure 1: The behavior of approximate solutions of $S(t)$, $I(t)$, $R(t)$, and $C(t)$ for $\alpha = 1$ and $\alpha = 0.98$.

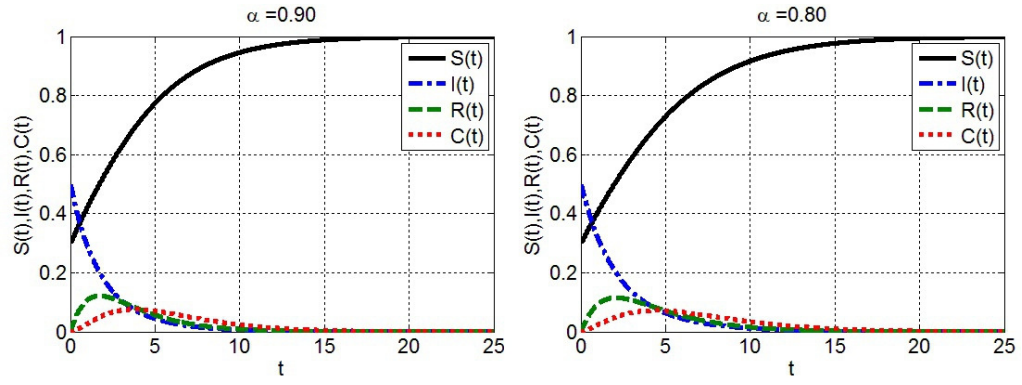

Figure 2: The behavior of approximate solutions of $S(t)$, $I(t)$, $R(t)$, and $C(t)$ for $\alpha = 0.90$ and $\alpha = 0.80$.

Simulation 2

Now, we simulate system (1.2) subject to initial conditions $(S(0), I(0), R(0), C(0))^T = (0.3, 0.5, 0, 0)$ with the following parameter values: $\mu = 0.4$, $\beta = 0.5$, $\gamma = 0.35$, $\sigma = 0.05$, $\theta = 0.05$, $\delta = 0.05$ [23] for $\alpha = 0.98$, $\alpha = 0.90$, $\alpha = 0.80$, and $\alpha = 0.70$. In this case we find that $R_0 = 1.11 > 1$. Hence, system (1.2) has a unique endemic equilibrium point E_1.

Figure.22-24 show the steady-state solution when influenza persists in the population. In this situation, both susceptible and infected individuals are expected to survive. As seen in simulation1, when the value of α increases, system (1.2) quickly converges to the endemic equilibrium point. That is to say, as α decreases, the disease's peak postpones.

3.4 Conclusion

In this paper, the fractional SIRC model of influenza A viruses in the Caputo-Fabrizio sense was investigated. The well-known Banach contraction has been employed to prove the existence and uniqueness of the solution of the

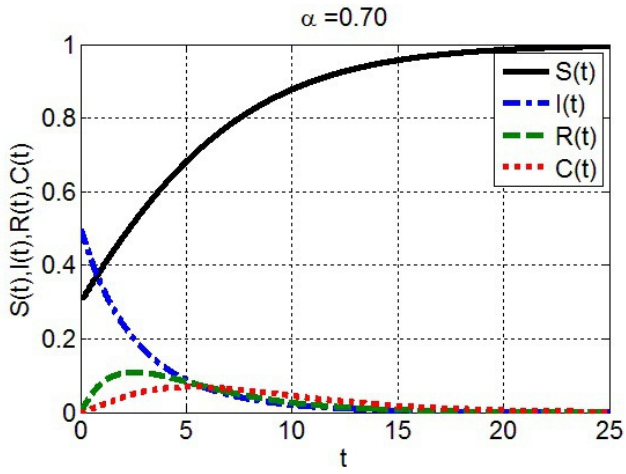

Figure 3: The behavior of approximate solutions of $S(t)$, $I(t)$, $R(t)$, and $C(t)$ for $\alpha = 0.70$.

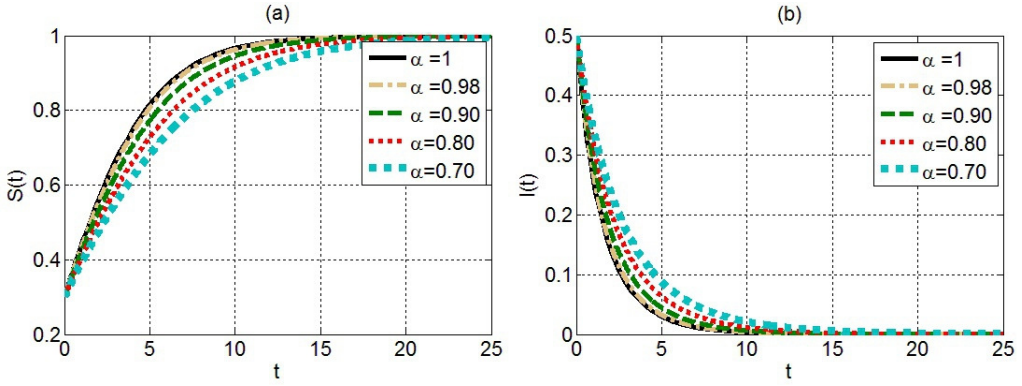

Figure 4: The behavior of approximate solutions of susceptible $S(t)$ and infected $I(t)$ individuals for different values of α ($\alpha = 0.98,\ 0.90,\ 0.80,\ 0.70$).

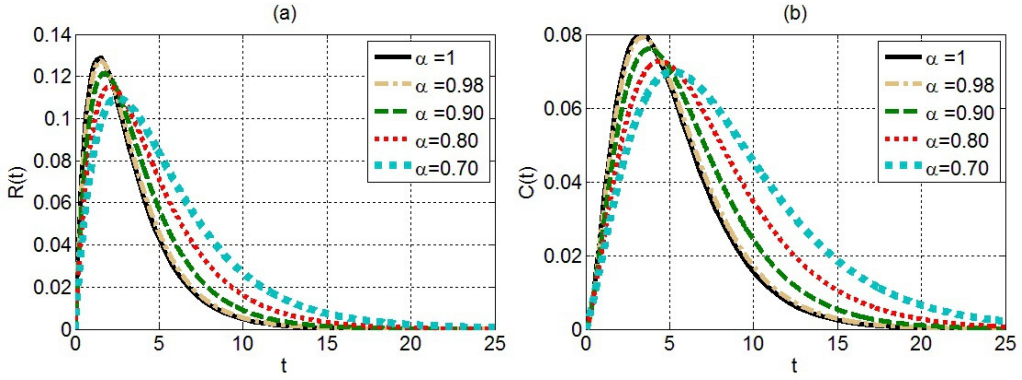

Figure 5: The behavior of approximate solutions of recovered $R(t)$ and cross-immune $C(t)$ individuals for different values of α ($\alpha = 0.98,\ 0.90,\ 0.80,\ 0.70$).

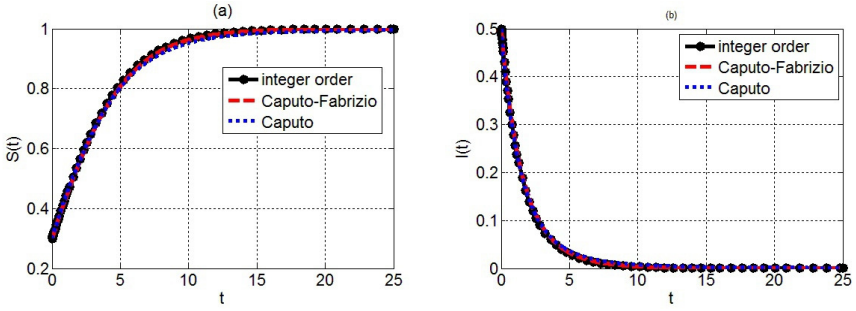

Figure 6: The stability of the disease-free steady-state for $\alpha = 0.98$ via Caputo and Caputo-Fabrizio ($(a)S(t)$, $(b)I(t)$).

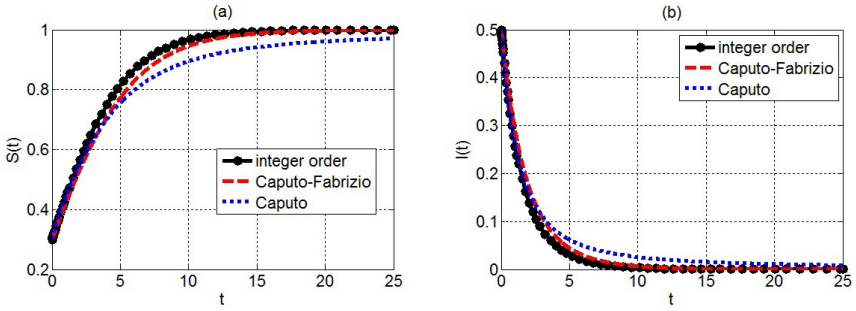

Figure 7: The stability of the disease-free steady-state for $\alpha = 0.90$ via Caputo and Caputo-Fabrizio ($(a)S(t)$, $(b)I(t)$).

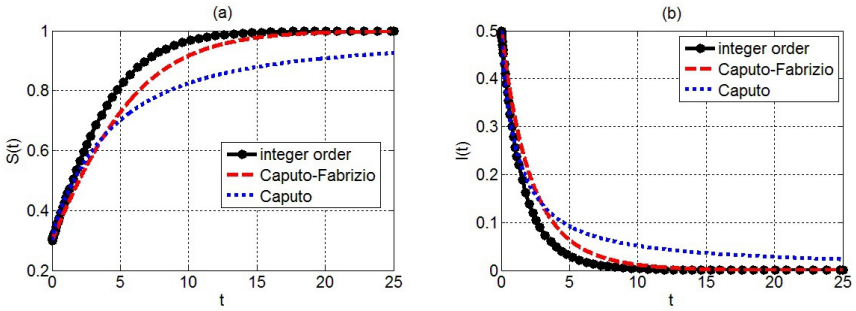

Figure 8: The stability of the disease-free steady-state for $\alpha = 0.80$ via Caputo and Caputo-Fabrizio ($(a)S(t)$, $(b)I(t)$).

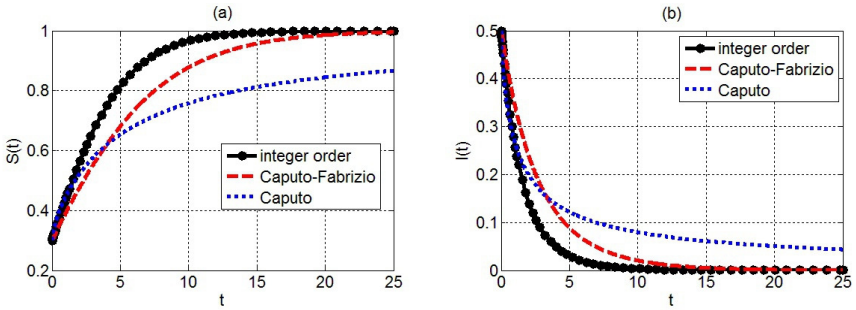

Figure 9: The stability of the disease-free steady-state for $\alpha = 0.70$ via Caputo and Caputo-Fabrizio ($(a)S(t)$, $(b)I(t)$).

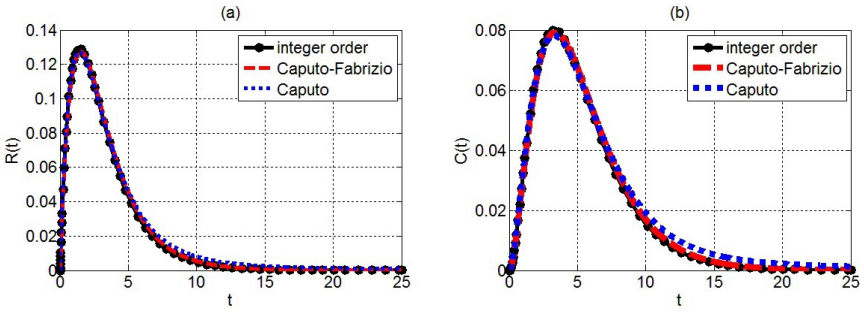

Figure 10: The stability of the disease-free steady-state for $\alpha = 0.98$ via Caputo and Caputo-Fabrizio ($(a)R(t)$, $(b)C(t)$).

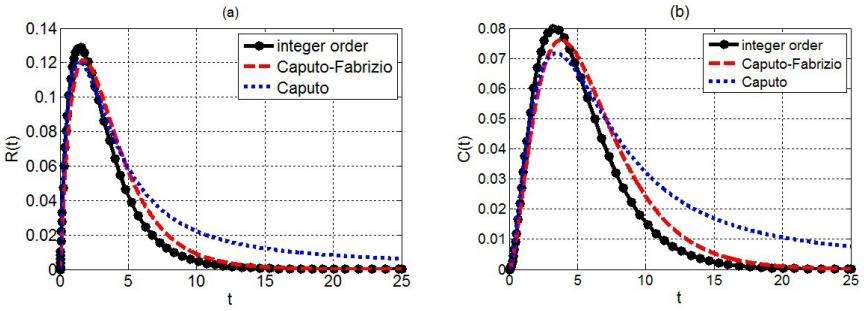

Figure 11: The stability of the disease-free steady-state for $\alpha = 0.90$ via Caputo and Caputo-Fabrizio ($(a)R(t)$, $(b)C(t)$).

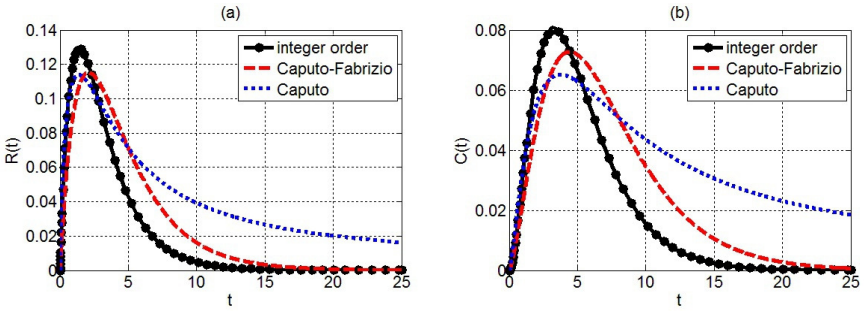

Figure 12: The stability of the disease-free steady-state for $\alpha = 0.80$ via Caputo and Caputo-Fabrizio ($(a)R(t)$, $(b)C(t)$).

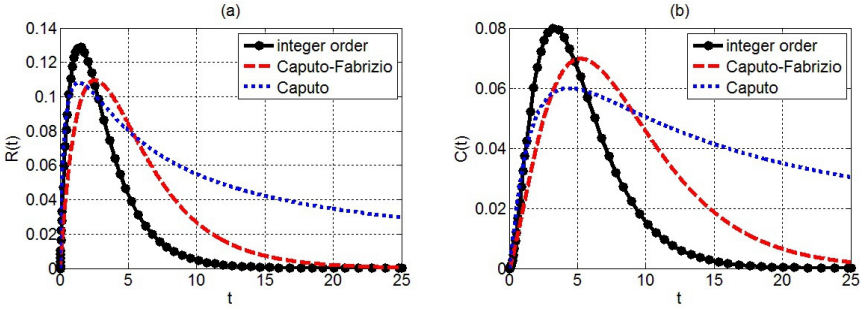

Figure 13: The stability of the disease-free steady-state for $\alpha = 0.70$ via Caputo and Caputo-Fabrizio ($(a)R(t)$, $(b)C(t)$).

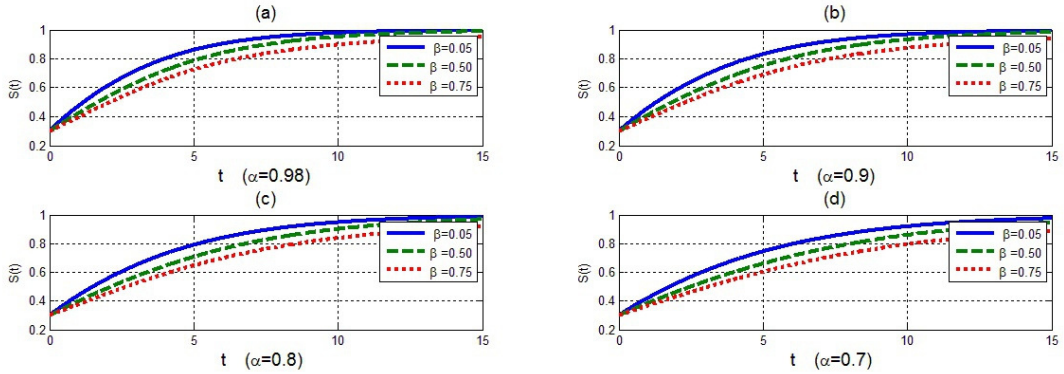

Figure 14: Influence of β on the number of susceptible individuals $S(t)$ for different values of α ($\alpha = 0.98,\ 0.90,\ 0.80,\ 0.70$).

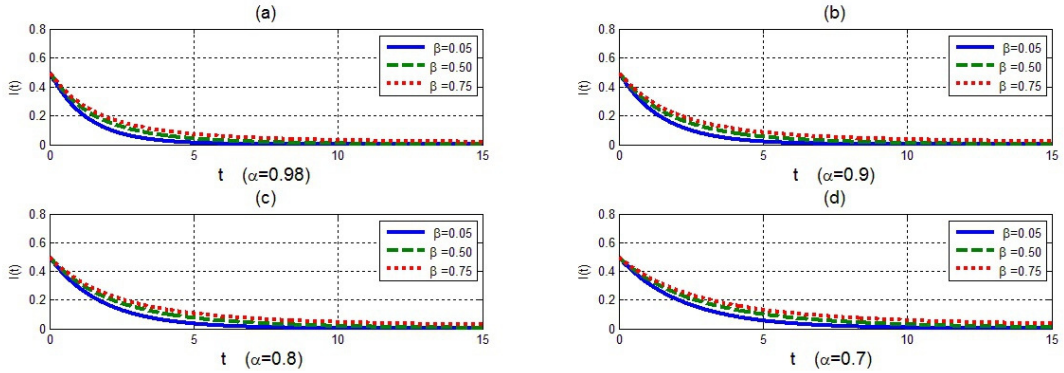

Figure 15: Influence of β on the number of infected individuals $I(t)$ for different values of α ($\alpha = 0.98,\ 0.90,\ 0.80,\ 0.70$).

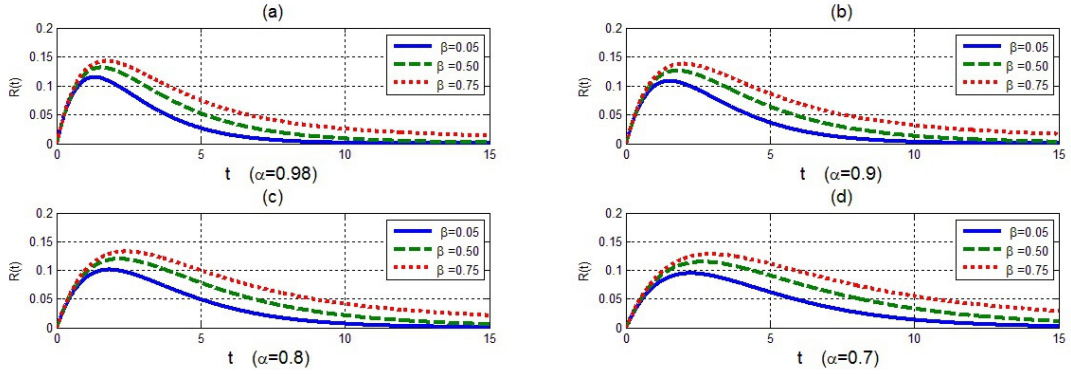

Figure 16: Influence of β on the number of recovered individuals $R(t)$ for different values of α ($\alpha = 0.98,\ 0.90,\ 0.80,\ 0.70$).

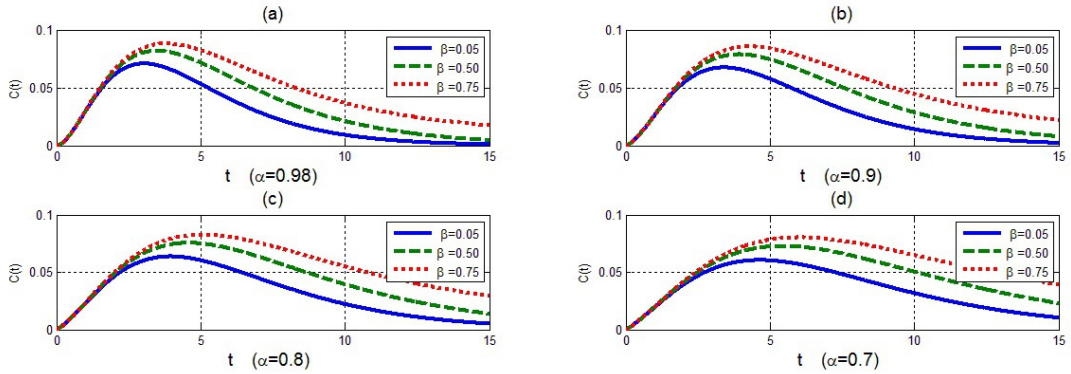

Figure 17: Influence of β on the number of cross-immune individuals $C(t)$ for different values of α ($\alpha = 0.98,\ 0.90,\ 0.80,\ 0.70$).

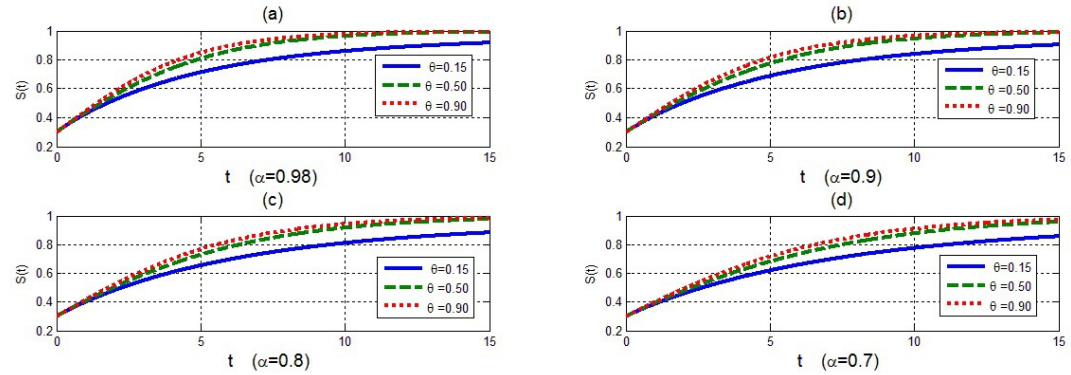

Figure 18: Influence of θ on the number of suseptible individuals $S(t)$ for different values of α ($\alpha = 0.98,\ 0.90,\ 0.80,\ 0.70$).

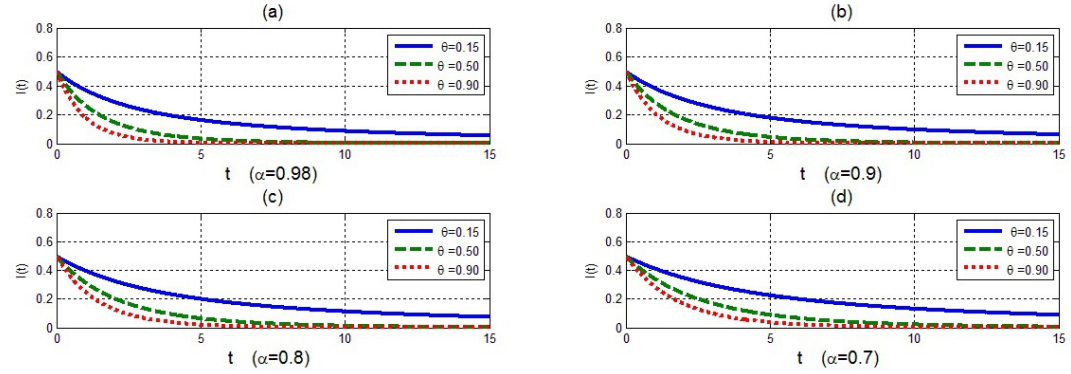

Figure 19: Influence of θ on the number of infected individuals $I(t)$ for different values of α ($\alpha = 0.98,\ 0.90,\ 0.80,\ 0.70$).

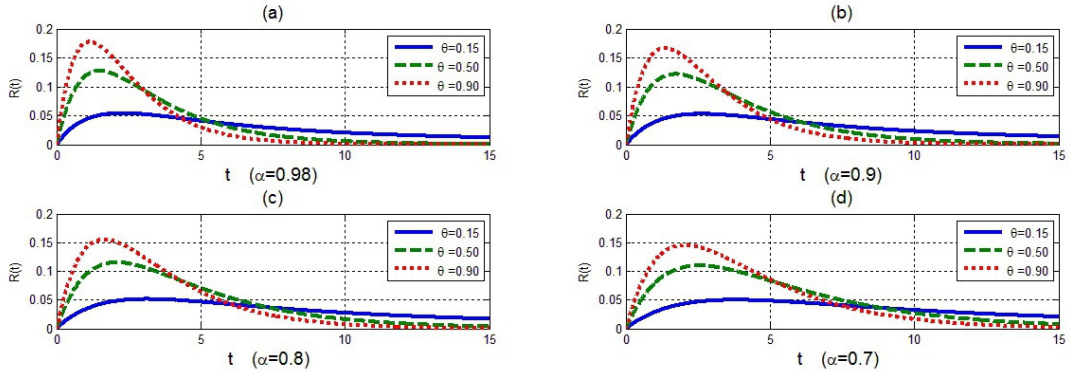

Figure 20: Influence of θ on the number of recovered individuals $R(t)$ for different values of α ($\alpha = 0.98,\ 0.90,\ 0.80,\ 0.70$).

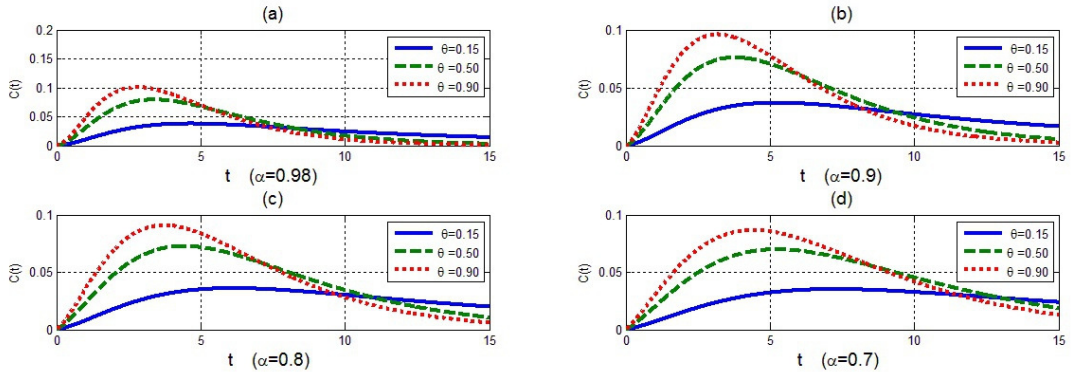

Figure 21: Influence of θ on the number of cross-immune individuals $C(t)$ for different values of α ($\alpha = 0.98,\ 0.90,\ 0.80,\ 0.70$).

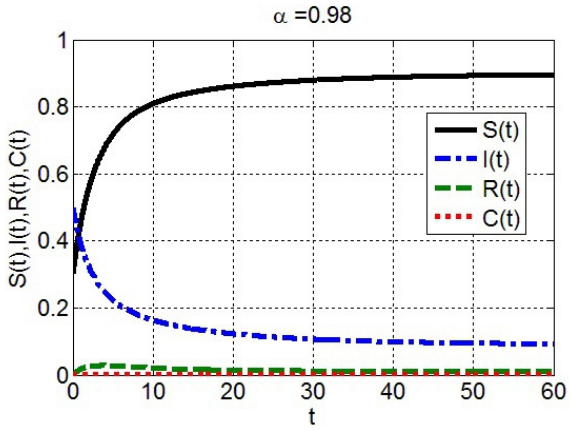

Figure 22: The behavior of approximate solutions of $S(t)$, $I(t)$, $R(t)$, and $C(t)$ (stability of the endemic equilibrium point).

proposed model. The reproduction number of the disease was calculated and the stability of equilibrium points was established. In addition, we gave more conditions for the endemic equilibrium point than those presented in the literature. An efficient numerical method based on the three-step Adams-Bashforth scheme has been used for getting approximate solutions. Many plots were given for various values of α for checking the behavior of approximative solutions. The obtained results were found to be extremely close to the classical solutions when $\alpha \to 1$ and the convergence to the equilibrium states was more slowly when α decreases. From this study, it was indeed understood also that increasing θ the inverse of infections period and reducing β the contact rate of susceptible and infected people had a great contribution to controlling and stopping the spread of influenza in the population. Furthermore, the presented comparative analysis with Caputo derivative shows the accuracy and efficiency of our approach.

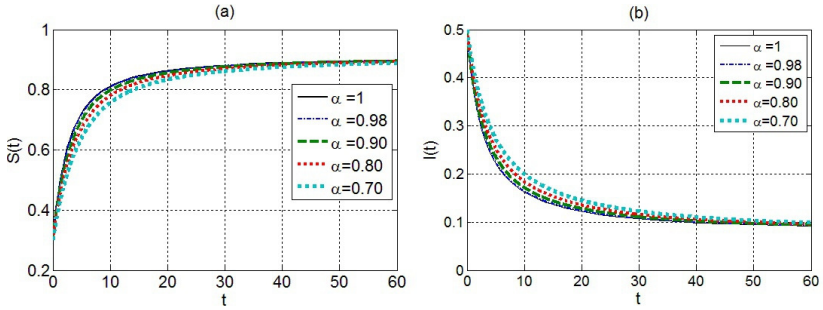

Figure 23: The behavior of approximate solutions of susceptible $S(t)$ and infected $I(t)$ individuals (the case of endemic equilibrium point).

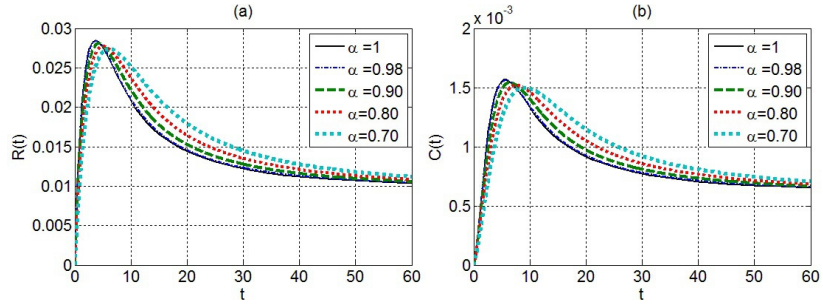

Figure 24: The behavior of approximate solutions of recovered $R(t)$ and cross-immune $C(t)$ individuals (the case of endemic equilibrium point).

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