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New type graph with respect to ideal near ring

Ameer A.J. AL-Swidi*, Ahmed A. Omran

Department of Mathematics, College of Education for Pure Science, University of Babylon, Iraq

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Abstract

Weakly completely prime graph of a near ring $(W_I(N))$ was defined in this paper, the relationship between the elements of a near ring was determined by a definition of weakly completely prime ideal, so we studied many concepts related to the statements of the given near ring. We also found some theories related to previous graphs of our papers.

Keywords: weakly c-prime ideal, c-prime ideal, c-equiprime ideal, edge summation, domination number, chromatic number, uniquely colorable
2020 MSC: 11R44

1 Introduction

Pilz [16] is considered one of the founders of the near ring theory, he presented many basic relationships and theories in this field. Many of our concepts and definitions were based on Groenwald [11] such as completely prime and semi prime ideals. Anderson and Smith [6] defined weakly prime ideal in commutative ring. Al-swidi and Omran [4, 5] defined the completely equiprime graph of a near ring and studied all relationships and theories in this regard, the were also studied uniquely colorable completely equiprime graph of a near ring, which we referred to them in many relationships with these definitions. While Beck [7] studied the simple graph which is the type of zero divisor graph, also Bhavanari, Kuncham and Dasari [9] studied the definition of prime graph of a ring. Our definitions in this paper were to generalize some theories and concepts, for example: domination in graph [13, 17], algebraic coding [3], soft theory [1, 2], general graph [12, 15]. And for the rest of the mathematical concepts.

This part provides some basic definitions that we need:

Let the ideal I of R. Then is called completely prime ideal (c-prime ideal), if $m, n \in R$, $m.n \in I$ then $m \in I$ or $n \in I$ and completely semi prime ideal if $m = n$ [11].

Let the ideal(left ideal) I of a near ring (right near ring), then I is called completely equiprime ideal (c-equiprime ideal) if $a \in N \backslash I$ and $b, d \in N$ with $ab - ad \in I$ implies $b - d \in I$ [14].

Let the ideal I of N. Then is called radical of an ideal if $rad(I) = \{m \in N : m^n \in I, \text{ for some } n \in Z^+\}$ [16].

The non-trivial near ring is called symmetric near ring if $x.0 = 0$ for all $x \in N$ [8].

Let the ideal I of N with for all $a \in N$ a graph $CEQ^a(N)$ the vertices set are elements of N with the pair of vertices $a \neq (b-c)$ are adjacent if and only if $ab-ac \in I$ or $(b-c)a-(b-c)0 \in I$, for all $b, c \in N$ then $U_{a\in N} CEQ^a(N)$ is called completely equiprime graph of N and denoted by $CEQ_I(N)$ [4, 5].

*Corresponding author
Email addresses: ameer.alswidi@yahoo.com (Ameer A.J. AL-Swidi), pure.ahmed.omran@uobabylon.edu.iq (Ahmed A. Omran)

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Most of the definitions below are known by researchers as concepts of the ring R, while in our paper we dealt with as a near ring N.

Let I be an proper ideal of a commutative ring R with identity then is called weakly completely prime, if $0 \neq m.n \in I$ implies $m \in I$ or $n \in I$ and weakly completely semi prime ideal if $m = n$ [6].

We say that it is integral if $a.b = 0$ implies $a = 0$ or $b = 0(a, b \in N)$ [19]. We define, set of right zero divisors Z_r and left zero divisors Z_l of a near ring by $Z_r := \{m \in N | \exists n \in N \backslash \{0\} : n.m = 0\}$ and similarly $Z_l := \{m \in N | \exists n \in N \backslash \{0\} : m.n = 0\}$. Certainly, the zero divisors is a right and left zero divisors.

A graph $\Gamma_I(R)$ define on a commutative R where the vertices sets are elements of R with the pair of vertices $a \neq b$ are adjacent if and only if $a.b = 0$ [7]. Instead the ring we work on a near ring N and denoted by $\Gamma_I(N)$.

Let R be a ring. A graph is called a prime graph of R if $V = R$ with $E = \{\overline{xy} | aRb = 0 \text{ or } bRa = 0, \text{ and } a \neq b\}$, instead the ring we work on N and denoted by $PG(N)$ [9].

A graph G is called a simple if there is no loop and no parallel edges [10, 18]. Two vertices are adjacent if both of them were the ends of the edge. Where the number of incident edges in vertex is defined the degree of vertex. The sample δ is representing the minimum degree and Δ is the maximum degree of G. A graph is connected if there is a path between each pair of vertices, otherwise the graph is disconnected. The distance is the length of the shortest path between the vertices a_1, a_2 in G and denoted by $d(a_1, a_2)$ while the diameter of G is the maximum distance of all pairs and denoted by diam (G). A graph is called Bipartite if can be partition into two sets V_m and V_n such that one vertex in V_m joined to one vertex in V_n and denoted by $B_{m,n}$ and a complete Bipartite graph is a graph such that every vertex in V_m is joined to every vertices in V_n and denoted by $K_{m,n}$ and Bipartite graph of the form $K_{1,n-1}$ is called star graph and denoted by S_n. Let $F \subseteq G$, where F is the set of vertices of G is said domination set if each vertices of F is adjacent or every vertex of G is adjacent to F, the minimum cardinality of S denoted by $\gamma(G)$. A proper coloring of G is a coloring of vertices in G such that no two adjacent vertices have same color,the chromatic number $\chi(G)$ of the minimum number of a proper colors needed to coloring G. A graph with n vertices then achromatic partition of G is a smallest partition of V into disjoint independent sets if G has only one chromatic partition then we say that G is uniquely colorable.

2 Main results

we are presenting some new definitions with proof of theorems, as well as finding relationships with each other and with the rest of the previous concepts and definitions.

Definition 2.1. Let the ideal $I \neq 0$ of N with the vertices set of graph $W_I(N)$ are elements of N and the pair of vertices $a \neq b$ are adjacent if and only if $0 \neq a.b \in I(a.b \in I - \{0\})$ or $0 \neq b.a \in I(b.a \in I - \{0\})$, for all $a, b \in N$ then $W_I(N)$ is called weakly completely prime(weakly c-prime) graph of N.

Example 2.2. Let $N = \{0, 1, 2, 3, a, b, c, d\}$ be a near ring defined in Table 1. And let the ideal $I = \{0, 2, c, d\}$ be an ideal defining on $W_I(N)$, see Figure 1.

Table 1: Multiplication and Addition table of $N = \{0, 1, 2, 3, a, b, c, d\}$

+	0	1	2	3	a	b	c	d		·	0	1	2	3	a	b	c	d
0	0	1	2	3	a	b	c	d		0	0	0	0	0	0	0	0	0
1	1	2	3	0	d	c	a	b		1	0	1	2	3	a	b	c	d
2	2	3	0	1	b	a	d	c		2	0	2	0	2	0	2	0	2
3	3	0	1	2	c	d	b	a		3	0	3	2	1	b	a	c	d
a	a	d	b	c	2	0	1	3		a	0	a	2	b	a	b	c	d
b	b	c	a	d	0	2	3	1		b	0	b	2	a	b	a	c	d
c	c	a	d	b	1	3	0	2		c	0	c	0	c	0	0	0	0
d	d	b	c	a	3	1	2	0		d	0	d	0	d	2	2	0	0

observation 2.3. Let the ideal $I \neq \{0\}$ of N, then

1. $W_I(N)$ is a simple graph.
2. $W_I(N)$ is a finite graph.

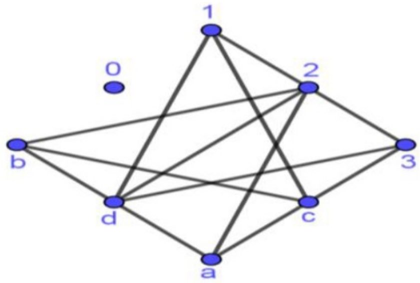

Figure 1: $W_I(N)$

3. The degree of all vertices in $W_I(N)$ is finite.

Proof .

1. As for every $\overline{xy} \in E(W_I(N))$ and for $x, y \in N$ are distinct. Then $W_I(N)$ does not have self-loop or multiple edges and so that is a simple graph.
2. As $V(W_I(N)) = N$ for any ideals of N (finite near ring), so that $V(W_I(N))$ is finite and by (1) as $W_I(N)$ is a simple graph that mean it has no multiple edges, thus $E(W_I(N))$ is finite. Therefore, $W_I(N)$ is a finite graph.
3. Directly from (2).

□

Proposition 2.4. Let $I = \{0, a\}$ be a weakly c-prime ideal of N, then $W_I(N)$ is a star graph, whenever $\{0\}$ be a c-prime ideal in N.

Proof . Let $I = \{0, a\}$ and $\overline{xa} \in E(W_I(N))$, for $x \in N$

then $0 \neq a.\ x \in I$ or $a.x \in I - \{0\}$ (as I is weakly c-prime ideal)

since $\{0\}$ is c-prime ideal then there is not $a.x \in \{0\}$ such that $a \notin \{0\}$ or $x \notin \{0\}$

and we get that there exist edges between a and all elements of N in $W_I(N)$, then $W_I(N)$ is a star graph S_n where $|N| = n$. □

Example 2.5. Let $N = \{0, a, b, c\}$ be a near ring defined in Table 2. And let the ideal $I = \{0, a\}$ be a weakly c-prime defining in $W_I(N)$ is a star graph(see Figure 2) and $I = \{0\}$ is a c-prime ideal in N.

Table 2: Multiplication and Addition table of $N = \{0, a, b, c\}$

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c

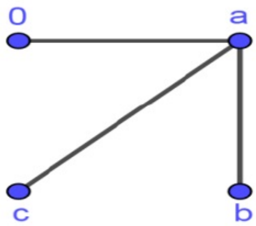

Figure 2: $W_I(N)$ with I is weakly c-prime ideal

Corollary 2.6. Let $I = \{0, a\}$ be a weakly c-prime ideal of N, then $W_I(N)$ is uniquely colorable, whenever $\{0\}$ be a c-prime ideal in N.

Proof . From the Proposition 2.4, the $W_I(N) = S_n$ and the star graph S_n is uniquely colorable then $W_I(N)$ is uniquely colorable. □

Proposition 2.7. Let the two ideals $I \neq \{0\}$ and $K \neq \{0\}$ of N with $I \subseteq K$ then $W_I(N) \subseteq W_K(N)$.

Proof . As $V(W_I(N)) = N = V(W_K(N))$ for any ideals of N

Now, let the edge $\overline{ax} \in E(W_I(N))$ then a is adjacent to x in $W_I(N)$, so that $0 \neq a.x \in I \subseteq K$, therefore $\overline{ax} \in E(W_K(N))$, we get $E(W_I(N)) \subseteq E(W_K(N))$ so that $W_I(N) \subseteq W_K(N)$. □

Proposition 2.8. Let the ideal $I \neq \{0\}$ of N then $W_I(N) \subseteq W_{rad(I)}(N)$.

Proof . As $I \subseteq rad(I)$, then from Proposition 2.7, thus $W_I(N) \subseteq W_{rad(I)}(N)$. □

Example 2.9. In Example 2.2, the radical of the ideal $I = \{0, c\}$ is $rad(I) = \{0, 2, c, d\}$ (see Figure 1) so that $W_I(N) \subseteq W_{rad(I)}(N)$ (see Figure 3).

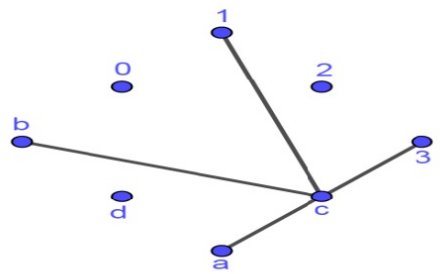

Figure 3: $W_I(N)$ with $I = \{0, c\}$

Proposition 2.10. Let $I \neq \{0\}$ be an ideal of N, then $1 \leq \chi(W_I(N)) < |N|$.

Proof . If $W_I(N)$ is null graph then $\chi(W_I(N)) = 1$ and $W_I(N)$ is impossible to become complete graph since 0 is isolated vertex or not adjacent to some vertices in elements of N, so that $\chi(W_I(N)) < |N|$. □

Proposition 2.11. Let the ideal $I \neq \{0\}$ of a zero symmetric near ring N, then in $W_I(N)$ has at least one isolated vertex.

Proof . As for every $x \in N$, then $0 \neq x.0 \notin I$ since N is a zero symmetric, so that $\overline{x0} \notin E(W_I(N))$ and $0 \neq 0.x \notin I$ for all elements of N therefore $\overline{0x} \notin E(W_I(N))$, so that 0 is isolated vertex in $W_I(N)$. □

Corollary 2.12. Let the ideal $I \neq \{0\}$ of ring integer module $n(Z_n)$, then $W_I(N)$ has 0 is isolated vertex always.

Proof . Since Z_n is always zero symmetric near ring (ring), then by Proposition 2.11, $W_I(N)$ has 0 is isolated vertex. □

Proposition 2.13. Let $I \neq \{0\}$ be a weakly c-prime ideal of N, then $deg(a) = \Delta(W_I(N)) \leq n - 1 \ \forall a \in I - \{0\}$, where $|N| = n$.

Proof . Let $I \neq \{0\}$ be a weakly c-prime ideal and $a \in I$, then $0 \neq a.x \in I$ for $x \in N$, as I is right ideal then we get a is adjacent to the elements of N and is the maximum degree with $\leq n - 1$ since $W_I(N)$ is simple graph and depend on the number of isolated vertex, so that $deg(a) = \Delta(W_I(N)) \leq n - 1$. □

Example 2.14. In Example 2.2, as $I = \{0, 2, c, d\}$ is weakly c-prime ideal and the degree a is a maximum degree in $W_I(N)$ for all $a \in I - \{0\}$ with less than $n - 1 = 8 - 1 = 7$.

Corollary 2.15. Let $I \neq \{0\}$ be a weakly c-prime ideal of a zero symmetric near ring N, then $deg(a) = \Delta(W_I(N)) < n - 1$ $\forall a \in I - \{0\}$, where $|N| = n$.

Proof . By Proposition 2.11, then $W_I(N)$ have at least one isolated vertex and Proposition 2.13, so that $deg(a) = \Delta(W_I(N)) < n - 1$. □

Proposition 2.16. Let $I \neq \{0\}$ be a proper weakly c-prime ideal of N then:

1. $W_I(N\backslash I)$ is null graph.
2. $\chi(W_I(N\backslash I)) = 1$.
3. $W_I(N\backslash I)$ is uniquely colorable.

Proof .

1. Same proof Proposition 3.21 in [4].
2. Directly from (1).
3. As $W_I(N\backslash I)$ is null graph, then $W_I(N\backslash I)$ is uniquely colorable.

□

Proposition 2.17. Let the ideal $I \neq \{0\}$ of N, then $W_I(N\backslash I) \subseteq W_I(N)$.

Proof . Same proof Proposition 3.27 in [4]. □

observation 2.18. Let the ideal $I \neq \{0\}$ of N then:

1. $\chi(W_I(N\backslash I)) \leq \chi(W_I(N))$.
2. $\chi(W_I(N\backslash\{x\})) \leq \chi(W_I(N))$, for $x \in N$.
3. $\chi(W_I(N\backslash\{\overline{xy}\})) \leq \chi(W_I(N))$, for $x, y \in N$ and $\overline{xy} \in E(W_I(N))$.

Proposition 2.19. Let the ideal $I \neq \{0\}$ of N then in $W_I(N)$ the following are hold:

1. If I is c-equiprime ideal in $W_I(N)$, then I is weakly c-prime ideal.
2. If I is c-prime ideal in $W_I(N)$, then I is weakly c-prime ideal.

Proof .

1. Let $a \in N\backslash I$ and $\overline{ax} \in E(W_I(N))$ with $0 \neq 0.x \in I$ for $x \in N$. Then $a.x - a.0 \in I$ so that $x - 0 \in I$ as I is c-equiprime ideal therefore $x \in I$. Then I is weakly c-prime ideal.
2. Let $\overline{xy} \in E(W_I(N))$ with $0 \neq x.y \in I$ for $x, y \in N$, since I is c-prime ideal, then $x \in I$ or $y \in I$ so that I is weakly c-prime ideal of N.

□

Corollary 2.20. Let the ideal $I \neq \{0\}$ of N, then in $W_I(N)$ the following are hold:

1. If I is c-equiprime ideal in $W_I(N)$, then is weakly c-semiprime ideal
2. If I is c-semiprime ideal in $W_I(N)$, then is weakly c-semiprime ideal.

Proof . Same proof Proposition 2.19. □

Definition 2.21. A simple graph G is called edge summation if there exist subgraphs K and H such that $V(G) = V(K) \cup V(H)$ and $E(G) = E(K) \cup E(H)$ and denoted by $G = K \oplus H$.

Lemma 2.22. Let I be an ideal of N then:

1. $W_I(N)$ is induced subgraph of $CEQ_I(N)$, whenever the ideal $I \neq \{0\}$.
2. $\Gamma_I(N)$ is induced subgraph of $CEQ_I(N)$.
3. $PG(N)$ is induced subgraph of $CEQ_I(N)$.

Proof . As $V(W_I(N)) = V(\Gamma_I(N)) = V(PG(N)) = V(CEQ_I(N)) = N$.

1. Let $\overline{ax} \in E(W_I(N))$, then a is adjacent to x in $W_I(N)$ and $0 \neq a.x \in I$, so that for $a \neq x \in N$ is adjacent, let $a \in N\backslash I$ with $a.x - a.0 \in I$, therefore a is adjacent to $x-0$ in $CEQ_I(N)$, that mean a is adjacent to x and 0 already in $CEQ_I(N)$ then $\overline{ax} \in E(CEQ_I(N))$ so that $E(W_I(N)) \subseteq E(CEQ_I(N))$, therefore $W_I(N) \subseteq CEQ_I(N)$.
2. same proof (1).
3. same proof (1).

□

Proposition 2.23. Let the ideal $I \neq \{0\}$ of a zero symmetric near ring N then in $W_I(N)$ the following are holds:

1. disconnected.
2. $diam(W_I(N)) \leq 2$.
3. $2 \leq \gamma(W_I(N)) \leq |N|$.

Proof .

1. From Proposition 2.11, 0 is always isolated vertex then $W_I(N)$ is disconnected.
2. As $W_I(N)$ is induced subgraph of $CEQ_I(N)$ by Lemma 2.22, then by Proposition 3.14 in [4] we get the required.
3. From Proposition 2.11, then $\gamma(W_I(N)) \geq 2$ and $\gamma(W_I(N)) = |N|$ whenever $W_I(N)$ is null graph.

□

Theorem 2.24. Let $I \neq \{0\}$ be a c-equiprime ideal of N, then

1. $CEQ_I(N) = \Gamma_I(N) \oplus W_I(N)$.
2. $CEQ_I(N) = PG(N) \oplus W_I(N)$.

Proof . As $V(CEQ_I(N)) = V(\Gamma_I(N)) \cup V(W_I(N)) = N$.

1. Now for $a \in N\backslash I$, let $\overline{ax} \in E(CEQ_I(N))$ for $x \in N$, then we have two cases:
   Case1: if $ax = 0$ then a is adjacent to x in $\Gamma_I(N)$ and $\overline{ax} \in E(\Gamma_I(N))$, then $E(CEQ_I(N)) \subseteq E(\Gamma_I(N))$
   Case2: if $ax \neq 0$ then a is adjacent to x in $W_I(N)$ and $\overline{ax} \in E(W_I(N))$, then $E(CEQ_I(N)) \subseteq E(W_I(N))$
   Therefore, from case (1) and case (2), we get $E(CEQ_I(N)) \subseteq E(\Gamma_I(N)) \cup E(W_I(N))$
   From Lemma 2.22, we get $E(\Gamma_I(N)) \cup E(W_I(N)) \subseteq E(CEQ_I(N))$
   Then, $E(CEQ_I(N)) = E(\Gamma_I(N)) \cup E(W_I(N))$
   Therefore, $CEQ_I(N) = \Gamma_I(N) \oplus W_I(N)$.
2. same proof (1).

□

Example 2.25. In Example 2.2 as $I = \{0, 2, c, d\}$ is c-equiprime ideal of N and from Figures 1,4,5 we can illustrate the Theorem 2.24.

Proposition 2.26. Let $I \neq \{0\}$ be an ideal of N, then

1. $\chi(CEQ_I(N)) \leq \chi(\Gamma_I(N)) + \chi(W_I(N))$.
2. $\chi(CEQ_I(N)) \leq \chi(PG(N)) + \chi(W_I(N))$.

Proof .

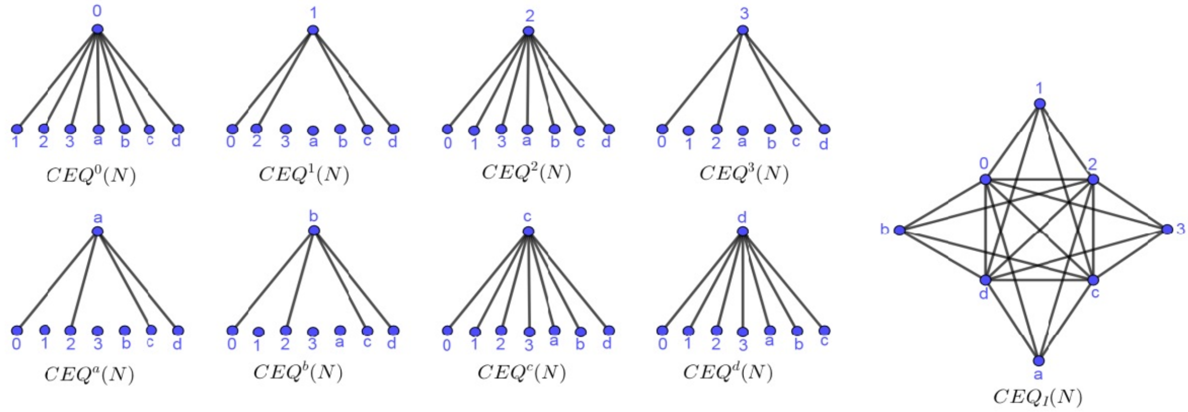

Figure 4: $CEQ^{a\in N}(N)$ and $CEQ_I(N)$

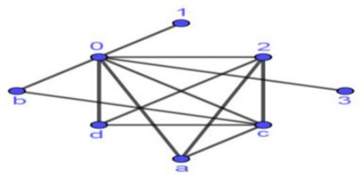

Figure 5: $\Gamma_I(N) \equiv PG(N)$

1. As $\Gamma_I(N)$ and $W_I(N)$ are induced subgraphs of $CEQ_I(N)$ (by Lemma 2.22), then $\chi(CEQ_I(N)) \leq \chi(\Gamma_I(N)) + \chi(W_I(N))$.
2. same proof (1).

□

Example 2.27. For Example on Proposition 2.26, see Examples 2.2, 2.25.

Proposition 2.28. Let $I \neq \{0\}$ be an ideal of N, then

1. $\gamma(CEQ_I(N)) \leq \gamma(\Gamma_I(N))$
2. $\gamma(CEQ_I(N)) \leq \gamma(W_I(N))$
3. $\gamma(CEQ_I(N)) \leq \gamma(PG(N))$.

Proof .

1. directly from definition of c-equiprime graph of N, then 0 is adjacent to all elements of N, so that $\gamma(CEQ_I(N)) = 1$ then $\gamma(CEQ_I(N)) \leq \gamma(\Gamma_I(N))$.
2. same proof (1).
3. same proof (1).

□

3 Conclusion

We conclude that ideals of a near ring N in graph $W_I(N)$ does not specify the adjacent vertices in comparison with graph $CEQ_I(N)$,the same is in the case of uniquely colorable, so there are no specific restrictions and $W_I(N)$ is induced subgraph of $CEQ_I(N)$.

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