Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 1143–1150
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.20527.2163

Second-order abstract Cauchy problem of conformable fractional type

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(Communicated by Javad Damirchi)

Abstract

In this paper, we discuss atomic solutions of the second-order abstract Cauchy problem of conformable fractional type

$$\begin{aligned} u^{(2\alpha)}(t) + Bu^{(\alpha)}(t) + Au(t) &= f(t) \\ u(0) &= u_0, \\ u^{(\alpha)}(0) &= u_0^{(\alpha)}, \end{aligned}$$

where A, B are closed linear operators on a Banach space X, $f : [0, \infty) \to X$ is continuous and u is a continuously differentiable function on $[0, \infty)$. Some new results on atomic solutions using tensor product technique are obtained.

Keywords: Inverse problem; fractional derivative; tensor product of Banach spaces; atomic solution
2020 MSC: 26A33, 34G10, 34A55

1 Introduction

Many mathematical models in applied sciences involve the study of what is called the Abstract Cauchy problem which has the form

$$\begin{aligned} Bu'(t) + Au(t) &= f(t), \ t \in [0,1] \text{ or } [0,\infty) \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

where A, B are densely defined closed linear operators on a Banach space X, and f is an $X-$valued continuous function while u is a continuously differentiable X valued function. Problem (1.1) is called degenerate problem if B is not invertible, otherwise it is called non-degenerate. If $f = 0$, then Problem (1.1) is called a homogenous problem.

Many researchers were interested in studying the homogeneous and degenerate form of such problem using variety of methods such as semigroups or Factorization technique, see [8, 12, 20]. In [4], the inverse form of Problem 1.1 was studied under certain conditions on the operators A and B to convert the problem to a degenerate one.

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Received: June 2020 *Accepted:* September 2020

Fractional order differential equations received a great attention in the last years since it plays a fundamental role in modeling real life problems with applications in many branches of science, such as biology, physics, finance, engineering, etc. One of the most important problems of fractional order type is the fractional Abstract Cauchy problem which has the form

$$\begin{aligned} Bu^{\alpha}(t)+Au(t) &= f(t),\ t\in[0,a] \text{ or } [0,\infty) \\ u(0) &= u_0, \end{aligned} \tag{1.2}$$

where A, B are densely defined closed linear operators on a Banach space X, $u \in C^{(\alpha)}(I,X)$, $f \in C(I,X)$ and $u_0 \in X$, where $C(I,X)$ denotes the Banach space of all continuous functions from the compact Hausdorff space I into X.

It should be noted that up to now, there are many different definitions of fractional derivatives, such as Caputo, Hadamard, Riemann, Caputo-Frabrizio, and others. Most of these definitions use the integral form see [17, 19]. Unfortunately all the existing fractional derivatives do not satisfy the classical properties of the usual derivatives: product rule, quotient rule and chain rule for the derivative of two functions and most of them except Caputo derivative don't satisfy that the derivative of the constant function is zero. To find a solution for some of these difficulties an interesting definition for fractional derivative that uses limit approach is given by Khalil et. all, [15, 5] as an extension of the usual definition of derivatives as follows:

Definition 1.1. [15] Let $f:[0,\infty)\to\mathbb{R}$ be a function. The $\alpha-$conformable fractional derivative' of f is defined by

$$D^{\alpha}(f)(t)=\lim_{\epsilon\to 0}\frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon}$$

for all $t>0$ and $\alpha\in(0,1)$. If f is $\alpha-$ differentiable in some $(0,a)$, $a>0$, and $\lim_{t\to 0^+} f^{\alpha}(t)$ exists, then $f^{\alpha}(0)=\lim_{t\to 0^+} f^{\alpha}(t)$. Let $f^{(\alpha)}(t)$ stands for $D^{\alpha}(f)(t)$ and by $f^{(2\alpha)}(t)$ we mean $D^{\alpha}D^{\alpha}(f)(t)$.

In 2010, a new technique based on tensor product of Banach spaces was used to find a unique solution for the Abstract Cauchy problem under certain conditions on the operators A and B, see[21, 22]. In [16], the tensor product technique is used to give a unique two rank solution for the homogenous Abstract Cauchy problem of conformable type (1.2) . While an atomic solution for certain degenerate and non-degenerate inverse problem is obtained in [14].

In this paper we focus on finding an atomic solution of the second order non-homogeneous Abstract Cauchy problem of conformable fractional type:

$$\begin{aligned} u^{(2\alpha)}(t)+Au^{(\alpha)}(t)+Bu(t) &= f(t) \\ u(0) &= u_0, \\ u^{(\alpha)}(0) &= u_0^{(\alpha)}, \end{aligned} \tag{1.3}$$

using tensor product technique, where A and B are densely defined closed linear operators on a Banach space X, $u\in C^{(2\alpha)}(I,X)$, $f\in C(I,X)$ and $u_0, u_0^{(\alpha)}\in X$.

2 Tensor Product

Let X^*, Y^* be the dual of the two Banach spaces X and Y respectively. For $(x,y)\in X\times Y$, the linear operator $x\otimes y: X^*\to Y$ defined by $x\otimes y\,(x^*)=x^*(x)\,y$ is called an **atom**. It is easy to see that $x\otimes y$ is a bounded linear operator with norm $\|x\otimes y\|=\|x\|\,\|y\|$. The linear space spanned by the set $\{x\otimes y,(x,y)\in X\times Y\}$ in $L(X^*,Y)$ is denoted by $X\otimes Y$. There are many norms that one can put on $X\otimes Y$. One of most popular ones is the injective norm $||.||_{\vee}$, see[18]. For $T=\sum_{i=1}^{n}x_i\otimes y_i\in X\otimes Y$.

$$||T||_{\vee}=\sup\left\{\sum_{i=1}^{n}|\langle x,x^*\rangle\langle y,y^*\rangle|,\ x^*\otimes y^*\in X^*\times Y^*,\ ||x^*||=||y^*||=1\right\}.$$

The space $(X\otimes Y,||.||_{\vee})$ need not be complete. We let $X\overset{\vee}{\otimes}Y$ denote the completion of $X\otimes Y$ in $L(X^*,Y)$ with respect to the injective norm.

One of the nice results in tensor product is that, $C(I,X)$ isometrically isomorphic to $C(I)\overset{\vee}{\otimes}X$, For more on tensor product and the use of atoms we refer the reader to [9, 10, 18, 13].

We begin our section by the following lemma which we need in our work.

Lemma 2.1. Let $g_1\otimes y_1$ and $g_2\otimes y_2$ be two non zero atoms in $C(I)\overset{\vee}{\otimes}X$. Then the following are equivalent:

(1) $g_1\otimes y_1+g_2\otimes y_2=g_3\otimes y_3$, a non zero atom.

(2) g_1,g_2 or y_1,y_2 are linearly dependent.

Proof . (2) → (1). Clear.

(1) → (2). Assume $g_1\neq g_2$. Then, by a consequence of the Hahn-Banach Theorem, [11] there exists a continuous linear functional μ on $C(I)$, such that $\mu(g_1)\neq\mu(g_2)\neq 0$ and $\mu(g_3)=0$. This implies that

$$y_1=\frac{-\mu(g_2)}{\mu(g_1)}y_2$$

and so, y_1,y_2 are linearly dependent.

Similarly, if $y_1\neq y_2$, we use the same idea but on the adjoint operators, noting that the adjoint of $x\otimes y$ is $y\otimes x$, when we are dealing with real Banach spaces, which is our case. **This ends the proof.** □

Lemma 2.2. Let $g_1\otimes y_1$, $g_2\otimes y_2$, and $g_3\otimes y_3$ be three non zero atoms in $C(I)\overset{\vee}{\otimes}X$. Assume $g_1\otimes y_1+g_2\otimes y_2+g_3\otimes y_3$ $=g\otimes y\neq 0$. Then the atoms $g_1\otimes y_1$, $g_2\otimes y_2$, and $g_3\otimes y_3$ are linearly dependent.

Proof . If possible assume that such atoms are linearly independent. Then g_1,g_2,g_3 are linearly independent and y_1,y_2,y_3 are linearly independent. But, by a consequence of the Hahn Banach Theorem, [11] there exists a continuous linear functional μ on $C(I)$, such that $\mu(g_1)\neq 0$, $\mu(g_2)\neq 0,\mu(g_3)\neq 0$ and $\mu(g)=0$. This implies that, y_1,y_2 and y_3 are linearly dependent, contradicting the assumption. **This ends the proof.** □

3 Atomic Solution

In this paper we concentrate on finding an atomic solution $u=u_1\otimes x$ to the non-homogeneous second order fractional Abstract Cauchy problem of the form

$$\begin{aligned} u^{(2\alpha)}(t)+Au^{(\alpha)}(t)+Bu(t) &= f(t) \\ u(0) &= u_0, \\ u^{(\alpha)}(0) &= u_0^{(\alpha)}, \end{aligned} \tag{3.1}$$

using tensor product technique, where A and B are densely defined closed linear operators on the Banach space X, $u_1\in C^{(2\alpha)}(I),f\in C(I,X)$ and $u_0,u_0^{(\alpha)}$ and $x\in X$.

If $u=u_1\otimes x$, then we can write (3.1) in tensor product as follows:

$$u_1^{(2\alpha)}\otimes x+u_1^{(\alpha)}\otimes Ax+u_1\otimes Bx=f\otimes z.$$

Here, the unknowns are u_1 and x, while A,B,f and z are given. With no loss of generality we can assume that $f(0)=1$.

Since the sum of three atoms is an atom, then by the use of Lemma 2.2, either $u_1^{(2\alpha)}\otimes x+u_1^{(\alpha)}\otimes Ax$ is an atom or $u_1^{(2\alpha)}\otimes x+u_1\otimes Bx$ is an atom or $u_1^{(\alpha)}\otimes Ax+u_1\otimes Bx$ is an atom. All these cases

are discussed in details in the following three theorems.

Theorem 3.1. Let A,B be densely defined closed linear operators on a Banach space X, $x\in Domain(A\cap B)$, $u_1(t)$ is $(2\alpha)-$differentiable function on I. If $u_1^{(2\alpha)}\otimes x+u_1^{(\alpha)}\otimes Ax$ is an atom, then the fractional differential equation (3.1) has a unique atomic solution if the following conditions are satisfied

(i)There exists some $x^* \in X^*$ and $g \in C(I,R)$, such that g is $(2\alpha)-$differentiable function on I, where $g^{(2\alpha)}(0)$ exist, and $u_1(t)\langle x, x^*\rangle = g(t)$.

(ii) x is uniquely imaged by the operators $I+A+B$, $I+B$

Proof . Without loss of generality assume that $f(0) = u_1(0) = u_1^{(\alpha)}(0) = 1$. Write (3.1) in tensor product form, we get

$$u_1^{(2\alpha)} \otimes x + u_1^{(\alpha)} \otimes Ax + u_1 \otimes Bx = f \otimes z. \tag{4.1}$$

Since $u_1^{(2\alpha)} \otimes x + u_1^{(\alpha)} \otimes Ax$ is an atom, by Lemma 2.1, either $u_1^{(2\alpha)}(t) = u_1^{(\alpha)}(t)$ or $Ax = x$.

Case (1)

$$u_1^{(2\alpha)}(t) = u_1^{(\alpha)}(t). \tag{4.2}$$

Solving (4.2), we get $u_1(t) = c_1 + c_2 e^{\frac{t^\alpha}{\alpha}}$. Using the initial conditions $u_1(0) = 1$ and $u_1^{(\alpha)}(0) = 1$, we get $c_1 = 0$ and $c_2 = 1$. Hence $u_1(t) = e^{\frac{t^\alpha}{\alpha}}$. Since $g(t) = u_1(t)\langle x, x^*\rangle$, it follows that $g(0) = \langle x, x^*\rangle$ and $g^{(\alpha)}(t) = e^{\frac{\beta}{\alpha}t^\alpha} g(0) = u_1^{(\alpha)}(t)g(0)$. Thus, $u_1(t)$ is uniquely determined.

Now, substitute $u_1(t)$ in (4.1), we get

$$e^{\frac{t^\alpha}{\alpha}}(x + Ax + Bx) = f(t)z.$$

This is true for all t. In particular take $t = 0$ and use the assumption on f to get

$$x + Ax + Bx = (I + A + B)x = z.$$

By the assumption on z, we get x to be uniquely determined. Thus, (4.1) has a unique solution.

Case (2)

$$Ax = x. \tag{4.3}$$

Now, substitute (4.3) in (4.1), we get

$$(u_1^{(2\alpha)} + u_1^{(\alpha)}) \otimes x + u_1 \otimes Bx = f \otimes z. \tag{4.4}$$

Since the sum of two atoms equals one atom, using Lemma 2.1, we have two sub-cases, $u_1^{(2\alpha)} + u_1^{(\alpha)} = u_1$ or $Bx = x$.

Case (a):

$$u_1^{(2\alpha)} + u_1^{(\alpha)} = u_1. \tag{4.5}$$

Write (4.5) in characteristic form, we get

$$r^2 + r - 1 = 0. \tag{4.6}$$

Solving (4.6), we get

$$r = \frac{-1 \pm \sqrt{5}}{2}.$$

Thus,

$$u_1(t) = e^{\frac{-t^\alpha}{2\alpha}}(c_1 e^{\frac{-\sqrt{5}}{2\alpha}t^\alpha} + c_2 e^{\frac{\sqrt{5}}{2\alpha}t^\alpha}).$$

Using the initial conditions $u_1(0) = 1$ and $u_1^{(\alpha)}(0) = 1$, we get

$$c_1 = -\frac{3-\sqrt{5}}{2\sqrt{5}}, \qquad c_2 = \frac{3+\sqrt{5}}{2\sqrt{5}}.$$

Thus, $u_1(t)$ is uniquely determined.

Now, substitute (4.5) in (4.4), we get

$$u_1(t)(x + Bx) = f(t)z.$$

This is true for all t. Hence $(I + B)x = z$. By the assumption on z, we get x is uniquely determined.

Case (b)

$$Bx = x. \tag{4.7}$$

Substitute (4.7) in (4.4), we get,

$$(u_1^{(2\alpha)} + u_1^{(\alpha)} + u_1) \otimes x = f \otimes z.$$

Solve the homogeneous equation

$$u_1^{(2\alpha)} + u_1^{(\alpha)} + u_1 = 0. \tag{4.8}$$

Using characteristic equation of (4.8) , we get that

$$u_1(t) = e^{\frac{-t^\alpha}{2\alpha}} \cos(\frac{\sqrt{3}}{2\alpha}t^\alpha), \ \sin(\frac{\sqrt{3}}{2\alpha}t^\alpha$$

are two independent solutions of (4.8) . Using these two solutions by the method of variation of parameters, [3], a particular solution u_p of the non homogeneous equation

$$u_1^{(2\alpha)} + u_1^{(\alpha)} + u_1 = f \tag{4.9}$$

can be obtained. Hence the general solution of 4.9 is

$$u_1(t) = e^{\frac{-t^\alpha}{2\alpha}} \left(c_1 \cos(\frac{\sqrt{3}}{2\alpha}t^\alpha) + c_2 \sin(\frac{\sqrt{3}}{2\alpha}t^\alpha \right) + u_p.$$

Using the initial conditions $u_1(0) = 1$ and $u_1^{(\alpha)}(0) = 1$, c_1 and c_2 could be determined. Since $z = x$ it follows that x is uniquely determined and hence a unique solution of 4.1 is obtained. □

Theorem 3.2. Let A, B be densely defined closed linear operators on a Banach space X, $x \in Domain(A \cap B)$, $u(t)$ is $(2\alpha)-$differentiable function on I, and $u = u_1 \otimes x$. If $u_1^{(2\alpha)} \otimes x + u_1(t) \otimes Bx$ is an atom, then the fractional differential equation (3.1) has a unique solution if the following conditions are satisfied

(i)There exists some $x^* \in X^*$ and $g \in C(I, R)$, such that g is $(2\alpha)-$differentiable function on I, where $g^{(2\alpha)}(0)$ exist, and $u_1(t)\langle x, x^*\rangle = g(t)$.

(ii) x is uniquely imaged by the operators $I + A + B$, $I + A$.

Proof . Without loss of generality assume that $f(0) = u_1(0) = u_1^{(\alpha)}(0) = 1$. Write (3.1) in tensor product form, we get

$$u_1^{(2\alpha)} \otimes x + u_1^{(\alpha)} \otimes Ax + u_1 \otimes Bx = f \otimes z. \tag{5.1}$$

Since $u_1^{(2\alpha)} \otimes x + u_1(t) \otimes Bx$ is an atom, by Lemma 2.1, either $u_1^{(2\alpha)}(t) = \lambda u_1(t)$ or $Bx = \beta x$. With no loss of generality we can take $\beta = \lambda = 1$.

Case (1) :

$$u_1^{(2\alpha)}(t) = u_1(t), \tag{5.2}$$

Solving (5.2), we get $u_1(t) = c_1 e^{\frac{t^\alpha}{\alpha}} + c_2 e^{\frac{-t^\alpha}{\alpha}}$. Since $u_1(0) = 1$, $u_1^{(\alpha)}(0) = 1$, we get $c_1 = 1$ and $c_2 = 0$. Consequently, $u_1(t) = e^{\frac{t^\alpha}{\alpha}}$. Since $g(t) = u_1(t)\langle x, x^*\rangle$, it follows that $g(0) = \langle x, x^*\rangle$. Hence, $g^{(\alpha)}(t) = e^{\frac{t^\alpha}{\alpha}} g(0)$. Thus $u_1(t)$ is uniquely determined. Now, substitute $u_1(t)$ in (5.1), we get

$$u_1(t)\,(x + Ax + Bx) = f(t)z. \tag{5.3}$$

Thus

$$(I + A + B)\,x = \frac{f(t)}{u_1(t)}z. \tag{5.4}$$

Since this is true for every t, we have

$$(I + A + B)\,x = \frac{f(0)}{u_1(0)}z = z. \tag{5.5}$$

By the assumption on z, we get x uniquely determined. Thus, (5.1) has a unique solution

Case (2)

$$Bx = x. \tag{5.6}$$

Now, substitute (5.6) in (5.1), we get

$$\begin{aligned} f \otimes z &= u_1^{(2\alpha)} \otimes x + u_1^{(\alpha)} \otimes Ax + u_1 \otimes x \\ &= \left(u_1^{(2\alpha)} + u_1\right) \otimes x + u_1^{(\alpha)} \otimes Ax. \end{aligned} \tag{3.2}$$

Since the sum of two atoms equal one atom, using Lemma 2.1, we have the following two sub-cases:

Case (a):

$$u_1^{(2\alpha)}(t) + u_1(t) = u_1^{(\alpha)}(t). \tag{5.8}$$

Write (5.8) in characteristic form, we get

$$r^2 - r + 1 = 0, \tag{5.9}$$

Solving (5.9), we get

$$r = \frac{1 \pm \sqrt{3}i}{2}.$$

Thus,

$$u_1(t) = e^{\frac{t^\alpha}{2\alpha}} \left(c_1 \cos(\frac{\sqrt{3}}{2\alpha} t^\alpha) + c_2 \sin(\frac{\sqrt{3}}{2\alpha} t^\alpha \right).$$

Since $u_1(0) = 1$, we get $c_1 = 1$ and since $u_1^{(\alpha)}(0) = 1$, we get $c_2 = \frac{1}{\sqrt{3}}$. Thus,

$$u_1(t) = e^{\frac{t^\alpha}{2\alpha}} \left(\cos(\frac{\sqrt{3}}{2\alpha} t^\alpha) + \frac{1}{\sqrt{3}} \sin(\frac{\sqrt{3}}{2\alpha} t^\alpha) \right).$$

Consequently, $u_1(t)$ is uniquely determined.

Now, substitute (5.8) in (3.2), we get

$$u_1^{(\alpha)}(t)(x + Ax) = f(t)z. \tag{5.10}$$

Since this is true for every t, we have

$$(I + A)\, x = \frac{f(0)}{u_1(0)} z = z. \tag{5.11}$$

By the assumption on z, we get x uniquely determined. Thus, (5.1) has a unique solution

Case (b)

$$Ax = x. \tag{5.12}$$

Substitute (5.12) in (3.2), we get

$$(u_1^{(2\alpha)} + u_1^{(\alpha)} + u_1) \otimes x = f \otimes z. \tag{5.13}$$

Similarly as in Theorem 3.1 case 2(b) we get $u_1(t)$ and x are uniquely determined and hence (5.1) has a unique solution. □

Theorem 3.3. Let A, B be densely defined closed linear operators on a Banach space X, $x \in Domain(A \cap B)$, $u(t)$ is $(2\alpha)-$differentiable function on I, and $u = u_1 \otimes x$. If $u_1^{(\alpha)}(t) \otimes Ax + u_1(t) \otimes Bx$ is an atom, then the fractional differential equation (3.1) has a unique solution if the following conditions are satisfied

(i)There exists some $x^* \in X^*$ and $g \in C(I, R)$, such that g is $(2\alpha)-$differentiable function on I, where $g^{(2\alpha)}(0)$ exist, and $u_1(t)\langle x, x^*\rangle = g(t)$.

(ii) x is uniquely imaged by the operators $I + A + B$, $I + B$ and B

Proof . Without loss of generality assume that $f(0) = u_1(0) = u_1^{(\alpha)}(0) = 1$. Write (3.1) in tensor product form, we get

$$u_1^{(2\alpha)} \otimes x + u_1^{(\alpha)} \otimes Ax + u_1 \otimes Bx = f \otimes z. \tag{6.1}$$

Since $u_1^{(\alpha)}(t) \otimes Ax + u_1(t) \otimes Bx$ is an atom, by Lemma 2.1, we have the two cases: either $u_1^{(\alpha)}(t) = u_1(t)$ or $Ax = Bx$.

Case (1)

$$u_1^{(\alpha)}(t) = u(t). \tag{6.2}$$

Solving (6.2), we get $u_1(t) = ce^{\frac{t^\alpha}{\alpha}}$. Since $u_1^{(\alpha)}(0) = 1$, we get $c = 1$. Consequently, $u_1(t) = e^{\frac{t^\alpha}{\alpha}}$. Since $g(t) = u_1(t)\langle x, x^*\rangle$. Thus, $g(0) = \langle x, x^*\rangle$. Also, $g^{(\alpha)}(t) = e^{\frac{t^\alpha}{\alpha}} g(0)$. Now, substitute $u_1(t)$ in (6.1), we get

$$e^{\frac{t^\alpha}{\alpha}} (x + Ax + Bx) = f(t)z. \tag{6.3}$$

Since (6.3) is true for all t, we get

$$(I + A + B)\, x = f(0)z. \tag{6.4}$$

By the assumption on z, we get x uniquely determined. Thus, (6.1) has a unique solution.

Case (2)

$$Ax = Bx. \tag{6.5}$$

Now, substitute (6.5) in (6.1), we get

$$u_1^{(2\alpha)} \otimes x + \left(u_1^{(\alpha)} + u_1\right) \otimes Bx = f \otimes z. \tag{6.6}$$

Since the sum of two atoms equals one atom, using Lemma 2.1, we have the following two sub-cases:

Case (a):

$$u_1^{(\alpha)} + u_1 = u_1^{(2\alpha)}(t). \tag{6.7}$$

From (6.7) and since $u_1^{(\alpha)}(0) = u_1(0) = 1$. Write (6.7) in characteristic form, we get

$$r^2 - r - 1 = 0. \tag{6.8}$$

Solving (6.8), we get

$$r = \frac{1 \pm \sqrt{5}}{2}.$$

Thus

$$u_1(t) = e^{\frac{t^\alpha}{2\alpha}} \left(c_1 e^{\frac{-\sqrt{5}}{2\alpha} t^\alpha} + c_2 e^{\frac{\sqrt{5}}{2\alpha} t^\alpha}\right).$$

Using the initial conditions $u_1(0) = 1$ and $u_1^{(\alpha)}(0) = 1$, we get

$$c_1 = \frac{-1 + \sqrt{5}}{2\sqrt{5}}, \qquad c_2 = \frac{1 + \sqrt{5}}{2\sqrt{5}}.$$

Thus, $u_1(t)$ is uniquely determined.

Now, substitute (6.7) in (6.6), we get

$$u_1^{(\alpha)}(t) + u_1(t)(x + Bx) = f(t)z.$$

This is true for all t. Hence $(I + B)x = z$. By the assumption on z, x is uniquely determined. Thus, (6.1) has a unique solution.

Case (b)

$$Bx = x. \tag{6.9}$$

Substitute (6.9) in (6.6), we get

$$\left(u_1^{(2\alpha)} + u_1^{(\alpha)} + u_1\right) \otimes x = f \otimes z. \tag{6.10}$$

Similarly as in Theorem 3.1 case 2(b) we get $u_1(t)$ and x are uniquely determined and hence (5.1) has a unique solution. □

acknowledgement

The authors thank the referee for his valuable comments.

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