Int. J. Nonlinear Anal. Appl. 13 (Special issue for ICDACT-2021), 31–42
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.6328

Results on coupled common fixed point by applying a new approach of Y-cone metric spaces

Richa Sharma[a], Virendra Singh Chouhan[b,*], Rahul Shukla[c]

[a]*Department of Mathematics, Chandigarh University, Mohali, India*

[b]*Department of Mathematics and Statistics, Manipal University Jaipur, India*

[c]*Department of Mathematics and Applied Mathematics, University of Johannesburg, Auckland Park 2006, South Africa*

(Communicated by Mayank Sharma)

Abstract

The main motive of this paper is to discuss coupled coincidence points in the setting of the newly established concept Y-cone metric spaces. We obtain coupled coincidence point theorems through mixed monotone mappings in ordered Y-cone metric spaces. We give an illustrative example, which constitutes the main theorem.

Keywords: Coupled common fixed point, mixed g-monotone maps, Y-cone metric space.
2020 MSC: 54H25; 47H10

1 Introduction

Banach contraction principle (BCT) provides uniqueness and existence of a solution of an operator equation $Gx = x$. Many authors have established contractive mappings to obtain the fixed points of this class of mappings in complete metric spaces. A number of extensions and generalizations of BCT have appeared in the literature, for more results, we refer to ([3, 4, 5, 6, 9, 10, 11, 15]).

On the other hand, metric spaces present a significant idea to the study of Functional Analysis and Topology. To establish an appropriate concept of a metric space, numerous techniques are available in this phenomena. A number of extension of the concept of metric spaces have then turned up in other papers (see [2, 7, 8, 14]). Recently, cone metric spaces were introduced by Huang and Zhang [8], they described convergence in cone metric spaces and presented the completeness. To apply this concept, the role of cone metric spaces have developed by number of authors (see [1, 12, 13, 16]).

The study of a coupled fixed point results were initiated by Bhaskar and Lakshmikantham [4] in ordered metric spaces, they execute their concept to show the existence and uniqueness of a solution of boundary value problem which is periodic in nature. It is observed that the interplay between the metrical structure of the space and order is very fruitful. Due to this significance, many researchers have obtained results for different contractive conditions [2, 5, 6, 10, 13, 15].

*Corresponding author
Email addresses: richa.tuknait@yahoo.in (Richa Sharma), darbarvsingh@yahoo.com (Virendra Singh Chouhan), rshukla.vnit@gmail.com (Rahul Shukla)

Received: August 2021 *Accepted:* January 2022

Motivated by above results, we present some coupled common fixed point results in the setting of Y-cone metric spaces. To illustrate the usability of our results, we furnish an example.

2 Preliminaries

The concept of b-metric space were initiated by Czerwik [7]. For more details about the following definitions, reader see [7].

Definition 2.1. [7] Presume S be a set which is nonempty and $s \geq 1$ be a real number. A mapping $d : S \times S \to \mathbb{R}^+$ is a b-metric on S if, the following conditions satisfy, for all $s_1, s_2, s_3 \in S$:

(1) $d(s_1, s_2) = 0 \iff s_1 = s_2$,

(2) $d(s_1, s_2) = d(s_2, s_1)$,

(3) $d(s_1, s_2) \leq s[d(s_1, s_3) + d(s_3, s_2)]$.

Here, the pair (S, d) is known as a b-metric space.

Definition 2.2. [2] Let S be a set which is nonempty. A mapping $A : S^n \to [0, \infty)$ is known as an A-metric on S if, for all $s_i, a \in S, i = 1, 2, \cdots, n$, conditions mentioned below hold:

(A1) $A(s_1, s_2, s_3, \cdots, s_{n-1}, s_n) \geq 0$,

(A2) $A(s_1, s_2, s_3, \cdots, s_{n-1}, s_n) = 0 \iff s_1 = s_2 = s_3 = \cdots = s_{n-1} = s_n$,

(A3)

$$\begin{aligned} A(s_1, s_2, s_3, \cdots, s_{n-1}, s_n) \leq\ & A(s_1, s_1, s_1, \cdots, (s_1)_{n-1}, a) \\ & + A(s_2, s_2, s_2, \cdots, (s_2)_{n-1}, a) \\ & \vdots \\ & + A(s_{n-1}, s_{n-1}, s_{n-1}, \cdots, (s_{(n-1)})_{n-1}, a) \\ & + A(s_n, s_n, s_n, \cdots, (s_n)_{n-1}, a). \end{aligned}$$

The pair (S, A) is known as an A-metric space.

3 Y-cone Metric Spaces

Throughout this paper, we take E is a Banach space and P is a cone in E together $intP \neq \emptyset$ and $\leq$ with respect to P is a partial ordering .

Definition 3.1. [14] Let S be a set which is nonempty and $k \geq 1$ be a real number. Suppose a function $Y : S^n \to E$ is called a Y-cone metric on S if, for all $s_i, a \in S, i = 1, 2, \cdots, n$, the conditions mentioned below hold:

(Y1) $Y(s_1, s_2, s_3, \cdots, s_{n-1}, s_n) \geq \theta$,

(Y2) $Y(s_1, s_2, s_3, \cdots, s_{n-1}, s_n) = \theta \iff s_1 = s_2 = s_3 = \cdots = s_{n-1} = s_n$,

(Y3)

$$\begin{aligned} Y(s_1, s_2, s_3, \cdots, s_{n-1}, s_n) \leq\ & k[Y(s_1, s_1, s_1, \cdots, (s_1)_{n-1}, a) \\ & + Y(s_2, s_2, s_2, \cdots, (s_2)_{n-1}, a) \\ & \vdots \\ & + Y(s_{n-1}, s_{n-1}, s_{n-1}, \cdots, (s_{(n-1)})_{n-1}, a) \\ & + Y(s_n, s_n, s_n, \cdots, (s_n)_{n-1}, a)]. \end{aligned}$$

The pair (S, Y) is called an Y-cone metric space.

It is noted that cone b- metric space becomes a special case of Y-cone metric space with $n = 2$.

Proposition 3.2. [14] If (S, Y) is Y-cone metric space, then for each $u, v \in S$, we have

$$Y(u, u, \cdots, u, v) = Y(v, v, \cdots, v, u)$$

Example 3.3. [14] Presume $S = \{1, 2, 3, 4, 5\}$, $P = \{s \in E : s \geq 0\}$ where, $E = \mathbb{R}$ and. Define $Y : S^n \to E$ by:

$$Y(s_1, s_2, s_3, \cdots, s_{n-1}, s_n) = \begin{cases} |s_1 - s_2|^{-1} + |s_2 - s_3|^{-1} + \cdots + |s_{n-1} - s_n|^{-1} & \text{if} \quad s_i \neq s_j, \\ \theta & \text{if} \quad s_i = s_j. \end{cases}$$

$\forall i, j = 1, 2, \cdots, n$. Thus (S, Y) is a Y-cone metric space together coefficient $k = \frac{12}{7}$.

Example 3.4. [14] Let $S = [0, 1]$ and $E = C^1_{\mathbb{R}}[0, 1]$ with $\| v \| = \| v \|_\infty + \|v' \|_\infty$, $v \in E$ and Suppose $P = \{v \in E : v(t) \geq 0 \text{ on } [0, 1]\}$. It is already known that cone is solid but it is not normal. A Y-cone metric $Y : S^n \to E$ defined by

$$\begin{aligned} Y(s_1, s_2, s_3, \cdots, s_{n-1}, s_n) =& [|s_1 - s_2|^2 + |s_1 - s_3|^2 + \cdots + |s_1 - s_n|^2 \\ &+ |s_2 - s_3|^2 + |s_2 - s_4|^2 + \cdots + |s_2 - s_n|^2 \\ &+ \cdots + |s_{n-1} - s_n|^2]e^t \\ &= \sum_{i=1}^{n} \sum_{i<j} |s_i - s_j|^2 e^t. \end{aligned}$$

Thus (S, Y) is a complete Y-cone metric space together the coefficient $k = 2$.

Lemma 3.5. [14] Presume S be a Y-cone metric space, for every $s, u \in S$ we have, $Y(s, s, \cdots, s, z) \leq k[(n - 1)Y(s, s, \cdots, s, u) + Y(z, z, \cdots, z, u)]$ and $Y(s, s, \cdots, s, z) \leq k[(n - 1)Y(s, s, \cdots, s, u) + Y(u, u, \cdots, u, z)]$.

Definition 3.6. [14] Presume (S, Y) be a Y-cone metric space along with coefficient $k \geq 1$. For every $s \in S$ and $\theta \ll p$, take $B_Y(s, p) = \{w \in S : Y(s, s, \cdots, s, w) \ll p\}$ and take $B = \{B_Y(s, p) : s \in S \text{ and } \theta \ll p\}$. Therefore, B is a subbase for some topology τ on S.

Remark 3.7. [14] Presuppose (S, Y) be a Y-cone metric space. Here, τ represents the topology on S, B represents a subbase for the topology on τ and $B_Y(s, p)$ represents the Y-ball in (S, Y), which are expressed in Definition 3.6. Also, Y represents the base generated by subbase B.

Definition 3.8. [14] Let (S, Y) be a Y-cone metric space. A sequence $\{s_n\}$ in S converges to s if for every $c \in E$ with $\theta \ll c$, there is a natural number N such that for all $n > \mathbb{N}$, $Y(s_n, s_n, \cdots, s_n, s) \ll c$ for some fixed s in X. Hence s is called the limit of a sequence $\{s_n\}$ and is denoted by $\lim_{n\to\infty} s_n = s$ or $s_n \to s$ as $n \to \infty$.

Definition 3.9. [14] Let (S, Y) be a Y-cone metric space. A sequence $\{s_n\}$ in S is called a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is a natural number $\mathbb{N}$ such that for all $n, m > \mathbb{N}$, we have $Y(s_n, s_n, \cdots, s_n, s_m) \ll c$.

Definition 3.10. [14] The Y-cone metric space S is said to be complete if every Cauchy sequence in S is convergent in S.

Lemma 3.11. [14] Presuppose (S, Y) be a Y-cone metric space. Existence of sequences $\{s_n\}, \{w_n\}$ such that $s_n \to s, w_n \to w$, then $\lim_{n\to\infty} Y(s_n, s_n, \cdots, s_n, w_n) = Y(s, s, \cdots, s, w)$.

Remark 3.12. [14] Let (S, Y) be a Y-cone metric over the ordered Banach space E defined on R with a cone P. Now the subsequent attributes are often used:

(1) If $b_1 \leq b_2$ and $b_2 \ll b_3$, then $b_1 \ll b_3$.

(2) If $\theta \leq v \ll c$ for every $c \in intP$, thus $v = \theta$.

(3) If $c \in intP$, $\theta \leq b_n$ and $b_n \to \theta$, then there $\exists\, n_0$ for every $n > n_0$ we have $b_n \ll c$.

(4) If E is a Banach space on real R with cone P and if $b \leq \lambda\, b$ where $b \in P$ and $\theta \leq \lambda < 1$, then $b = \theta$.

Definition 3.13. [4] An element $(s, w) \in S \times S$ is known as coupled fixed point for the map $F : S \times S \to S$ if $F(s, w) = s,\ F(w, s) = w$.

Definition 3.14. [6] An element $(s, w) \in S \times S$ is known as coupled coincident point for the map $F : S \times S \to S$ and $g : S \to S$ if $F(s, w) = gs,\ F(w, s) = gw$.

Definition 3.15. [6] Let $(S, \leq)$ be a partially ordered set and let $F : S \times S \to S$ and $g : S \to S$ be two mappings. We say F has the mixed g-monotone property if $F(s, w)$ is g-non-decreasing in its first argument and is g-non-increasing in its second argument, for any $s, w \in S$

$$\begin{aligned} s_1, s_2 \in S, gs_1 \leq gs_2 &\implies F(s_1, w) \leq F(s_2, w) \\ w_1, w_2 \in S, gw_1 \leq gw_2 &\implies F(s, w_1) \geq F(s, w_2). \end{aligned}$$

4 Coupled Common Fixed Point Results

Now, we provide the results on coupled coincidence point fulfilling in the framework of partially ordered Y-cone metric spaces for more general contractive conditions. We start with the subsequent result.

Theorem 4.1. Presume $(S, \leq, Y)$ be a partially ordered complete Y-cone metric space together the coefficient $k \geq 1$ relative to a solid cone P. Presuppose $F : S\times S \to S$ and $g : S \to S$ be the continuous mappings possesses the condition of mixed g monotone on S. Assuming that $\exists\, a_m \geq 0,\ m = 1, 2, \cdots, 10$ with $a_1+a_2+a_3+a_4+a_5+a_6+2k(a_7+a_8) < 1$ and $\sum_{m=1}^{10} a_m < 1$ such that

$$\begin{aligned} &Y(F(s, w), F(s, w), \cdots, F(s, w), F(u, v)) \\ &\quad \leq a_1 Y(gs, gs, \cdots, gs, gu) + a_2 Y(gw, gw, \cdots, gw, gv) \\ &\qquad + a_3 Y(gs, gs, \cdots, gs, F(s, w)) + a_4 Y(gw, gw, \cdots, gw, F(w, s)) \\ &\qquad + a_5 Y(gu, gu, \cdots, gu, F(u, v)) + a_6 Y(gv, gv, \cdots, gv, F(v, u)) \\ &\qquad + a_7 Y(gs, gs, \cdots, gs, F(u, v)) + a_8 Y(gw, gw, \cdots, gw, F(v, u)), \\ &\qquad + a_9 Y(gu, gu, \cdots, gu, F(s, w)) + a_{10} Y(gv, gv, \cdots, gv, F(w, s)) \end{aligned} \tag{4.1}$$

$\forall\ s, w, u, v \in S$ with $gs \leq gu$ and $gw \geq gv$. Suppose that there exists $s_0, w_0 \in S$ such that $gs_0 \leq F(s_0, w_0)$, $gw_0 \geq F(w_0, s_0)$, furthermore $F(S \times S) \subset g(S)$ then F and g in S have a coupled coincidence point.

Proof . Choose $s_0, w_0 \in S$, one can construct the sequences $\{s_n\}$ and $\{w_n\}$ such that $gs_{2n+1} = F(s_{2n}, w_{2n}),\ gw_{2n+1} = F(w_{2n}, s_{2n})$ and $gs_{2n+2} = F(s_{2n+1}, w_{2n+1}),\ gw_{2n+2} = F(w_{2n+1}, s_{2n+1})$ for all $n \geq 0$.

Observing that F posses the property of mixed g-monotone on S. We have

$$gs_0 \leq gs_1 \leq \cdots \leq gs_n \leq gs_{n+1} \leq \cdots \text{ and } gw_0 \geq gw_1 \geq \cdots \geq gw_n \geq gw_{n+1} \geq \cdots.$$

Then by (4.1), we have

$$\begin{aligned}
Y\ & (gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) \\
= \ & Y\big(F(s_{2n}, w_{2n}), F(s_{2n}, w_{2n}), \cdots, F(s_{2n}, w_{2n}), F(s_{2n+1}, w_{2n+1})\big) \\
\leq \ & a_1 Y\big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+1}\big) + a_2 Y\big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+1}\big) \\
& + a_3 Y\Big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, F(s_{2n}, w_{2n})\Big) + a_4 Y\Big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, F(w_{2n}, s_{2n})\Big) \\
& + a_5 Y\Big(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, F(s_{2n+1}, w_{2n+1})\Big) \\
& + a_6 Y\Big(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, F(w_{2n+1}, s_{2n+1})\Big) \\
& + a_7 Y\Big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, F(s_{2n+1}, w_{2n+1})\Big) + a_8 Y\Big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, F(w_{2n+1}, s_{2n+1})\Big) \\
& + a_9 Y\Big(gs_{2n+1}, , gs_{2n+1}, \cdots, gs_{2n+1}, F(s_{2n}, w_{2n})\Big) \\
& + a_{10} Y\Big(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, F(w_{2n}, s_{2n})\Big).
\end{aligned} \tag{4.2}$$

In similar way, we get

$$\begin{aligned}
& Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2}) \\
& = Y\big(F(w_{2n}, s_{2n}), F(w_{2n}, s_{2n}), \cdots, F(w_{2n}, s_{2n}), F(w_{2n+1}, s_{2n+1})\big) \\
& \leq a_1 Y\big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+1}\big) + a_2 Y\big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+1}\big) \\
& \quad + a_3 Y\Big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, F(w_{2n}, s_{2n})\Big) + a_4 Y\Big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, F(s_{2n}, w_{2n})\Big) \\
& \quad + a_5 Y\Big(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, F(w_{2n+1}, s_{2n+1})\Big) + a_6 Y\Big(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, F(s_{2n+1}, w_{2n+1})\Big) \\
& \quad + a_7 Y\Big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, F(w_{2n+1}, s_{2n+1})\Big) + a_8 Y\Big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, F(s_{2n+1}, w_{2n+1})\Big) \\
& \quad + a_9 Y\Big(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, F(w_{2n}, s_{2n})\Big) + a_{10} Y\Big(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, F(s_{2n}, w_{2n})\Big)
\end{aligned} \tag{4.3}$$

Adding (4.2) and (4.3), we get

$$\begin{aligned}
& Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) + Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2}) \\
& \leq a_1 Y\big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+1}\big) + a_2 Y\big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+1}\big) \\
& \quad + a_3 Y\big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+1}\big) + a_4 Y\big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+1}\big) \\
& \quad + a_5 Y\big(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}\big) + a_6 Y\big(gw_{2n+1}, gy_{2n+1}, \cdots, gy_{2n+1}, gy_{2n+2}\big) \\
& \quad + a_7 Y\big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+2}\big) + a_8 Y\big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+2}\big) \\
& \quad + a_9 Y\big(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+1}\big) + a_{10} Y\big(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+1}\big) \\
& \quad + a_1 Y\big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+1}\big) + a_2 Y\big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+1}\big) \\
& \quad + a_3 Y\big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+1}\big) + a_4 Y\big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+1}\big) \\
& \quad + a_5 Y\big(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2}\big) + a_6 Y\big(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}\big) \\
& \quad + a_7 Y\big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+2}\big) + a_8 Y\big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+2}\big) \\
& \quad + a_9 Y\big(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+1}\big) + a_{10} Y\big(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+1}\big) \\
& \leq (a_1 + a_2 + a_3 + a_4) Y\big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+1}\big) + (a_1 + a_2 + a_3 + a_4) Y\big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+1}\big) \\
& \quad + (a_5 + a_6) Y\big(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}\big) + (a_5 + a_6) Y\big(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2}\big) \\
& \quad + (ka_7 + ka_8) Y\big(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+1}\big) + (ka_7 + ka_8) Y\big(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+1}\big) \\
& \quad + (ka_7 + ka_8) Y\big(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}\big) + (ka_7 + ka_8) Y\big(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2}\big) \\
& = (a_1 + a_2 + a_3 + a_4 + ka_7 + ka_8)\big(Y(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+1}) + Y(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+1})\big) \\
& \quad + (a_5 + a_6 + ka_7 + ka_8)\big(Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) + Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2})\big)
\end{aligned}$$

Thus, we have

$$\begin{aligned}&Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) + Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2})\\ &\le (a_1 + a_2 + a_3 + a_4 + ka_7 + ka_8)\big(Y(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+1}) + Y(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+1})\big)\\ &\quad + (a_5 + a_6 + ka_7 + ka_8)\big(Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) + Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2})\big)\end{aligned} \tag{4.4}$$

It follows from (4.4) that

$$\begin{aligned}&[Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) + Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2})]\\ &\le \frac{(a_1 + a_2 + a_3 + a_4 + ka_7 + ka_8)}{1-(a_5 + a_6 + ka_7 + ka_8)}[Y(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+1}) + Y(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+1})].\end{aligned}$$

Let $\delta = \frac{(a_1+a_2+a_3+a_4+ka_7+ka_8)}{1-(a_5+ka_6+ka_7+ka_8)}$, then $0 \le \delta < 1$ and

$$\begin{aligned}&[Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) + Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2})]\\ &\le \delta[Y(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+1}) + Y(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+1})].\end{aligned} \tag{4.5}$$

It follows from (4.5) that

$$\begin{aligned}&[Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) + Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2})]\\ &\le \delta[Y(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2n+1}) + Y(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2n+1})]\\ &\le \delta\Big(\delta\big(Y(gs_{2n-1}, gs_{2n-1}, \cdots, gs_{2n-1}, gs_{2n}) + Y(gw_{2n-1}, gw_{2n-1}, \cdots, gw_{2n-1}, gw_{2n})\big)\Big)\\ &\le \delta\Big(\delta\big(\delta(Y(gs_{2n-2}, gs_{2n-2}, \cdots, gs_{2n-2}, gs_{2n-1}) + Y(gw_{2n-2}, gw_{2n-2}, \cdots, gw_{2n-2}, gw_{2n-1}))\big)\Big).\end{aligned}$$

This implies

$$\begin{aligned}&[Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) + Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2})]\\ &\le \delta^3[Y(gs_{2n-2}, gs_{2n-2}, \cdots, gs_{2n-2}, gs_{2n-1}) + Y(gw_{2n-2}, gw_{2n-2}, \cdots, gw_{2n-2}, gw_{2n-1})]\\ &\quad \vdots\\ &\le \delta^{2n+1}(Y(gs_0, gs_0, \cdots, gs_0, gs_1) + Y(gw_0, gw_0, \cdots, gw_0, gw_1).\end{aligned} \tag{4.6}$$

By Lemma 3.5 we have for all $n, m \in N$ with $n \le m$

$$\begin{aligned}&Y(g_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2m+1}) + Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2m+1})\\ &\le k[(n-1)Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) + Y(gs_{2n+2}, gs_{2n+2}, \cdots, gs_{2n+2}, gs_{2m+1})]\\ &\quad + k[(n-1)Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2})) + Y(gw_{2n+2}, gw_{2n+2}, \cdots, gw_{2n+2}, gw_{2m+1})]\\ &\le (k(n-1)Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) + k(n-1)Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2}))\\ &\quad + (k^2(n-1)Y(gs_{2n+2}, gs_{2n+2}, \cdots, gs_{2n+2}, gs_{2n+3})\\ &\quad + k^2(n-1)Y(gw_{2n+2}, gw_{2n+2}, \cdots, gw_{2n+2}, gw_{2n+3}))\\ &\quad \cdots + (k^{2m-1}(n-1)Y(gs_{2m-1}, gs_{2m-1}, \cdots, gs_{2m-1}, gs_{2m})\\ &\quad + k^{2m-1}(n-1)Y(gw_{2m-1}, gw_{2m-1}, \cdots, gw_{2m-1}, gw_{2m}))\\ &\quad + k^{2m-1}(Y(gs_{2m}, gs_{2m}, \cdots, gs_{2m}, gs_{2m+1}) + Y(gw_{2m}, gw_{2m}, \cdots, gw_{2m}, gw_{2m+1}))\\ &\le (k(n-1)Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) + k(n-1)Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2}))\\ &\quad + (k^2(n-1)Y(gs_{2n+2}, gs_{2n+2}, \cdots, gs_{2n+2}, gs_{2n+3}) + k^2(n-1)Y(gw_{2n+2}, gw_{2n+2}, \cdots, gw_{2n+2}, gw_{2n+3}))\\ &\quad + \vdots\\ &\quad + (k^{2m}(n-1)Y(gs_{2m}, gs_{2m}, \cdots, gs_{2m}, gs_{2m+1}) + k^{2m}(n-1)Y(gw_{2m}, gw_{2m}, \cdots, gw_{2m}, gw_{2m+1}))\\ &\le k(n-1)\delta^{2n+1}(1 + k\delta + k^2\delta^2 + \cdots)(Y(gs_0, gs_0, \cdots, gs_0, gs_1) + Y(gw_0, gw_0, \cdots, gw_0, gw_1)).\end{aligned}$$

This implies

$$\begin{aligned}&Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2m+1}) + Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2m+1})\\ &\le (n-1)\frac{k\delta^{2n+1}}{1-k\delta}(Y(gs_0, gs_0, \cdots, gs_0, gs_1) + Y(gw_0, gw_0, \cdots, gw_0, gw_1)).\end{aligned}$$

Similarly, we get

$$\begin{aligned}&Y(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2m+1}) + Y(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2m+1})\\&\quad\leq (n-1)\frac{k\delta^{2n}}{1-k\delta}(Y(gs_0, gs_0, \cdots, gs_0, gs_1) + Y(gw_0, gw_0, \cdots, gw_0, gw_1)),\end{aligned}$$

and

$$\begin{aligned}&Y(gs_{2n}, gs_{2n}, \cdots, gs_{2n}, gs_{2m}) + Y(gw_{2n}, gw_{2n}, \cdots, gw_{2n}, gw_{2m})\\&\quad\leq (n-1)\frac{k\delta^{2n}}{1-k\delta}(Y(gs_0, gs_0, \cdots, gs_0, gs_1) + Y(gw_0, gw_0, \cdots, gw_0, gw_1))\end{aligned}$$

and

$$\begin{aligned}&Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2m}) + Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2m})\\&\quad\leq (n-1)\frac{k\delta^{2n+1}}{1-k\delta}(Y(gs_0, gs_0, \cdots, gs_0, gs_1) + Y(gw_0, gw_0, \cdots, gw_0, gw_1)).\end{aligned}$$

Hence, for all $n, m \in N$ with $n \leq m$ and $k\ \delta < 1$ imply that

$$\begin{aligned}&Y(gs_n, gs_n, \cdots, gs_n, gs_m) + Y(gw_n, gw_n, \cdots, gw_n, gw_m)\\&\leq (n-1)\frac{k\delta^{n}}{1-k\delta}[Y(gs_0, gs_0, \cdots, gs_0, gs_1) + Y(gw_0, gw_0, \cdots, gw_0, gw_1)] \ \to\ \theta \text{ as } n\ \to\ \infty\end{aligned}$$

According to Remark 3.10(3), and for any $c \in E$ with $\theta \ll c$, there esists n_0 such that for any

$$n > n_0,\ \ (n-1)\frac{k\delta^{n}}{1-k\delta}(Y(gs_0, gs_0, \cdots, gs_0, gs_1) + Y(gw_0, gw_0, \cdots, gw_0, gw_1)) \ll c.$$

Furthermore, for any $m > n > n_0$, Remark (3.12) (1) shows that $Y(gs_0, gs_0, \cdots, gs_0, gs_1)\ +\ Y(gw_0, gw_0, \cdots, gw_0, gw_1) \ll c$. Hence, by Definition (3.9), $\{gs_n\}$ and $\{gw_n\}$ are Cauchy sequences in S. By the completeness of S, $\exists\ s, w \in S$ such that

$$\lim_{n\to\infty} gs_n = s \ \text{ and } \lim_{n\to\infty} gw_n = w.$$

Now we show that (s, w) is a coupled coincidence point of F and g.

Suppose F is continuous, then we have

$$\begin{aligned}gs &= \lim_{n\to\infty} gs_{n+1} = \lim_{n\to\infty} F(s_n, w_n)\\&= F(\lim_{n\to\infty} s_n, \lim_{n\to\infty} w_n) = F(s, w).\end{aligned}$$

Again,

$$\begin{aligned}gw &= \lim_{n\to\infty} gw_{n+1} = \lim_{n\to\infty} F(w_n, s_n)\\&= F(\lim_{n\to\infty} w_n, \lim_{n\to\infty} s_n) = F(w, s).\end{aligned}$$

This completes the proof. □

Theorem 4.2. Suppose all the conditions of Theorem 4.1 are satisfied. Moreover, assume that S has the following properties

(a) if a sequence$\{s_n\}$ in S which is non-decreasing converges to some point $s \in S$, then $s_n \leq s$, $\forall n$,

(b) if a sequence $\{w_n\}$ in S which is non-increasing converges to some point $w \in S$, then $w_n \geq w$, $\forall n$.

Then the conclusion of Theorem 4.1 also hold.

Proof . Applying the proof of Theorem 4.1 it is required only to prove $gs = F(s,w),\ gw = F(w,s)$.

In fact, since $\{s_n\}$ is non-decreasing and $s_n \to s$ and $\{w_n\}$ is non-increasing and $w_n \to w$, by our assumption, $s_n \leq s$ and $w_n \geq w\ \forall\ n$.

Applying the contractive condition we have

$$\begin{aligned} &Y(gs, gs, \cdots, gs, F(s,w)) \\ &\quad \leq k[(n-1)Y(gs, gs, \cdots, gs, gs_{2n+2}) + Y(gs_{2n+2}, gs_{2n+2}, \cdots, gs_{2n+2}, F(s,w))] \end{aligned} \tag{4.7}$$

Similarly

$$\begin{aligned} &Y(gw, gw, \cdots, gw, F(w,s)) \\ &\quad \leq k[(n-1)Y(gw, gw, \cdots, gw, gw_{2n+2}) + Y(gw_{2n+2}, gw_{2n+2}, \cdots, gw_{2n+2}, F(w,s))] \end{aligned} \tag{4.8}$$

From (4.7) and (4.8), we have

$$\begin{aligned} &Y(gs, gs, \cdots, gs, F(s,w)) + Y(gw, gw, \cdots, gw, F(w,s)) \\ &\quad \leq k[(n-1)Y(gs, gs, \cdots, gs, gs_{2n+2}) + (n-1)Y(gw, gw, \cdots, gw, gw_{2n+2}) \\ &\qquad + Y(gs_{2n+2}, gs_{2n+2}, \cdots, gs_{2n+2}, F(s,w)) + Y(gw_{2n+2}, gw_{2n+2}, \cdots, gw_{2n+2}, F(w,s))] \\ &\quad = k[(n-1)Y(gs, gs, \cdots, gs, gs_{2n+2}) + (n-1)Y(gw, gw, \cdots, gw, gw_{2n+2}) \\ &\qquad + Y(F(s_{2n+1}, w_{2n+1}), F(s_{2n+1}, w_{2n+1}), \cdots, F(s_{2n+1}, w_{2n+1}), F(s,w)) \\ &\qquad + Y(F(w_{2n+1}, s_{2n+1}), F(w_{2n+1}, s_{2n+1}), \cdots, F(w_{2n+1}, s_{2n+1}), F(w,s))]. \end{aligned} \tag{4.9}$$

By using (4.1), we obtain

$$\begin{aligned} &Y(F(s_{2n+1}, w_{2n+1}), F(s_{2n+1}, w_{2n+1}), \cdots, F(s_{2n+1}, w_{2n+1}), F(s,w)) \\ &\quad \leq a_1 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs) + a_2 Y(gw_{2n+1}), gw_{2n+1}), \cdots, gw_{2n+1}), gw) \\ &\qquad + a_3 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, G(s_{2n+1}, w_{2n+1})) + a_4 Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, G(w_{2n+1}, s_{2n+1})) \\ &\qquad + a_5 Y(gs, gs, \cdots, gs, F(s,w))) + a_6 Y(gw, gw, \cdots, gw, F(w,s)) \\ &\qquad + a_7 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, F(s,w)) + a_8 Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, F(w,s)) \\ &\qquad + a_9 Y(gs, gs, \cdots, gs, G(s_{2n+1}, w_{2n+1})) + a_{10} Y(gw, gw, \cdots, gw, G(w_{2n+1}, s_{2n+1})) \\ &\quad = a_1 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs) + a_2 Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, w) \\ &\qquad + a_3 Y((gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) + a_4 Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2}) \\ &\qquad + a_5 Y(gs, gs, \cdots, gs, F(s,w)) + a_6 Y(gw, gw, \cdots, gw, F(w,s)) \\ &\qquad + a_7 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, F(s,w)) + a_8 Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, F(w,s)) \\ &\qquad + a_9 Y(gs, gs, \cdots, gs, gs_{2n+2}) + a_{10} Y(gw, gw, \cdots, gw, gw_{2n+2}) \end{aligned} \tag{4.10}$$

Similarly, we have

$$\begin{aligned} &Y(F(w_{2n+1}, s_{2n+1}), F(w_{2n+1}, s_{2n+1}), \cdots, F(w_{2n+1}, s_{2n+1}), F(w,s)) \\ &\quad = a_1 Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw) + a_2 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs) \\ &\qquad + a_3 Y((gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2}) + a_4 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) \\ &\qquad + a_5 Y(gw, gw, \cdots, gw, F(w,s)) + a_6 Y(gs, gs, \cdots, gs, F(s,w)) \\ &\qquad + a_7 Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, F(w,s)) + a_8 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, F(s,w)) \\ &\qquad + a_9 Y(gw, gw, \cdots, gw, gw_{2n+2}) + a_{10} Y(gs, gs, \cdots, gs_{2n+2}) \end{aligned} \tag{4.11}$$

It follows (4.9), (4.10) and (4.11) that

$$\begin{aligned} &Y(gs, gs, \cdots, gs, F(s,w)) + Y(gw, gw, \cdots, gw, F(w,s)) \\ &\quad \leq k[(n-1)Y(gs, gs, \cdots, gs, gs_{2n+2}) + (n-1)Y(gw, gw, \cdots, gw, gw_{2n+2}) \\ &\quad + a_1 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs) + a_2 Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw) \\ &\quad + a_3 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) + a_4 Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2}) \\ &\quad + a_5 Y(gs, gs, \cdots, gs, F(s,w)) + a_6 Y(gw, gw, \cdots, gw, F(w,s)) \\ &\quad + a_7 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, F(s,w)) + a_8 Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, F(w,s)) \\ &\quad + a_9 Y(gs, gs, \cdots, gs, gs_{2n+2}) + a_{10} Y(gw, gw, \cdots, gw, gw_{2n+2}) \\ &\quad + a_1 Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw) + a_2 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs) \\ &\quad + a_3 Y((gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, gw_{2n+2}) + a_4 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, gs_{2n+2}) \\ &\quad + a_5 Y(gw, gw, \cdots, gw, F(w,s)) + a_6 Y(gs, gs, \cdots, gs, F(s,w)) \\ &\quad + a_7 Y(gw_{2n+1}, gw_{2n+1}, \cdots, gw_{2n+1}, F(w,s)) + a_8 Y(gs_{2n+1}, gs_{2n+1}, \cdots, gs_{2n+1}, F(s,w)) \\ &\quad + a_9 Y(gw, gw, \cdots, gw, gw_{2n+2}) + a_{10} Y(gs, gs, \cdots, gs, gs_{2n+2})]. \end{aligned} \tag{4.12}$$

Taking the limit as $n \to \infty$ in above inequality, we have

$$\begin{aligned} &Y(gs, gs, \cdots, gs, F(s,w)) + Y(gw, gw, \cdots, gw, F(w,s)) \\ &\quad \leq k[a_5 Y(gs, gs, \cdots, gs, F(s,w)) + a_6 Y(gw, gw, \cdots, gw, F(w,s)) \\ &\quad + a_7 Y(gs, gs, \cdots, gs, F(s,w)) + a_8 Y(gw, gw, \cdots, gw, F(w,s)) \\ &\quad + a_5 Y(gw, gw, \cdots, gw, F(w,s)) + a_6 Y(gs, gs, \cdots, gs, F(s,w)) \\ &\quad + a_7 Y(gw, gw, \cdots, gw, F(w,s)) + a_8 Y(gs, gs, \cdots, gs, F(s,w))]. \end{aligned}$$

$$\begin{aligned} &Y(gs, gs, \cdots, gs, F(s,w)) + Y(gw, gw, \cdots, gw, F(w,s)) \\ &\quad \leq k(a_5 + a_6 + a_7 + a_8)[Y(gs, gs, \cdots, gs, F(s,w)) + Y(gw, gw, \cdots, gw, F(w,s))]. \end{aligned}$$

Since, $0 \leq k(a_5 + a_6 + a_7 + a_8) < 1$, Remark 3.12(4) shows

$$Y(gs, gs, \cdots, gs, F(s,w)) + Y(gw, gw, \cdots, gw, F(w,s)) = \theta$$

that is, $F(s,w) = gs$ and $F(w,s) = gw$. This proves that (s,w) is a coupled common fixed point of F and g and this finishes the proof. □

$$\text{for } (s,w),(u,v) \in S \times S \text{ there exists } (z,t) \in S \times S \text{ which is comparable to } (s,w) \text{ and } (u,v). \tag{4.13}$$

Note that in $S \times S$ we consider the partial order relation given by

$$(s,w) \leq (u,v) \iff s \leq u \text{ and } w \geq v.$$

Theorem 4.3. Using condition (4.13) to the hypotheses of Theorem 4.1 (resp. Theorem 4.2) we get uniqueness of the coupled coincidence point of F and g. Furthermore, any fixed point is common for F and g.

Proof . Suppose F has (s,w) and (s',w') coupled coincidence points , that is, $F(s,w) = gs, F(w,s) = gw, F(s',w') = gs'$ and $F(w',s') = gw'$. We shall prove that $gs = gs', gw = gw'$.

Let (s,w) and (s',w') are not comparable. Then by assumption there exist $(z,t) \in S \times S$ comparable with both of them. Suppose that $(gs, gw) \leq (gs', gw')$ without loss of generality, it follows from Theorem 4.1.

$$\begin{aligned}
&Y(gs, gs, \cdots, gs, gs') + Y(gw, gw, \cdots, gw, gw') \\
&\quad = Y(F(s,w), F(s,w), \cdots, F(s,w), F(s',w')) + Y(F(w,s), F(w,s), \cdots, F(s,w), F(w',s')) \\
&\quad \leq a_1 Y(gs, gs, \cdots, gs, gs') + a_2 Y(gw, gw, \cdots, gw, gw') \\
&\qquad + a_3 Y(gs, gs, \cdots, gs, F(s,w)) + a_4 Y(gw, gw, \cdots, gw, F(w,s)) \\
&\qquad + a_5 Y(gs', gs', \cdots, gs', F(s',w')) + a_6 Y(gw', gw' \cdots, gw', F(w',s')) \\
&\qquad + a_7 Y(gs, gs, \cdots, gs, F(s',w')) + a_8 Y(gw, gw, \cdots, gw, F(w',s')) \\
&\qquad + a_9 Y(gs', gs', \cdots, gs', F(s,w)) + a_{10} Y(gw', gw' \cdots, gw', F(w,s)) \\
&\quad = a_1 Y(gs, gs, \cdots, gs, gs') + a_2 Y(gw, gw, \cdots, gw, gw') \\
&\qquad + a_3 Y(gs, gs, \cdots, gs, gs) + a_4 Y(gw, gw, \cdots, gw, gw) \\
&\qquad + a_5 Y(gs', gs', \cdots, gs', gs') + a_6 Y(gw', gw', \cdots, gw', gw') \\
&\qquad + a_7 Y(gs, gs, \cdots, gs, gs') + a_8 Y(gw, gw, \cdots, gw, gw') \\
&\qquad + a_9 Y(gs', gs', \cdots, gs', gs) + a_{10} Y(gw', gw', \cdots, gw', gw)
\end{aligned}$$

Thus,

$$\begin{aligned}
&Y(gs, gs, \cdots, gs, gs') \\
&\quad = (a_1 + a_7 + a_9) Y(gs, gs, \cdots, gs, gs') + (a_2 + a_8 + a_{10}) Y(gw, gw, \cdots, gw, gw').
\end{aligned}$$

Similarly,

$$\begin{aligned}
&Y(gw, gw, \cdots, gw, gw') \\
&\quad = (a_1 + a_7 + a_9) Y(gw, gw, \cdots, gw, gw') + (a_2 + a_8 + a_{10}) Y(gs, gs, \cdots, gs, gs').
\end{aligned}$$

From above two inequalities, we have

$$\begin{aligned}
&Y(gs, gs, \cdots, gs, gs') + Y(gw, gw, \cdots, gw, gw') \\
&\quad = (a_1 + a_2 + a_7 + a_8 + a_9 + a_{10})[Y(gw, gw, \cdots, gw, gw') + Y(gs, gs, \cdots, gs, gs')].
\end{aligned}$$

Since, $0 \leq (a_1 + + a_2 + a_7 + a_8 + a_9 + a_{10}) < 1$, Remark 3.12(4) shows that $Y(gs, gs, \cdots, gs, gs') + Y(gw, gw, \cdots, gw, gw') = \theta$, which implies $gs = gs'$ and $gw = gw'$.

Now, we show that any fixed point of F is a fixed point of g. Applying Theorem 4.1, we get

$$\begin{aligned}
Y(gs, gs, \cdots, gs, gw) &= Y(F(s,w), F(s,w), \cdots, F(s,w), F(w,s)) \\
&\leq a_1 Y(gs, gs, \cdots, gs, gw) + a_2 Y(gw, gw, \cdots, gw, gs) \\
&\quad + a_3 Y(gs, gs, \cdots, gs, F(s,w)) + a_4 Y(gw, gw, \cdots, gw, F(w,s)) \\
&\quad + a_5 Y(gw, gw, \cdots, gw, F(w,s)) + a_6 Y(gs, gs, \cdots, gs, F(s,w)) \\
&\quad + a_7 Y(gs, gs, \cdots, gs, F(w,s)) + a_8 Y(gw, gw, \cdots, gw, F(s,w)) \\
&\quad + a_9 Y(gw, gw, \cdots, gw, F(s,w)) + a_{10} Y(gs, gs, \cdots, gs, F(w,s)) \\
&= (a_1 + a_7 + a_{10}) Y(gs, gs, \cdots, gs, gw) + (a_2 + a_8 + a_9) Y(gw, gw, \cdots, gw, gs).
\end{aligned} \tag{4.14}$$

In similar way

$$\begin{aligned}
Y(gw, gw, \cdots, gw, gs) &= Y(F(w,s), F(w,s), \cdots, F(w,s), F(s,w)) \\
&= (a_2 + a_8 + a_9) Y(gs, gs, \cdots, gs, gw) + (a_1 + a_7 + a_{10}) Y(gw, gw, \cdots, gw, gs).
\end{aligned} \tag{4.15}$$

Adding (4.14) and (4.15), we obtain

$$\begin{aligned}
&Y(gs, gs, \cdots, gs, gw) + Y(gw, gw, \cdots, gw, gs) \\
&\quad = (a_1 + a_2 + a_7 + a_8 + a_9 + a_{10})[Y(gs, gs, \cdots, gs, gw) + Y(gw, gw, \cdots, gw, gs)].
\end{aligned}$$

Since, $0 \leq (a_1 + a_2 + a_7 + a_8 + a_9 + a_{10}) < 1$, Remark 3.12(4) shows $Y(gs, gs, \cdots, gs, gw) + Y(gw, gw, \cdots, gw, gs) = \theta$, which implies $gs = gw$. The coupled common fixed point of F and g is unique. This finishes the proof. □

Example 4.4. Let $(S, \leq, Y)$ be a totally ordered complete Y-cone metric space with Y-cone metric defined as in Example 3.4. Let $F : S \times S \to S$ as $F(s,w) = \frac{(s+3w)}{5}$ for all $s, w \in S$.

Suppose that $g : S \to S$ as $gs = s$

$$\begin{aligned}
&Y(F(s,w), F(s,w), \cdots, F(s,w), F(u,v)) \\
&= \Big[(n-1)|F(s,w) - F(u,v)|^2 + (n-1)|F(w,s) - F(v,u)|^2\Big]e^t \\
&= \Big[(n-1)|\frac{s+3w}{5} - \frac{u+3v}{5}|^2 + (n-1)|\frac{w+3s}{5} - \frac{v+3u}{5}|^2\Big]e^t \\
&= \Big[(n-1)|(\frac{s}{5} - \frac{u}{5}) + (\frac{3w}{5} - \frac{3v}{5})|^2 + (n-1)|(\frac{w}{5} - \frac{v}{5}) + (\frac{3u}{5} - \frac{3s}{5})|^2\Big]e^t \\
&\leq 2(n-1)\Big[|\frac{s}{5} - \frac{u}{5}|^2 + |\frac{3w}{5} - \frac{3v}{5}|^2 + |\frac{w}{5} - \frac{v}{5}|^2 + |\frac{3u}{5} - \frac{3s}{5}|^2\Big]e^t \\
&= \frac{2(n-1)}{25}\Big[|s-u|^2 + |3w - 3v|^2 + |w-v|^2 + |3u - 3s|^2\Big]e^t \\
&\leq \frac{2(n-1)}{5}\Big[|s-u|^2 + |w-v|^2\Big]e^t \\
&= \frac{2}{5}[Y(gs, gs, \cdots, gs, gu) + Y(gw, gw, \cdots, gw, gv)].
\end{aligned}$$

where $a_1 = \frac{2}{5} = a_2, a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = 0$. Hence, the properties of Theorem (4.1) are satisfied. Further, $(0,0)$ is the unique coupled coincidence point of F and g.

If g is an identity mapping,we have the results.

Corollary 4.5. Presume $(S, \leq, Y)$ be a partially ordered complete Y-cone metric space with the coefficient $k \geq 1$ relative to a solid cone P. Presuppose $F : S \times S \to S$ be the continuous mappings posseses the property of mixed monotone on S. Suppose that $\exists\ a_m \geq 0,\ m = 1, 2, \cdots, 10$ with $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + 2k(a_7 + a_8) < 1$ and $\sum_{m=1}^{10} a_m < 1$ such that

$$\begin{aligned}
&Y(F(s,w), F(s,w), \cdots, F(s,w), F(u,v)) \\
&\quad \leq a_1 Y(s, s, \cdots, s, u) + a_2 Y(w, w, \cdots, w, v) \\
&\qquad + a_3 Y(s, s, \cdots, s, F(s,w)) + a_4 Y(w, w, \cdots, w, F(w,s)) \\
&\qquad + a_5 Y(u, u, \cdots, u, F(u,v)) + a_6 Y(v, v, \cdots, v, F(v,u)) \\
&\qquad + a_7 Y(s, s, \cdots, s, F(u,v)) + a_8 Y(w, w, \cdots, w, F(v,u)), \\
&\qquad + a_9 Y(u, u, \cdots, u, F(s,w)) + a_{10} Y(v, v, \cdots, v, F(s,w))
\end{aligned} \tag{4.16}$$

$\forall\ s, w, u, v \in S$ with $s \leq u$ and $w \geq v$. Suppose either F is continous or S has the following properties

(a) if a sequence$\{s_n\}$ in S which is non-decreasing converges to some point $s \in S$, then $s_n \leq s$, $\forall n$,

(b) if a sequence $\{w_n\}$ in S which is non-increasing converges to some point $w \in S$, then $w_n \geq w$, $\forall n$.

If there exists $s_0, w_0 \in S$ such that $s_0 \leq F(s_0, w_0)$ and $y_0 \geq F(w_0, s_0)$, then F has a coupled fixed point.

Corollary 4.6. Let $(S, \leq, Y)$ be a partially ordered complete Y-cone metric space with the coefficient $k \geq 1$ relative to a solid cone P. Let $F : S \times S \to S$ be the mappings such that F has the mixed monotone property on S. Suppose that there exist $K \in [0,1)$ such that

$$\begin{aligned}
&Y(F(s,w), F(s,w), \cdots, F(s,w), F(u,v)) \\
&\quad + Y(F(w,s), F(w,s), \cdots, F(w,s), F(v,u)) \\
&\quad \leq K(Y(s, s, \cdots, s, u) + Y(w, w, \cdots, w, v))
\end{aligned} \tag{4.17}$$

$\forall\ s, w, u, v \in S$ with $s \leq u$ and $w \geq v$. Suppose either F is continuous or S has the following properties

(a) if a sequence $\{s_n\}$ in S which is non-decreasing converges to some point $s \in S$, then $s_n \leq s$, $\forall n$,

(b) if a sequence $\{w_n\}$ in S which is non-increasing converges to some point $y \in S$, then $w_n \geq w$, $\forall n$.

If there exists $s_0, w_0 \in S$ such that $s_0 \leq F(s_0, w_0)$ and $w_0 \geq F(w_0, s_0)$, then F has a coupled fixed point.

Proof . Applying $a_1 = K, a_2 = a_3 = a_4 = a_5 = a_6 = 0$ in Theorems (4.1) and (4.2), we obtain the corollary. □

5 Conclusions

In this paper, we establish the existence and uniqueness of coupled coincidence theorems on complete Y-cone metric spaces. Lastly, we provide the example to support our result.

References

[1] M. Abbas, M.A. Khan and S. Radenovic, *Common coupled fixed point theorem in cone metric space for w-compatible mappings*, Appl. Math. Comput. **217** (2010), 195-202.

[2] M. Abbas, B. Ali and Y.I. Suleiman, *Generalized coupled common fixed point results in A-metric spaces*, Fixed Point Theory Appl. **2015** (2015), Article ID 64.

[3] R.P. Agarwal, D. O'Regan, D.R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications*, Vol. 6. New York, Springer, 2009.

[4] T.G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65** (2006), no. 7, 1379–1393.

[5] B.S. Choudhury and A. Kundu, *A coupled coincidence point result in partially ordered metric spaces for compatible mappings*, Nonlinear Anal. **73** (2010), no. 8, 2524–2531

[6] L. Ćirić and V. Lakshmikantham, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009), no. 12, 4341–4349.

[7] S. Czerwik, *Nonlinear set-valued contraction mappings in b- metric spaces*, Atti Sem. Mat. Fiz. Univ. Modena **46** (1998), no. 2, 263–276.

[8] L.G. Haung and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), no. 2, 1468–1476.

[9] D.H. Hyers, G. Isac and T.M. Rassias, *Topics in nonlinear analysis and applications*, World Scientific Publishing Co., Inc., River Edge, NJ, 1997.

[10] N.V. Luong and N.X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal. **74** (2011), no. 3, 983–992.

[11] R. Pant, Rajendra, R. Shukla, H.K. Nashine and R. Panicker, *Some new fixed point theorems in partial metric spaces with applications*, J. Funct. Spaces **2017** (2017), Art. ID 1072750, 1-13.

[12] A. Razani, H.H. Zadeh, A. Jabbari, *Coupled fixed point theorems in partially ordered metric spaces which endowed with vector-valued metrics*, Aust. J. Basic Appl. Sci. **6** (2012), no. 2, 124–129.

[13] F. Sabetghadam, H.P. Masiha and A.H. Sanatpour, *Some coupled fixed point theorems in cone metric spaces*, Fixed Point Theory Appl. **2009** (2009), Art. ID 125426, 8 pp.

[14] R. Sharma, V.S. Chouhan and S. Mishra, *Y-cone metric spaces and coupled fixed point results with applications to integral equation*, TWMS J. Appl. Engin. Math. **12** (2022), no. 1, 388–405.

[15] W. Shatanawi, B. Samet and M. Abbas, *Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces*, Math. Comput. Model. **55** (2012), no. 3-4, 680–687.

[16] D. Turkoglu and M. Abuloha, *Cone metric spaces and fixed point theorems in diametrically contractive mappings*, Acta Math. Sin. (Engl. Ser.) **26** (2010), no. 3, 489–496.