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On weak B-bi-regular bi-near subtraction semigroup

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Abstract

We introduce weak B-bi-regular in bi-near subtraction semigroup and obtain properties of the same in a certain class of bi-near subtraction semigroup of X.

Keywords: weak B-bi-regular, B-bi-regular, bi-regular, $\bar{S}$-bi-near subtraction semigroup, left (right) X-bi-algebra.
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1 Introduction

In this paper we can see bi-near subtraction semigroup by the algebraic structure of $(X, -, \cdot)$ where $X = X_1 \cup X_2$. For basic definition we may refer to Pilz [6] by near-rings. Zelinka [8] discussed a problem proposed by schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras. Motivated by the study of "On weak B-regular near-rings" by Jayalakshmi [2]. Further Jayalakshmi, Mahalakshmi and Maharasi in [3] introduced weak k-regular "On strong bi-ideals and weak k-regular in near subtraction semigroup". With this in mind, we introduce the notion of weak B-bi-regular bi-near subtraction of semigroup and study the properties of a bi-near subtraction semigroup with property (α)

2 Preliminaries

Definition 2.1. Let $(X, -, \cdot)$ be nonempty set. Where $X = X_1 \cup X_2$, where X_1 and X_2 are proper subsets of X. (i.e.,) $X_1 \not\subset X_2$ (or) $X_2 \not\subset X_1$ satisfying the following conditions.

- $(X_1, -, \cdot)$ near subtraction semi group (right).
- $(X_2, -, \cdot)$ is subtraction semi group.

Definition 2.2. Let S be nonempty subset. Then a near subtraction semigroup is sub algebra of X, if $u - v \in S$, for every $u, v \in S$.

Definition 2.3. Let S be nonempty subset. Then S be bi-sub algebra of $(X, -)$ if

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- $S = S_1 \cup S_2$.
- $(S_1, -)$ is near sub algebra of $(X_1, -)$.
- $(S_2, -)$ is sub algebra of $(X_2, -)$.

Definition 2.4. An element of X is idempotent if $x^2 = x$ for all $x \in X_1 \cup X_2$.

Definition 2.5. A bi-sub algebra A of $(X, -)$ is said to be left (right) X-bi sub algebra of X if $XA \subseteq A(AX \subseteq A)$. Let X is said to be two sided, if every left X-bi sub algebra of X is a right X-bi sub algebra of X and vice versa.

Definition 2.6. If X is said to be bi-regular. Then for all $a \in X_1 \cup X_2$, there exists $b \in X_1 \cup X_2$, such that $a = aba$.If X is said to be strongly bi-regular, if for each $a \in X_1 \cup X_2$, there exist $b \in X_1 \cup X_2$ such that $a = ba^2$.

Lemma 2.7. Let X is zero-symmetric bi-near subtraction semigroup and if X is strongly bi-regular, then X is a bi-regular.

Definition 2.8. A S-bi-near subtraction semigroup is said to be $\overline{S}$-bi-near subtraction semigroup, if $a \in aX$ for all $a \in X_1 \bigcup X_2$.

Definition 2.9. If X is said to have property (α), then aX is bi-sub algebra of $(X, -)$ for every $a \in X_1 \bigcup X_2$.

Remark 2.10. If X is -bi-near subtraction semigroup with property (α), then $(a)_r = aX$ and $a_1 = Xa$ for all $a \in X_1 \bigcup X_2$.

Lemma 2.11. A $\overline{S}$-bi-near subtraction semigroup X with property (α) is bi-regular if and only if $B = BXB$, for every bi-ideals B of X.

Definition 2.12. If X is called strictly Commuting Principal X-bi-sub algebra(CPXBS) if $y(Xx) = (Xx)y$ for all $x, y \in X_1 \bigcup \mathbb{X}_2$.

Theorem 2.13. The following subdivisions are equivalent.

(i) A bi-near subtraction semigroup of X is GNF.
(ii) A bi-near subtraction semigroup of X is bi-regular and each idempotent is central.
(iii) A bi-near subtraction semigroup of X is bi-regular and sub commutative.

3 Weak B-bi regular

Definition 3.1. Let X is B-biregular, if for every element of X is B-regular for every $a \in (a)_r X (a)_1$ for every $a \in X_1 \bigcup \mathbb{X}_2$. Here $(a)_r (a)_1$ is right and left X-bi-sub algebra generated by $a \in X_1 \bigcup \mathbb{X}_2$.

Definition 3.2. Let X is left (right) weak B-bi-regular, if for all $a \in X_1 \bigcup X_2$(i.e.,)$a \in X(a)_1$ for all $a \in X_1 \bigcup \mathbb{X}_2$. Here $(a)_r (a)_1$ is right and left X-bi-sub algebra generated by $a \in X_1 \bigcup X_2$. If X is called weak B-bi-regular if X is both left and right weak B-bi-regular.

Remark 3.3. Every bi-regular bi-near subtraction semigroup is also a B-bi-regular bi-near subtraction semigroup.

Example 3.4. Let X is nonempty set. Where X_1 is proper subset in X. Then be near subtraction semi group (right) choose from Klein's four group scheme $(7, 8, 1, 2)$ $(P.408, Pilz[6])$.
Table 1. bi-regular as well as $B-$bi regular near subtraction semigroup

$-$	0	X	y	z
0	0	0	0	0
x	x	0	x	0
y	y	Y	0	0
z	z	Y	x	0

$\cdot$	0	x	y	z
0	0	0	0	0
X	x	x	x	x
Y	0	x	y	z
Z	x	0	y	z

One can check that is bi-regular near subtraction semigroup $x \in xX_1x$, and also a B-bi-regular, since $x \in (x)_rX_1(x)_1$ Or X_2 is proper subset in X. Then be $X_2 = \{0, a, b, c\}$ be subtraction semi group choose from Klein's four group scheme $(7, 7, 1, 1)(P.408, Pilz[6])$.
Table 2. bi-regular as well as B-bi-regular subtraction semigroup

−	0	A	b	c
0	0	0	0	0
a	a	0	a	0
b	b	B	0	0
c	c	B	a	0

·	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	b	b
c	a	a	c	c

One can check that X_2 is bi-regular subtraction semigroup, $a \in aX_2$ and also a B-bi-regular, since $a \in (a)_rX_2(a)_1$ Clearly every bi-regular bi-near subtraction semigroup is also a B-bi-regular bi-near subtraction semigroup.

Remark 3.5. Every B-bi regular bi-near subtraction semigroup is also a weak B-bi-regular bi-near subtraction semigroup of X.

Example 3.6. Let X is nonempty set. Where X_1 is proper subset in X. Then $X_1 = \{0, x, y, z\}$ be near subtraction semigroup (right) choose from Klein's four group scheme $(1, 1, 1, 1)(P.408, Pilz[6])$.
Table 3. $B-$ bi-regular as well as weak $B-$ bi regular near subtraction semigroup

−	0	X	y	z
0	0	0	0	0
X	x	0	x	0
Y	y	Y	0	0
Z	z	Y	x	0

·	0	x	y	z
0	0	0	0	0
x	x	x	x	x
y	y	y	y	y
z	z	z	z	z

One can check that is B-bi-regular near subtraction semigroup, $x \in (x)_rX_1(x)_1$ and also a weak B-bi-regular, since $x \in X_1(x)_1(x \in (x)_rX_1)$. Or X_2 is proper subset in X. Then be subtraction semigroup choose from klein's four group scheme $(0, 1, 1, 1)(P.108, Pilz[6])$.
Table 4. B-bi-regular as well as weak B-bi-regular subtraction semirgroup

−	0	A	b	c
0	0	0	0	0
a	a	0	a	0
b	b	B	0	0
c	c	B	a	0

·	0	a	b	c
0	0	0	0	0
a	0	a	a	A
b	0	b	b	B
c	0	c	c	C

One can check that X_2 is B-bi-regular subtraction semigroup, $a \in (a)_rX_2(a)_1$ and also a weak B-bi-regular subtraction semigroup, since $a \in X_2(a)_1(a \in (a)_rX_2)$ Clearly every B-bi regular bi-near subtraction semigroup is also a weak B-bi-regular bi-near subtraction semigroup.

Remark 3.7. Every weak B-bi-regular bi-near subtraction semigroup is different from B-bi regular bi-near subtraction semigroup.

Example 3.8. Let X is nonempty set. Where X_1 is proper subset in X. Then $X_1 = \{0, x, y, z\}$ be near subtraction semi group (right) choose from Klein's four group scheme $(0, 14, 2, 1)(P.408, Pilz[6])$.
Table 5. Weak B-bi-regular not a B-bi-regular near subtraction semigroup.

−	0	x	y	Z
0	0	0	0	0
X	x	0	x	0
Y	y	y	0	0
Z	z	y	x	0

·	0	x	y	z
0	0	0	0	0
x	0	0	x	x
y	0	x	z	y
z	0	x	y	z

One can check that X_1 is weak B-bi-regular near subtraction semigroup $x \in X_1(x)_1(x \in (x)_r X_1)$. But not a B-bi-regular, since $x \notin (x)_r X_1(x)_1$.
Or X_2 is proper subset in X. Then $X_2 = \{0, a, b, c\}$ be subtraction semi group choose from Klein's four group scheme $(7, 7, 1, 1)(P.408, Pilz[6])$.
Table 6. Weak B-bi-regular not a B-bi-regular subtraction semigroup

−	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	b	b
z	a	a	c	c

One can check that X_2 is weak B-bi-regular subtraction semigroup, $a \in X_2(a)_1(a \in (a)_r X_2)$. But X_2 is not B-bi-regular, since $a \in (a)_r X_2(a)_1$ .Clearly every weak B-bi-regular bi-near subtraction semigroup is different from B-bi regular bi-near subtraction semigroup.

Lemma 3.9. Every bi-regular bi-near subtraction semigroup of X is B-bi regular bi-near subtraction semigroup.

Proof . Consider X is bi-regular, then every element of X is regular, for every $a \in X_1 \bigcup X_2$, then there exists $b \in X_1 \bigcup X_2$ such that $a = aba$. To prove $a \in (a)_r X(a)_1$, here $(a)_r(a)_1$ is the right and left X-bi sub algebra generated by $a \in X_1 \bigcup X_2$. Obviously $a \in (a)_r$ and $a \in (a)_1$. Since X is bi-regular. Now, $a = aba \in (a)_r X(a)_1$ for all $a \in X_1 \bigcup X_2$. Therefore X is B-regular and so X is B-bi regular. □

Proposition 3.10. Let X be a bi-near subtraction semigroup. Then the following are equivalent.

(i) X is a weak B-bi-regular.
(ii) $RX \cap XL = R \cap L$, for every right X-bi sub algebra R and left X-bi sub algebra of X.
(iii) $(a)_r X \cap X(a)_l = (a)_r \cap (a)_l$ where $a \in X_1 \cup X_2$.

Proof . $(i) \Rightarrow (ii)$

Assume that X is a weak B-bi-regular bi-near subtraction semigroup. To prove $RX \cap XL = R \cap L$. Let $x \in R \cap L$ where R is X-bi sub algebra and L is left X-bi sub algebra of X. If $x \in R \cap L$, then $x \in R$ and $x \in L$. Since X is a weak B-bi-regular, then $x \in (x)_r X$ and $x \in X(x)_l$ for all $x \in X_1 \cup X_2$. In the sense, $x \in (x)_r X \cap X(x)_l \subseteq RX \cap XL$. Therefore $R \cap L \subseteq RX \cap XL$. Trivially $RX \cap XL \subseteq R \cap L$. Hence $RX \cap XL = R \cap L$.
$(ii) \Rightarrow (iii)$

Trivially true.
$(iii) \Rightarrow (i)$ For any $a \in X_1 \cup X_2$, let $a \in (a)_r$ and $a \in (a)_l$, then $a \in (a)_r \cap (a)_l = (a)_r X \cap X(a)_l$ implies $a \in (a)_r X$ and $a \in (a)_l X$ for all $a \in X_1 \cup X_2$. Hence X is both left and right weak B-bi-regular implies X is a weak B-bi-regular. □

Proposition 3.11. Let X be a bi-near subtraction semigroup. If X is weak B-bi-regular, then $YX \cap XY = Y$ for every invariant sub bi-near subtraction semigroup Y of X.

Proof . Let Y be a invariant sub bi-near subtraction semigroup, then $YX \cap XY \subseteq Y$. It is enough to prove $Y \subseteq YX \cap XY$. Let $y \in Y$ then $m \in X$. Now by Proposition 3.10 we have $y \in (y)_r \cap (y)_l = (y)_r X \cap X(y)_l \subseteq YX \cap XY$. Therefore $Y \subseteq MY \cap XY$. Hence $Y = YX \cap XY$. □

Proposition 3.12. Let X be a bi-near subtraction semigroup. If X is weak B-bi-regular, then $YX \cap XY = Y$ for every invariant X-bi sub algebra Y of X.

Proof . By Proposition 3.11, we can prove. □

Proposition 3.13. Let X be a bi-near subtraction semigroup of X. Then the following conditions are equivalent.

(i) Every Right X-bi sub algebra of X is idempotent and $X(x)_l = (x)_r X$
(ii) X is a weak B-bi-regular, two sided and $(x)_r X = (x)_r^2 X$.

(iii) X is B-bi-regular and two sided.

Proof . $(i) \Rightarrow (ii)$

Let assume that every right X-bi sub algebra is idempotent and $X(x)_l = (x)_r X$. Let $x \in (x)_r = (x)_r^2 \subseteq (x)_r X = X(x)_l$. Thus X is weak B-bi-regular. Since every right X-bi-sub algebra is idempotent, we acquire $(x)_r X = (x)_r^2 X$. Now to prove X is two sided. Let L be a left X-bi sub algebra of X. Let $x \in L$ then $xX \subseteq (x)_r X = (x)_r^2 X = (x)_r(x)_r X \subseteq X(x)_r X = XX(x)_l = X(x)_l \subseteq (x)_l \subseteq L$. Hence L is also a Right X-bi sub algebra of X.

On the other hand, let R be a Right X-bi sub algebra of X. For $x \in R$, let $x \in X_1 \cup X_2$. Now $x \in Xx \subseteq X(x)_r = X(x)_r^2 = X(x)_r(x)_r \subseteq X(x)_r X = XX(x)_l = (x)_r X \subseteq (x)_r \subseteq R$. Hence R is also a left X-bi sub algebra of X. Therefore X is two sided.

$(ii) \Rightarrow (iii)$

Let us assume that X is a weak B-bi-regular, two sided and $(x)_r X = (x)_r^2 X$. For $x \in X_1 \cup X_2$, $(x)_r = (x)_l$ (Since X is two sided). Since X is weak B-bi-regular, by the Proposition 3.10, we get $(x)_r \cap (x)_l = (x)_r X \cap X(x)_l = (x)_r 2X \cap X(x)_l$ (Since $(x)_r X = (x)_r^2 X) \subseteq (x)_r^2 X \subseteq (x)_r^2 = (x)_r(x)_l$. Therefore $(x)_r \cap (x)_l \subseteq (x)_r(x)_l$. Trivially $(x)_r(x)_l \subseteq (x)_r \cap (x)_l$ and so $(x)_r \cap (x)_l = (x)_r(x)_l$. By the Proposition 3.10, X is B-bi-regular bi-near subtraction semigroup.

$(iii) \Rightarrow (i)$

Let assume that X is B-bi-regular bi-near subtraction semigroup and two sided. Let Y be a Right X-bi sub algebra of X then Y is also a left X-bi sub algebra of X. (Since X is two sided). Thus $YXY \subseteq Y$. Since X is B-bi-regular, let $y \in Y$ then $y \in (y)_r X(y)_l \subseteq Y$. From this we get that $Y = YXY \subseteq YY = Y^2$. Therefore Y is idempotent, for $x \in X_1 \cup X_2$, taking Y as $(x)_l$ in the above argument we get that $(x)_l X(x)_l = (x)_l$. Now $(x)_r X \subseteq (x)_r = (x)_l = (x)_l X(x)_l \subseteq X(x)_l \subseteq (x)_l = (x)_r = (x)_r X(x)_l \subseteq (x)_r X$ and so $(x)_r X = X(x)_l$ implies $X(x)_l = (x)_r X$. Hence every right X-bi sub algebra of a bi-near subtraction semigroup is idempotent and $X(x)_l = (x)_r X$ for all $x \in X_1 \cup X_2$. □

Proposition 3.14. Let X be a left self-distributive and right permutable bi-near subtraction semigroup with property (α). Then X is a bi-regular bi-near subtraction semigroup of X if and only if X is $\bar{S}$ and weak B-bi-regular bi-near subtraction semigroup.

Proof . Consider X is both $\bar{S}$ and weak B-bi-regular bi-near subtraction semigroup. Let $x \in X_1 \cup X_2$ then $x \in (x)_r X \cap X(x)_l \subseteq xXX \cap XXx$. (i.e.,) $x = xx_1x_2$ for some $x_1, x_2 \in X_1 \cup X_2$. Now $x = xx_1xx_2$ (Since X is left self-distributive)$= xx_1x_2x$ (Since X is right permutable) $\in xXXx \subseteq xXx$. Therefore $x \in xXx$ for all $x \in X_1 \cup X_2$. (i.e.,) x is regular and so X is a bi-regular.

Converse part is trivially true. □

Theorem 3.15. Let X be a $\bar{S}$-bi-near subtraction semigroup with property (α). Then the following are equivalent.

(i) X is B-bi-regular and two sided.
(ii) X is weak B-bi-regular and $RL = RX \cap XL = LR$ for every left X-bi sub algebra L and Right X-bi sub algebra R of X.
(iii) X is B-bi-regular and sub commutative.
(iv) $XB^2 = B$, for each strong bi-ideal B of bi-near subtraction semigroup of X and X is sub commutative.
(v) $B = BXB$, for each bi-ideal B of X and X is sub commutative.
(vi) X is strictly CPXS and bi-regular.

Proof . $(i) \Rightarrow (ii)$ Consider X is B-bi-regular and two sided. To prove $RL = RX \cap XL = LR$. Since every B-bi-regular of X is weak B-bi-regular of X. By Proposition 3.10 $RX \cap XL = R \cap L$. By Proposition 3.12, $R \cap L = RL$. Since X is two sided, $R \cap L = LR$. Therefore X is weak B-bi-regular and $RX \cap XL = R \cap L = RL = LR$.
$(ii) \Rightarrow (iii)$ Consider X is weak B-bi-regular and $RL = RX \cap XL = LR$. To prove X is B-bi regular. Since X is weak B-bi-regular, by Proposition 3.10, $RX \cap XL = R \cap L$. Therefore $R \cap L = RX \cap XL = RL$. (i.e) $R \cap L = RL$. By Proposition 3.12, X is B-bi-regular. Since X is a $\overline{S}$-bi-near subtraction semigroup with property (α), by the Proposition 3.14, X is a bi-regular. Now $RL = R \cap L = LR$ then $R = R \cap X = NR$. So R is a left X-bi sub algebra. Similarly L is a right X-bi sub algebra. Thus X is two sided , $xX = (x)_r = (x)_l = Xx$ for all $x \in X_1 \cup X_2$. Thus X is sub commutative.
$(iii) \Rightarrow (iv)$ Consider X is B-bi-regular and sub commutative. To prove $XB^2 = B$. Let B be a Strong bi-ideal of X implies $XB^2 \subseteq B$. (i.e.,) $X_1B_1^2 \subseteq B_1$(or $X_2B_2^2 \subseteq B_2$). Now it is sufficient to prove $B \subseteq XB^2$. Let $b \in B$

then $b = bb_1b \in bXb$ implies $b \in bXb$. Now, $bXb = (Xb)b$. (Since X is sub commutative bi-near subtraction semigroup)$= Xb^2 \subseteq XB^2$. Therefore $b \in bXb \in XB^2$. Thus $B \subseteq XB^2$ and hence $B = XB^2$.
$(iv) \Rightarrow (v)$ Assume that $B = XB^2$. Let B be any bi-ideal of X implies $BXB \subseteq B$. (i.e.,) $B_1X_1B_1 \subseteq B_1$(or $B_2X_2B_2 \subseteq B_2$). Now $BXB = B_1X_1B_1 \subseteq B_1$(or $B_2X_2B_2 \subseteq B_2$)$= X_1B_1B_1$(or $X_2B_2B_2$). (Since X is sub commutative)$= X_1B_1^2$(or $X_2B_2^2$) $\subseteq XB^2$. Therefore $BXB \subseteq XB^2$ and $XB^2 \subseteq B$, i.e., B is a Strong bi-ideal of X. Hence $BXB = XB^2 = B$. Therefore $BXB = B$.
$(v) \Rightarrow (vi)$ Since X is $\bar{S}$-bi-near subtraction semigroup with property (α) and $B = BXB$, for each bi-ideal of X, X is bi-regular. By X is sub commutative, by Lemma 2.11, $E \subseteq C(X)$. Let $x = xax$ and $y = yby$. For any $c \in X_1 \cup X_2$, $ycx = ybycx = ycbyx = ycxby$= $ycxaxby = ycxbaxy = uxy$ where $u = ycxba \in X_1 \cup X_2$. Hence, $yXx \subseteq Xxy$. Also for $z \in X_1 \cup X_2$, $zxy = zxaxy = zxyax$= $zxybyax = ybzxyax = yvx$ where $v = bzxya \in X_1 \cup X_2$. (i.e) $Xxy \subseteq yXx$. From these two facts $Xxy = yXx$ for all $x, y \in X_1 \cup X_2$. (i.e.,) X is strictly $CPXS$.
$(vi) \Rightarrow (i)$ Assume that X is strictly $CPXS$ and bi-regular implies $Xxy = yXx$ for all $x, y \in X_1 \cup X_2$. Consider $y = x$ we acquire $xXx = Xx^2$. Since X is a bi-regular $x \in xXx = Xx^2$. (i.e.,) X is strongly bi-regular and hence Lemma 2.7, every bi-regular bi-near subtraction semigroup X is B-bi regular bi-near subtraction semigroup. Therefore X is a $\overline{S}$-bi-near subtraction semigroup with property (α), X is a two sided bi-near subtraction semigroup. □

4 Conclusion

We have concluded that every B-bi regular bi-near subtraction semigroup is also a weak B-bi-regular bi-near subtraction semigroup of X. But the converse is not true. And also we have discussed few of the properties of weak B-bi-regular bi-near subtraction semigroup.

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