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On generalized fractional inequalities for functions of bounded variation with two variables

Hüseyin Budak[a,*], Kubilay Özçelik[a], Artion Kashuri[b], Muhammad Aamir Ali[c], Zhiyue Zhang[c]

[a]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
[b]Department of Mathematics, Faculty of Technical Science, University Ismail Qemali, Vlora, Albania
[c]School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, China

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Abstract

In this paper, we firstly obtain some identities via generalized fractional integrals which generalize some important fractional integrals such as the Riemann-Liouville fractional integrals, the Hadamard fractional integrals, etc. Then by utilizing these equalities we establish some Ostrowski and Trapezoid type inequalities for functions of bounded variation with two variables. Moreover, we give some inequalities involving Hadamard fractional integrals as special cases of our main results.

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1 Introduction

The study of various types of integral inequalities has been the focus of great attention for well over a century by a number of mathematicians, interested both in pure and applied mathematics. One of the many fundamental mathematical discoveries of A. M. Ostrowski [24] is the following classical integral inequality associated with the differentiable mappings:

Theorem 1.1. Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a differentiable mapping on (κ_1, κ_2) whose derivative $\mathcal{F}' : (\kappa_1, \kappa_2) \to \mathbb{R}$ is bounded on (κ_1, κ_2), i.e. $\|\mathcal{F}'\|_\infty := \sup_{\tau \in (\kappa_1, \kappa_2)} |\mathcal{F}'(\tau)| < \infty$. Then, we have the inequality

$$\left| \mathcal{F}(\varkappa) - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\tau) d\tau \right| \leq \left[\frac{1}{4} + \frac{\left(\varkappa - \frac{\kappa_1 + \kappa_2}{2}\right)^2}{(\kappa_2 - \kappa_1)^2} \right] (\kappa_2 - \kappa_1) \|\mathcal{F}'\|_\infty ,$$

for all $\varkappa \in [\kappa_1, \kappa_2]$.

*Corresponding author
Email addresses: hsyn.budak@gmail.com (Hüseyin Budak), kubilayozcelik@windowslive.com (Kubilay Özçelik), artionkashuri@gmail.com (Artion Kashuri), mahr.muhammad.aamir@gmail.com (Muhammad Aamir Ali), 05298@njnu.edu.cn (Zhiyue Zhang)

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The constant $\frac{1}{4}$ is the best possible.

Ostrowski inequality has applications in quadrature, probability and optimization theory, stochastic, statistics, information and integral operator theory. Until now, a large number of research papers and books have been written on Ostrowski inequalities and their numerous applications.

The remainder of this work is organized as follows: In this Section 2, we present the definitions of fractional integrals and functions of bounded variation. In Section 3, we establish some identities involving generalized fractional integrals of two variables functions. Then, some new Ostrowski and Trapezoid type integral inequalities involving generalized fractional integrals are proved for functions of bounded variation with two variables in Section 4. Finally, in Section 5, we give some ineqalities for Hadamard fractional integrals as special cases of our main results.

2 Preliminaries

In the following, we give the definition of Riemann-Liouville fractional integrals:

Definition 2.1. Let $\mathcal{F} \in L_1[\kappa_1, \kappa_2]$. The Riemann-Liouville fractional integrals $J^{\alpha}_{\kappa_1+}\mathcal{F}$ and $J^{\alpha}_{\kappa_2-}\mathcal{F}$ of order $\alpha > 0$ with $\kappa_1 \geq 0$ are defined by

$$J^{\alpha}_{\kappa_1+}\mathcal{F}(\varkappa) = \frac{1}{\Gamma(\alpha)} \int\limits_{\kappa_1}^{\varkappa} (\varkappa - \tau)^{\alpha-1} \mathcal{F}(\tau) d\tau, \quad \varkappa > \kappa_1$$

and

$$J^{\alpha}_{\kappa_2-}\mathcal{F}(\varkappa) = \frac{1}{\Gamma(\alpha)} \int\limits_{\varkappa}^{\kappa_2} (\tau - \varkappa)^{\alpha-1} \mathcal{F}(\tau) d\tau, \quad \varkappa < \kappa_2$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^{0}_{\kappa_1+}\mathcal{F}(\varkappa) = J^{0}_{\kappa_2-}\mathcal{F}(\varkappa) = \mathcal{F}(\varkappa)$.

Hadamard fractional integrals given by as follows:

Definition 2.2. [21] Let $\mathcal{F} \in L_1([\kappa_1, \kappa_2])$. The Hadamard fractional integrals $\mathbf{H}^{\alpha}_{\kappa_1+}\mathcal{F}$, and $\mathbf{H}^{\alpha}_{\kappa_2-}\mathcal{F}$ of order $\alpha > 0$ with $\kappa_1 \geq 0$ are defined by

$$\mathbf{H}^{\alpha}_{\kappa_1+}\mathcal{F}(\varkappa) := \frac{1}{\Gamma(\alpha)} \int\limits_{\kappa_1}^{\varkappa} \left(\ln \frac{\varkappa}{\tau}\right)^{\alpha-1} \frac{\mathcal{F}(\tau)}{\tau} d\tau, \quad \varkappa > \kappa_1,$$

and

$$\mathbf{H}^{\alpha}_{\kappa_2-}\mathcal{F}(\varkappa) := \frac{1}{\Gamma(\alpha)} \int\limits_{\varkappa}^{\kappa_2} \left(\ln \frac{\tau}{\varkappa}\right)^{\alpha-1} \frac{\mathcal{F}(\tau)}{\tau} d\tau, \quad \varkappa < \kappa_2,$$

respectively.

Definition 2.3. [21] Let $\rho : [\kappa_1, \kappa_2] \to \mathbb{R}$ be an increasing and positive monotone function on $(\kappa_1, \kappa_2]$, having a continuous derivative $\rho'(\varkappa)$ on (κ_1, κ_2). The left-sides $(I^{\alpha}_{\kappa_1^+;\rho}\mathcal{F}(\varkappa))$ and right-sides $(I^{\alpha}_{\kappa_2^-;\rho}\mathcal{F}(\varkappa))$ fractional integral of $\mathcal{F}$ with respect to the function ρ on $[\kappa_1, \kappa_2]$ of order $\alpha < 0$ are defined by

$$I^{\alpha}_{\kappa_1^+;\rho}\mathcal{F}(\varkappa) = \frac{1}{\Gamma(\alpha)} \int\limits_{\kappa_1}^{\varkappa} \frac{\rho'(\tau)\mathcal{F}(\tau)}{[\rho(\varkappa) - \rho(\tau)]^{1-\alpha}} d\tau, \quad \varkappa > \kappa_1$$

and

$$I^{\alpha}_{\kappa_2^-;\rho}\mathcal{F}(\varkappa) = \frac{1}{\Gamma(\alpha)} \int\limits_{\varkappa}^{\kappa_2} \frac{\rho'(\tau)\mathcal{F}(\tau)}{[\rho(\tau) - \rho(\varkappa)]^{1-\alpha}} d\tau, \quad \varkappa < \kappa_2$$

respectively.

Hadamard fractional integrals of a function with two variables can be given as follows:

Definition 2.4. Let $\mathcal{F} \in L_1([\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4])$. The Hadamard fractional integrals $\mathbf{J}^{\alpha,\beta}_{\kappa_1+,\kappa_3+}\mathcal{F}$, $\mathbf{J}^{\alpha,\beta}_{\kappa_1+,\kappa_4-}\mathcal{F}$, $\mathbf{J}^{\alpha,\beta}_{\kappa_2-,\kappa_3+}\mathcal{F}$ and $\mathbf{J}^{\alpha,\beta}_{\kappa_2-,\kappa_4-}\mathcal{F}$ of order $\alpha, \beta > 0$ with $\kappa_1, \kappa_3 \geq 0$ are defined by

$$\mathbf{J}^{\alpha,\beta}_{\kappa_1+,\kappa_3+}\mathcal{F}(\varkappa,\gamma) := \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma} \left(\ln\frac{\varkappa}{\tau}\right)^{\alpha-1} \left(\ln\frac{\gamma}{\varsigma}\right)^{\beta-1} \frac{\mathcal{F}(\tau,\varsigma)}{\tau\varsigma} d\varsigma d\tau, \quad \varkappa > \kappa_1,\ \gamma > \kappa_3,$$

$$\mathbf{J}^{\alpha,\beta}_{\kappa_1+,\kappa_4-}\mathcal{F}(\varkappa,\gamma) := \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4} \left(\ln\frac{\varkappa}{\tau}\right)^{\alpha-1} \left(\ln\frac{\varsigma}{\gamma}\right)^{\beta-1} \frac{\mathcal{F}(\tau,\varsigma)}{\tau\varsigma} d\varsigma d\tau, \quad \varkappa > \kappa_1,\ \gamma < \kappa_4,$$

$$\mathbf{J}^{\alpha,\beta}_{\kappa_2-,\kappa_3+}\mathcal{F}(\varkappa,\gamma) := \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma} \left(\ln\frac{\tau}{\varkappa}\right)^{\alpha-1} \left(\ln\frac{\gamma}{\varsigma}\right)^{\beta-1} \frac{\mathcal{F}(\tau,\varsigma)}{\tau\varsigma} d\varsigma d\tau, \quad \varkappa < \kappa_2,\ \gamma > \kappa_3,$$

and

$$\mathbf{J}^{\alpha,\beta}_{\kappa_2-,\kappa_4-}\mathcal{F}(\varkappa,\gamma) := \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4} \left(\ln\frac{\tau}{\varkappa}\right)^{\alpha-1} \left(\ln\frac{\varsigma}{\gamma}\right)^{\beta-1} \frac{\mathcal{F}(\tau,\varsigma)}{\tau\varsigma} d\varsigma d\tau, \quad \varkappa < \kappa_2,\ \gamma < \kappa_4,$$

respectively.

Now, we give following generalized fractional integral operators:

Definition 2.5. [8] Let $\rho : [\kappa_1, \kappa_2] \to \mathbb{R}$ be an increasing and positive monotone function on $(\kappa_1, \kappa_2]$, having a continuous derivative $\rho'(\varkappa)$ on (κ_1, κ_2) and let $\varpi : [\kappa_3, \kappa_4] \to \mathbb{R}$ be an increasing and positive monotone function on $(\kappa_3, \kappa_4]$, having a continuous derivative $\varpi'(\gamma)$ on (κ_3, κ_4). Let $\mathcal{F} \in L_1([\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4])$. The generalized fractional integral operators for functions of two variables are defined by

$$\mathcal{J}^{\alpha,\beta}_{\kappa_1+,\kappa_3+;\rho,\varpi}\mathcal{F}(\varkappa,\gamma) := \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma} \frac{\rho'(\tau)}{[\rho(\varkappa)-\rho(\tau)]^{1-\alpha}} \frac{\varpi'(\varsigma)}{[\varpi(\gamma)-\varpi(\varsigma)]^{1-\beta}} \mathcal{F}(\tau,\varsigma) d\varsigma d\tau, \quad \varkappa > \kappa_1,\ \gamma > \kappa_3,$$

$$\mathcal{J}^{\alpha,\beta}_{\kappa_1+,\kappa_4-;\rho,\varpi}\mathcal{F}(\varkappa,\gamma) := \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4} \frac{\rho'(\tau)}{[\rho(\varkappa)-\rho(\tau)]^{1-\alpha}} \frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\gamma)]^{1-\beta}} \mathcal{F}(\tau,\varsigma) d\varsigma d\tau, \quad \varkappa > \kappa_1,\ \gamma < \kappa_4,$$

$$\mathcal{J}^{\alpha,\beta}_{\kappa_2-,\kappa_3+;\rho,\varpi}\mathcal{F}(\varkappa,\gamma) := \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma} \frac{\rho'(\tau)}{[\rho(\tau)-\rho(\varkappa)]^{1-\alpha}} \frac{\varpi'(\varsigma)}{[\varpi(\gamma)-\varpi(\varsigma)]^{1-\beta}} \mathcal{F}(\tau,\varsigma) d\varsigma d\tau, \quad \varkappa < \kappa_2,\ \gamma > \kappa_3,$$

and

$$\mathcal{J}^{\alpha,\beta}_{\kappa_2-,\kappa_4-;\rho,\varpi}\mathcal{F}(\varkappa,\gamma) := \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4} \frac{\rho'(\tau)}{[\rho(\tau)-\rho(\varkappa)]^{1-\alpha}} \frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\gamma)]^{1-\beta}} \mathcal{F}(\tau,\varsigma) d\varsigma d\tau, \quad \varkappa < \kappa_2,\ \gamma < \kappa_4.$$

More details for Riemann-Liouville fractional integrals, one can consult ([19], [21], [23], [25]).

Moreover, one can find some recent Hermite-Hadamard inequalities for function of one and two variables via Riemann-Liouville fractional integrals in ([1], [8], [18], [20], [22], [26]-[31]).

Functions of bounded variation of one variable are of great interest and usefulness because of their valuable properties, such as particularly with respect to additivity, decomposability into monotone functions, continuity, differentiability, measurability, integrability, and so on, have been much studied. There are many of papers on inequalities for functions of bounded variation of one variable, some of them please see ([2]-[4], [9], [14]-[16]). Moreover, Dragomir obtained some fractional inequalities involving functions of bounded variation ([11]-[13])

Functions of bounded variation with two variables are defined as follows:

Definition 2.6. [10] Assume that $\mathcal{F}(\varkappa,\gamma)$ is defined over the rectangle $Q=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]$. Let P be a partition of Q with

$$P:\kappa_1=\varkappa_0<\varkappa_1<...<\varkappa_{\mathcal{N}}=\kappa_2,\ \text{and}\ \kappa_3=\gamma_0<\gamma_1<...<\gamma_{\mathcal{M}}=\kappa_4;$$

and for all i,j let

$$\Delta_{11}\rho(\varkappa_i,\gamma_j)=\rho(\varkappa_{i-1},\gamma_{j-1})-\rho(\varkappa_{i-1},\gamma_j)-\rho(\varkappa_i,\gamma_{j-1})+\rho(\varkappa_i,\gamma_j).$$

The function $\mathcal{F}(\varkappa,\gamma)$ is said to be of bounded variation if the sum

$$\sum_{i=0}^{\mathcal{N}-1}\sum_{j=0}^{\mathcal{M}-1}|\Delta_{11}\mathcal{F}(\varkappa_i,\gamma_j)|$$

is bounded for all nets.

Therefore, one can define the concept of total variation of a function of variables, as follows:

Let $\mathcal{F}$ be of bounded variation on $Q=[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]$, and let $\sum(P)$ denote the sum $\sum\limits_{i=1}^{\mathcal{N}}\sum\limits_{j=1}^{\mathcal{M}}|\Delta_{11}\mathcal{F}(\varkappa_i,\gamma_j)|$ corresponding to the partition P of Q. The number

$$\bigvee_Q(\mathcal{F}):=\bigvee_{\kappa_1}^{\kappa_2}\bigvee_{\kappa_3}^{\kappa_4}(\mathcal{F}):=\sup\left\{\sum(P):P\in P(Q)\right\},$$

is called the total variation of $\mathcal{F}$ on Q.

There are also some paper on inequalities for functions of bounded variation with two variables ([5]-[7]). However there is a few papers fractional integral inequalities for functions of bounded variation with two variables. The aim of this paper establish some fractional Ostrowski, Midpoint and Trapezoid type inequalities for functions of bounded variation with two variables.

The aim of this study is to establish Hermite-Hadamard type integral inequalities for co-ordinated convex function involving generalized fractional integrals. The results presented in this paper provide extensions of those given in earlier works.

3 Some Identities for Generalized Fractional Integrals

Firstly, we define the following functions which will be used frequently:

$$\mathcal{M}_\rho^\alpha(\kappa_1,\kappa_2;\varkappa):=\frac{[\rho(\varkappa)-\rho(\kappa_1)]^\alpha+[\rho(\kappa_2)-\rho(\varkappa)]^\alpha}{\Gamma(\alpha+1)},$$

and

$$\mathcal{N}_\varpi^\beta(\kappa_3,\kappa_4;\gamma):=\frac{[\varpi(\gamma)-\varpi(\kappa_3)]^\beta+[\varpi(\kappa_4)-\varpi(\gamma)]^\beta}{\Gamma(\beta+1)}$$

for $(\varkappa,\gamma)\in[\kappa_1,\kappa_2]\times[\kappa_3,\kappa_4]$. We also denote $\mathcal{M}_\rho^\alpha(\kappa_1,\kappa_2)$ and $\mathcal{N}_\varpi^\beta(\kappa_3,\kappa_4)$ by

$$\mathcal{M}_\rho^\alpha(\kappa_1,\kappa_2):=\mathcal{M}_\rho^\alpha(\kappa_1,\kappa_2;\kappa_1)=\mathcal{M}_\rho^\alpha(\kappa_1,\kappa_2;\kappa_2)=\frac{[\rho(\kappa_2)-\rho(\kappa_1)]^\alpha}{\Gamma(\alpha+1)},$$

and

$$\mathcal{N}_\varpi^\beta(\kappa_3,\kappa_4):=\mathcal{N}_\varpi^\beta(\kappa_3,\kappa_4;\kappa_3)=\mathcal{N}_\varpi^\beta(\kappa_3,\kappa_4;\kappa_4)=\frac{[\varpi(\kappa_4)-\varpi(\kappa_3)]^\beta}{\Gamma(\beta+1)}.$$

In addition, by choosing $\rho(\tau)=\ln\tau$, $\tau\in[\kappa_1,\kappa_2]$ and $\varpi(\varsigma)=\ln\varsigma$, $\varsigma\in[\kappa_3,\kappa_4]$, we have the following representations

$$\mathcal{M}_{\ln}^\alpha(\kappa_1,\kappa_2;\varkappa):=\frac{\left[\ln\frac{\varkappa}{\kappa_1}\right]^\alpha+\left[\ln\frac{\kappa_2}{\varkappa}\right]^\alpha}{\Gamma(\alpha+1)}\ \text{and}\ \mathcal{N}_{\ln}^\beta(\kappa_3,\kappa_4;\gamma):=\frac{\left[\ln\frac{\gamma}{\kappa_3}\right]^\beta+\left[\ln\frac{\kappa_4}{\gamma}\right]^\beta}{\Gamma(\beta+1)}$$

and

$$\mathcal{M}_{\ln}^\alpha(\kappa_1,\kappa_2):=\frac{\left[\ln\frac{\kappa_2}{\kappa_1}\right]^\alpha}{\Gamma(\alpha+1)}\ \text{and}\ \mathcal{N}_{\ln}^\beta(\kappa_3,\kappa_4):=\frac{\left[\ln\frac{\kappa_4}{\kappa_3}\right]^\beta}{\Gamma(\beta+1)}.$$

Now we prove the following equalities:

Lemma 3.1. Let $\rho : [\kappa_1, \kappa_2] \to \mathbb{R}$ be an increasing and positive monotone function on $(\kappa_1, \kappa_2]$, having a continuous derivative $\rho'(\varkappa)$ on (κ_1, κ_2) and let $\varpi : [\kappa_3, \kappa_4] \to \mathbb{R}$ be an increasing and positive monotone function on $(\kappa_3, \kappa_4]$, having a continuous derivative $\varpi'(\gamma)$ on (κ_3, κ_4). If $\mathcal{F} : \Delta \to \mathbb{R}$ is integrable on Δ, then for $\alpha, \beta > 0$ we have the following equality

$$\begin{aligned}
&\mathcal{F}(\varkappa,\gamma) - \frac{1}{\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\left[\mathcal{J}_{\kappa_3+;\varpi}^{\beta}\mathcal{F}(\varkappa,\gamma) + \mathcal{J}_{\kappa_4-;\varpi}^{\beta}\mathcal{F}(\varkappa,\gamma)\right] \\
&-\frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)}\left[\mathcal{J}_{\kappa_1+;\rho}^{\alpha}\mathcal{F}(\varkappa,\gamma) + \mathcal{J}_{\kappa_2-;\rho}^{\alpha}\mathcal{F}(\varkappa,\gamma)\right] \\
&+\frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)} \\
&\times\left[\mathcal{J}_{\kappa_1+,\kappa_3+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma) + \mathcal{J}_{\kappa_1+,\kappa_4-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma) + \mathcal{J}_{\kappa_2-,\kappa_3+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma) + \mathcal{J}_{\kappa_2-,\kappa_4-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma)\right] \\
=\ &\frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\sum_{k=1}^{4} I_k
\end{aligned} \tag{3.1}$$

where

$$I_1 = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}\frac{\rho'(\tau)}{[\rho(\varkappa)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\gamma)-\varpi(\varsigma)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right]d\varsigma d\tau,$$

$$I_2 = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\varkappa)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\gamma)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right]d\varsigma d\tau,$$

$$I_3 = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\varkappa)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\gamma)-\varpi(\varsigma)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right]d\varsigma d\tau$$

and

$$I_4 = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\varkappa)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\gamma)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right]d\varsigma d\tau$$

for all $(\varkappa, \gamma) \in \Delta$.

Proof . We show that

$$\begin{aligned}
I_1 &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \\
&\quad\times\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}\frac{\rho'(\tau)}{[\rho(\varkappa)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\gamma)-\varpi(\varsigma)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right]d\varsigma d\tau \\
&= \mathcal{J}_{\kappa_1+,\kappa_3+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma) - \frac{[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{J}_{\kappa_1+;\rho}^{\alpha}\mathcal{F}(\varkappa,\gamma) - \frac{[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}_{\kappa_3+;\varpi}^{\beta}\mathcal{F}(\varkappa,\gamma) \\
&\quad+\frac{[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\frac{[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{F}(\varkappa,\gamma).
\end{aligned} \tag{3.2}$$

Similarly, we get

$$\begin{aligned}
&I_2 \\
=\ &\mathcal{J}_{\kappa_1+,\kappa_4-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma) - \frac{[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}}{\Gamma(\beta+1)}\mathcal{J}_{\kappa_1+;\rho}^{\alpha}\mathcal{F}(\varkappa,\gamma) - \frac{[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}_{\kappa_4-;\varpi}^{\beta}\mathcal{F}(\varkappa,\gamma)
\end{aligned} \tag{3.3}$$

$$+\frac{[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\frac{[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}}{\Gamma(\beta+1)}\mathcal{F}(\varkappa,\gamma),$$

$$\begin{aligned} & I_3 \\ = \ & \mathcal{J}^{\alpha,\beta}_{\kappa_2-,\kappa_3+;\rho,\varpi}\mathcal{F}(\varkappa,\gamma)-\frac{[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{J}^{\alpha}_{\kappa_2-;\rho}\mathcal{F}(\varkappa,\gamma)-\frac{[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\beta}_{\kappa_3+;\varpi}\mathcal{F}(\varkappa,\gamma) \\ & +\frac{[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}}{\Gamma(\alpha+1)}\frac{[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{F}(\varkappa,\gamma)\end{aligned} \tag{3.4}$$

and

$$\begin{aligned} & I_4 \\ = \ & \mathcal{J}^{\alpha,\beta}_{\kappa_2-,\kappa_4-;\rho,\varpi}\mathcal{F}(\varkappa,\gamma)-\frac{[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}}{\Gamma(\beta+1)}\mathcal{J}^{\alpha}_{\kappa_2-;\rho}\mathcal{F}(\varkappa,\gamma)-\frac{[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\beta}_{\kappa_4-;\varpi}\mathcal{F}(\varkappa,\gamma) \\ & +\frac{[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}}{\Gamma(\alpha+1)}\frac{[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}}{\Gamma(\beta+1)}\mathcal{F}(\varkappa,\gamma).\end{aligned} \tag{3.5}$$

If we add the equalities (3.2)-(3.5) and then divide the result by $\mathcal{M}^{\alpha}_{\rho}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}^{\beta}_{\varpi}(\kappa_3,\kappa_4;\gamma)$, then ve obtain the required identity (3.1). □

Lemma 3.2. Suppose that the assumptions of Lemma 3.1. Then we have the following equality

$$\begin{aligned} & \mathcal{F}(\varkappa,\gamma)-\frac{1}{\mathcal{N}^{\beta}_{\varpi}(\kappa_3,\kappa_4;\gamma)}\left[\mathcal{J}^{\beta}_{\gamma+;\varpi}\mathcal{F}(\varkappa,\kappa_4)+\mathcal{J}^{\beta}_{\gamma-;\varpi}\mathcal{F}(\varkappa,\kappa_3)\right] \\ & -\frac{1}{\mathcal{M}^{\alpha}_{\rho}(\kappa_1,\kappa_2;\varkappa)}\left[\mathcal{J}^{\alpha}_{\varkappa+;\rho}\mathcal{F}(\kappa_2,\gamma)+\mathcal{J}^{\alpha}_{\varkappa-;\rho}\mathcal{F}(\kappa_1,\gamma)\right] \\ & +\frac{1}{\mathcal{M}^{\alpha}_{\rho}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}^{\beta}_{\varpi}(\kappa_3,\kappa_4;\gamma)} \\ & \times\left[\mathcal{J}^{\alpha,\beta}_{\varkappa+,\gamma+;\rho,\varpi}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}^{\alpha,\beta}_{\varkappa+,\gamma-;\rho,\varpi}\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{J}^{\alpha,\beta}_{\varkappa-,\gamma+;\rho,\varpi}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}^{\alpha,\beta}_{\varkappa-,\gamma-;\rho,\varpi}\mathcal{F}(\kappa_1,\kappa_3)\right] \\ = \ & \frac{1}{\mathcal{M}^{\alpha}_{\rho}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}^{\beta}_{\varpi}(\kappa_3,\kappa_4;\gamma)}\sum_{k=5}^{8}I_k\end{aligned} \tag{3.6}$$

where

$$I_5=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right]d\varsigma d\tau,$$

$$I_6=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varkappa}^{\kappa_2}\int_{\kappa_1}^{\gamma}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right]d\varsigma d\tau,$$

$$I_7=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right]d\varsigma d\tau$$

and

$$I_8=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right]d\varsigma d\tau$$

for all $(\varkappa,\gamma)\in\Delta$.

Proof . By the Definition 2.5, we have

$$
\begin{aligned}
I_5 &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\\
&\quad\times[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)]\,d\varsigma d\tau\\
&= \mathcal{J}^{\alpha,\beta}_{\varkappa+,\gamma+;\rho,\varpi}\mathcal{F}(\kappa_2,\kappa_4)-\frac{[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}}{\Gamma(\beta+1)}\mathcal{J}^{\alpha}_{\varkappa+;\rho}\mathcal{F}(\kappa_2,\gamma)-\frac{[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\beta}_{\gamma+;\varpi}\mathcal{F}(\varkappa,\kappa_4)\\
&\quad+\frac{[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}}{\Gamma(\alpha+1)}\frac{[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}}{\Gamma(\beta+1)}\mathcal{F}(\varkappa,\gamma),
\end{aligned}
$$

and similarly

$$
\begin{aligned}
I_6 &= \mathcal{J}^{\alpha,\beta}_{\varkappa+,\gamma-;\rho,\varpi}\mathcal{F}(\kappa_2,\kappa_3)-\frac{[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{J}^{\alpha}_{\varkappa+;\rho}\mathcal{F}(\kappa_2,\gamma)-\frac{[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\beta}_{\gamma-;\varpi}\mathcal{F}(\varkappa,\kappa_3)\\
&\quad+\frac{[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}}{\Gamma(\alpha+1)}\frac{[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{F}(\varkappa,\gamma),
\end{aligned}
$$

$$
\begin{aligned}
I_7 &= \mathcal{J}^{\alpha,\beta}_{\varkappa-,\gamma+;\rho,\varpi}\mathcal{F}(\kappa_1,\kappa_4)-\frac{[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}}{\Gamma(\beta+1)}\mathcal{J}^{\alpha}_{\varkappa-;\rho}\mathcal{F}(\kappa_1,\gamma)-\frac{[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\beta}_{\gamma+;\varpi}\mathcal{F}(\varkappa,\kappa_4)\\
&\quad+\frac{[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\frac{[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}}{\Gamma(\beta+1)}\mathcal{F}(\varkappa,\gamma)
\end{aligned}
$$

and

$$
\begin{aligned}
I_8 &= \mathcal{J}^{\alpha,\beta}_{\varkappa-,\gamma-;\rho,\varpi}\mathcal{F}(\kappa_1,\kappa_3)-\frac{[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{J}^{\alpha}_{\varkappa-;\rho}\mathcal{F}(\kappa_1,\gamma)-\frac{[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\beta}_{\gamma-;\varpi}\mathcal{F}(\varkappa,\kappa_3)\\
&\quad+\frac{[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\frac{[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{F}(\varkappa,\gamma).
\end{aligned}
$$

This completes the proof. □

Lemma 3.3. Suppose that the assumptions of Lemma 3.1. Then we have the following equality

$$
\begin{aligned}
&\frac{\mathcal{F}(\kappa_1,\kappa_3)+\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{F}(\kappa_2,\kappa_4)}{4}\mathcal{M}^{\alpha}_{\rho}(\kappa_1,\kappa_2)\mathcal{N}^{\beta}_{\varpi}(\kappa_3,\kappa_4) \qquad (3.7)\\
&-\frac{\mathcal{N}^{\beta}_{\varpi}(\kappa_3,\kappa_4)}{4}\left[\mathcal{J}^{\alpha}_{\kappa_1+;\rho}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}^{\alpha}_{\kappa_1+;\rho}\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{J}^{\alpha}_{\kappa_2-;\rho}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}^{\alpha}_{\kappa_2-;\rho}\mathcal{F}(\kappa_1,\kappa_3)\right]\\
&-\frac{\mathcal{M}^{\alpha}_{\rho}(\kappa_1,\kappa_2)}{4}\left[\mathcal{J}^{\beta}_{\kappa_3+;\varpi}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}^{\beta}_{\kappa_4-;\varpi}\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{J}^{\beta}_{\kappa_3+;\varpi}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}^{\beta}_{\kappa_4-;\varpi}\mathcal{F}(\kappa_1,\kappa_3)\right]\\
&+\frac{1}{4}\Big[\mathcal{J}^{\alpha,\beta}_{\kappa_1+,\kappa_3+;\rho,\varpi}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}^{\alpha,\beta}_{\kappa_1+,\kappa_4-;\rho,\varpi}\mathcal{F}(\kappa_2,\kappa_3)\\
&+\mathcal{J}^{\alpha,\beta}_{\kappa_2-,\kappa_3+;\rho,\varpi}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}^{\alpha,\beta}_{\kappa_2-,\kappa_4-;\rho,\varpi}\mathcal{F}(\kappa_1,\kappa_3)\Big]\\
=\;&\frac{1}{4}\sum_{k=9}^{12}I_k
\end{aligned}
$$

where

$$
I_9=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_4)-\mathcal{F}(\kappa_2,\varsigma)+\mathcal{F}(\kappa_2,\kappa_4)\right]d\varsigma d\tau,
$$

$$I_{10}=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_3)-\mathcal{F}(\kappa_2,\varsigma)+\mathcal{F}(\kappa_2,\kappa_3)\right]d\varsigma d\tau$$

$$I_{11}=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_4)-\mathcal{F}(\kappa_1,\varsigma)+\mathcal{F}(\kappa_1,\kappa_4)\right]d\varsigma d\tau$$

and

$$I_{12}=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_3)-\mathcal{F}(\kappa_1,\varsigma)+\mathcal{F}(\kappa_1,\kappa_3)\right]d\varsigma d\tau$$

for all $(\varkappa,\gamma)\in\Delta$.

Proof . If we choose $(\varkappa,\gamma)=(\kappa_2,\kappa_4)$ in (3.2), then we have

$$\begin{aligned}I_9 &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \\ &\quad\times\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_4)-\mathcal{F}(\kappa_2,\varsigma)+\mathcal{F}(\kappa_2,\kappa_4)\right]d\varsigma d\tau \\ &= \mathcal{J}^{\alpha,\beta}_{\kappa_1+,\kappa_3+;\rho,\varpi}\mathcal{F}(\kappa_2,\kappa_4)-\frac{[\varpi(\kappa_4)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{J}^{\alpha}_{\kappa_1+;\rho}\mathcal{F}(\kappa_2,\kappa_4)-\frac{[\rho(\kappa_2)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\beta}_{\kappa_3+;\varpi}\mathcal{F}(\kappa_2,\kappa_4) \\ &\quad+\frac{[\rho(\kappa_2)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\frac{[\varpi(\kappa_4)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{F}(\kappa_2,\kappa_4).\end{aligned}\tag{3.8}$$

Similarly, we get

$$\begin{aligned}I_{10} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \\ &\quad\times\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_3)-\mathcal{F}(\kappa_2,\varsigma)+\mathcal{F}(\kappa_2,\kappa_3)\right]d\varsigma d\tau \\ &= \mathcal{J}^{\alpha,\beta}_{\kappa_1+,\kappa_4-;\rho,\varpi}\mathcal{F}(\kappa_2,\kappa_3)-\frac{[\varpi(\kappa_4)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{J}^{\alpha}_{\kappa_1+;\rho}\mathcal{F}(\kappa_2,\kappa_3)-\frac{[\rho(\kappa_2)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\beta}_{\kappa_4-;\varpi}\mathcal{F}(\kappa_2,\kappa_3) \\ &\quad+\frac{[\rho(\kappa_2)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\frac{[\varpi(\kappa_4)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{F}(\kappa_2,\kappa_3)\end{aligned}\tag{3.9}$$

for $(\varkappa,\gamma)=(\kappa_2,\kappa_3)$ in (3.3),

$$\begin{aligned}I_{11} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \\ &\quad\times\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\left[\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_4)-\mathcal{F}(\kappa_1,\varsigma)+\mathcal{F}(\kappa_1,\kappa_4)\right]d\varsigma d\tau \\ &= \mathcal{J}^{\alpha,\beta}_{\kappa_2-,\kappa_3+;\rho,\varpi}\mathcal{F}(\kappa_1,\kappa_4)-\frac{[\varpi(\kappa_4)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{J}^{\alpha}_{\kappa_2-;\rho}\mathcal{F}(\kappa_1,\kappa_4)-\frac{[\rho(\kappa_2)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\mathcal{J}^{\beta}_{\kappa_3+;\varpi}\mathcal{F}(\kappa_1,\kappa_4) \\ &\quad+\frac{[\rho(\kappa_2)-\rho(\kappa_1)]^{\alpha}}{\Gamma(\alpha+1)}\frac{[\varpi(\kappa_4)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\beta+1)}\mathcal{F}(\kappa_1,\kappa_4)\end{aligned}\tag{3.10}$$

for $(\varkappa,\gamma)=(\kappa_1,\kappa_4)$ in (3.4), and

$$I_{12} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\tag{3.11}$$

$$\times \int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} \frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}} \frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}} [\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_3)-\mathcal{F}(\kappa_1,\varsigma)+\mathcal{F}(\kappa_1,\kappa_3)]\, d\varsigma d\tau$$

$$= \mathcal{J}^{\alpha,\beta}_{\kappa_2-,\kappa_4-;\rho,\varpi}\mathcal{F}(\kappa_1,\kappa_3) - \frac{[\varpi(\kappa_4)-\varpi(\kappa_3)]^\beta}{\Gamma(\beta+1)}\mathcal{J}^{\alpha}_{\kappa_2-;\rho}\mathcal{F}(\kappa_1,\kappa_3) - \frac{[\rho(\kappa_2)-\rho(\kappa_1)]^\alpha}{\Gamma(\alpha+1)}\mathcal{J}^{\beta}_{\kappa_4-;\varpi}\mathcal{F}(\kappa_1,\kappa_3)$$

$$+\frac{[\rho(\kappa_2)-\rho(\kappa_1)]^\alpha}{\Gamma(\alpha+1)}\frac{[\varpi(\kappa_4)-\varpi(\kappa_3)]^\beta}{\Gamma(\beta+1)}\mathcal{F}(\kappa_1,\kappa_3)$$

for $(\varkappa,\gamma)=(\kappa_1,\kappa_3)$ in (3.5).

If we add the identities (3.8)-(3.11) and if we divide the result equality by 4, then we obtain the desired identity (3.7). □

4 Some Ostrowski Type Inequalities for Generalized Fractional Integrals

In this section, Ostrowski type inequalities involving generalized fractional integrals are obtained for functions of bounded variation with two variables.

Theorem 4.1. Let $\rho:[\kappa_1,\kappa_2]\to\mathbb{R}$ be an increasing and positive monotone function on $(\kappa_1,\kappa_2]$, having a continuous derivative $\rho'(\varkappa)$ on (κ_1,κ_2) and let $\varpi:[\kappa_3,\kappa_4]\to\mathbb{R}$ be an increasing and positive monotone function on $(\kappa_3,\kappa_4]$, having a continuous derivative $\varpi'(\gamma)$ on (κ_3,κ_4). If $\mathcal{F}:\Delta\to\mathbb{R}$ is of bounded variation on Δ, then we have the following inequalities

$$\begin{aligned}
&\Bigg|\mathcal{F}(\varkappa,\gamma) - \frac{1}{\mathcal{N}^\beta_\varpi(\kappa_3,\kappa_4;\gamma)}\left[\mathcal{J}^\beta_{\kappa_3+;\varpi}\mathcal{F}(\varkappa,\gamma)+\mathcal{J}^\beta_{\kappa_4-;\varpi}\mathcal{F}(\varkappa,\gamma)\right]\\
&-\frac{1}{\mathcal{M}^\alpha_\rho(\kappa_1,\kappa_2;\varkappa)}\left[\mathcal{J}^\alpha_{\kappa_1+;\rho}\mathcal{F}(\varkappa,\gamma)+\mathcal{J}^\alpha_{\kappa_2-;\rho}\mathcal{F}(\varkappa,\gamma)\right]\\
&+\frac{1}{\mathcal{M}^\alpha_\rho(\kappa_1,\kappa_2;\varkappa)\mathcal{N}^\beta_\varpi(\kappa_3,\kappa_4;\gamma)}\\
&\times\left[\mathcal{J}^{\alpha,\beta}_{\kappa_1+,\kappa_3+;\rho,\varpi}\mathcal{F}(\varkappa,\gamma)+\mathcal{J}^{\alpha,\beta}_{\kappa_1+,\kappa_4-;\rho,\varpi}\mathcal{F}(\varkappa,\gamma)+\mathcal{J}^{\alpha,\beta}_{\kappa_2-,\kappa_3+;\rho,\varpi}\mathcal{F}(\varkappa,\gamma)+\mathcal{J}^{\alpha,\beta}_{\kappa_2-,\kappa_4-;\rho,\varpi}\mathcal{F}(\varkappa,\gamma)\right]\Bigg|\\
\leq\ &\frac{1}{\mathcal{M}^\alpha_\rho(\kappa_1,\kappa_2;\varkappa)\mathcal{N}^\beta_\varpi(\kappa_3,\kappa_4;\gamma)}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\\
&\times\Bigg[\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}\frac{\rho'(\tau)}{[\rho(\varkappa)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\gamma)-\varpi(\varsigma)]^{1-\beta}}\bigvee_{\tau}^{\varkappa}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\,d\varsigma d\tau\\
&+\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\varkappa)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\gamma)]^{1-\beta}}\bigvee_{\tau}^{\varkappa}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\,d\varsigma d\tau\\
&+\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\varkappa)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\gamma)-\varpi(\varsigma)]^{1-\beta}}\bigvee_{\varkappa}^{\tau}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\,d\varsigma d\tau\\
&+\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\varkappa)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\gamma)]^{1-\beta}}\bigvee_{\varkappa}^{\tau}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\,d\varsigma d\tau\Bigg]\\
\leq\ &\frac{1}{\mathcal{M}^\alpha_\rho(\kappa_1,\kappa_2;\varkappa)\mathcal{N}^\beta_\varpi(\kappa_3,\kappa_4;\gamma)}\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg[[\rho(\varkappa)-\rho(\kappa_1)]^\alpha[\varpi(\gamma)-\varpi(\kappa_3)]^\beta\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F})\\
&+[\rho(\varkappa)-\rho(\kappa_1)]^\alpha[\varpi(\kappa_4)-\varpi(\gamma)]^\beta\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\gamma}^{\kappa_4}(\mathcal{F})+[\rho(\kappa_2)-\rho(\varkappa)]^\alpha[\varpi(\gamma)-\varpi(\kappa_3)]^\beta\bigvee_{\varkappa}^{\kappa_2}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F})
\end{aligned} \tag{4.1}$$

$$
\begin{aligned}
&+\left[\rho(\kappa_2)-\rho(\varkappa)\right]^{\alpha}\left[\varpi(\kappa_4)-\varpi(\gamma)\right]^{\beta}\bigvee_{\varkappa}^{\kappa_2}\bigvee_{\gamma}^{\kappa_4}(\mathcal{F})\Bigg]\\
\leq\quad & \frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left[\frac{1}{2}\left(\rho(\kappa_2)-\rho(\kappa_1)\right)+\left|\rho(\varkappa)-\frac{\rho(\kappa_1)+\rho(\kappa_2)}{2}\right|\right]^{\alpha}\\
&\times\left[\frac{1}{2}\left(\varpi(\kappa_4)-\varpi(\kappa_3)\right)+\left|\varpi(\gamma)-\frac{\varpi(\kappa_3)+\varpi(\kappa_4)}{2}\right|\right]^{\beta}\bigvee_{\kappa_1}^{\kappa_2}\bigvee_{\kappa_3}^{\kappa_2}(\mathcal{F})
\end{aligned}
$$

for all $(\varkappa,\gamma)\in\Delta$.

Proof . By the Lemma 3.1, we get

$$
\begin{aligned}
&\Bigg|\mathcal{F}(\varkappa,\gamma)-\frac{1}{\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\left[\mathcal{J}_{\kappa_3+;\varpi}^{\beta}\mathcal{F}(\varkappa,\gamma)+\mathcal{J}_{\kappa_4-;\varpi}^{\beta}\mathcal{F}(\varkappa,\gamma)\right]\\
&-\frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)}\left[\mathcal{J}_{\kappa_1+;\rho}^{\alpha}\mathcal{F}(\varkappa,\gamma)+\mathcal{J}_{\kappa_2-;\rho}^{\alpha}\mathcal{F}(\varkappa,\gamma)\right]\\
&+\frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\\
&\times\left[\mathcal{J}_{\kappa_1+,\kappa_3+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma)+\mathcal{J}_{\kappa_1+,\kappa_4-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma)+\mathcal{J}_{\kappa_2-,\kappa_3+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma)+\mathcal{J}_{\kappa_2-,\kappa_4-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma)\right]\Bigg|\\
\leq\quad & \frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\\
\times\quad & \Bigg[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}\frac{\rho'(\tau)}{\left[\rho(\varkappa)-\rho(\tau)\right]^{1-\alpha}}\frac{\varpi'(\varsigma)}{\left[\varpi(\gamma)-\varpi(\varsigma)\right]^{1-\beta}}\\
&\times\left|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right|d\varsigma d\tau\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{\left[\rho(\varkappa)-\rho(\tau)\right]^{1-\alpha}}\frac{\varpi'(\varsigma)}{\left[\varpi(\varsigma)-\varpi(\gamma)\right]^{1-\beta}}\\
&\times\left|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right|d\varsigma d\tau\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}\frac{\rho'(\tau)}{\left[\rho(\tau)-\rho(\varkappa)\right]^{1-\alpha}}\frac{\varpi'(\varsigma)}{\left[\varpi(\gamma)-\varpi(\varsigma)\right]^{1-\beta}}\\
&\times\left|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right|d\varsigma d\tau\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{\left[\rho(\tau)-\rho(\varkappa)\right]^{1-\alpha}}\frac{\varpi'(\varsigma)}{\left[\varpi(\varsigma)-\varpi(\gamma)\right]^{1-\beta}}\\
&\times\left|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right|d\varsigma d\tau\Bigg]\\
:\quad & =\kappa_1(\varkappa,\gamma)
\end{aligned}
\tag{4.2}
$$

Since $\mathcal{F}$ is of bounded variation on Δ, we have the following inequalities

$$
\left|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right|\leq\bigvee_{\tau}^{\varkappa}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\leq\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F})\ \text{ for }\ (\tau,\varsigma)\in[\kappa_1,\varkappa)\times[\kappa_3,\gamma),\tag{4.3}
$$

$$
\left|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right|\leq\bigvee_{\tau}^{\varkappa}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\leq\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\gamma}^{\kappa_4}(\mathcal{F})\ \text{ for }\ (\tau,\varsigma)\in[\kappa_1,\varkappa)\times[\gamma,\kappa_4],\tag{4.4}
$$

$$
\left|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)\right|\leq\bigvee_{\varkappa}^{\tau}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\leq\bigvee_{\varkappa}^{\kappa_2}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F})\ \text{ for }\ (\tau,\varsigma)\in[\varkappa,\kappa_2]\times[\kappa_3,\gamma)\tag{4.5}
$$

and

$$|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\gamma)-\mathcal{F}(\varkappa,\varsigma)+\mathcal{F}(\varkappa,\gamma)|\leq\bigvee_{\varkappa}^{\tau}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\leq\bigvee_{\varkappa}^{\kappa_2}\bigvee_{\gamma}^{\kappa_4}(\mathcal{F})\ \text{ for }\ (\tau,\varsigma)\in[\varkappa,\kappa_2]\times[\gamma,\kappa_4]. \tag{4.6}$$

By substituting the inequalities (4.3)-(4.6) in (4.2), then we obtain

$$\begin{aligned}
&\kappa_1(\varkappa,\gamma)\\
\leq\;& \frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\\
&\times\Bigg[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}\frac{\rho'(\tau)}{[\rho(\varkappa)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\gamma)-\varpi(\varsigma)]^{1-\beta}}\bigvee_{\tau}^{\varkappa}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\,d\varsigma d\tau\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\varkappa)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\gamma)]^{1-\beta}}\bigvee_{\tau}^{\varkappa}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\,d\varsigma d\tau\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\varkappa)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\gamma)-\varpi(\varsigma)]^{1-\beta}}\bigvee_{\varkappa}^{\tau}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\,d\varsigma d\tau\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\varkappa)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\gamma)]^{1-\beta}}\bigvee_{\varkappa}^{\tau}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\,d\varsigma d\tau\Bigg]\\
\leq\;& \frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\Bigg[\frac{[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F})\\
&+\frac{[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\gamma}^{\kappa_4}(\mathcal{F})+\frac{[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{\varkappa}^{\kappa_2}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F})\\
&+\frac{[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{\varkappa}^{\kappa_2}\bigvee_{\gamma}^{\kappa_4}(\mathcal{F})\Bigg]\\
:\;& \kappa_2(\varkappa,\gamma)
\end{aligned}$$

This proves the first and second inequality in (4.1).

The proof of last inequality in (4.1) is obvious from the facts that

$$\max\{\kappa_1\kappa_3,\kappa_1\kappa_4,\kappa_2\kappa_3,\kappa_2\kappa_4\} = \max\{\kappa_1,\kappa_2\}\max\{\kappa_3,\kappa_4\}, \tag{4.7}$$

$$\max\left\{\kappa_1^{\mathcal{N}},\kappa_2^{\mathcal{N}}\right\} = (\max\{\kappa_1,\kappa_2\})^{\mathcal{N}}=\left(\frac{\kappa_1+\kappa_2+|\kappa_1+\kappa_2|}{2}\right)^{\mathcal{N}}$$

for $\kappa_1,\kappa_2,\kappa_3,\kappa_4,\mathcal{N}>0$.

This completes completely the proof of theorem. □

Theorem 4.2. Suppose that the assumptions of Theorem 4.1, then we have the following inequalities

$$\begin{aligned}
&\Bigg|\mathcal{F}(\varkappa,\gamma)-\frac{1}{\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\left[\mathcal{J}_{\gamma+;\varpi}^{\beta}\mathcal{F}(\varkappa,\kappa_4)+\mathcal{J}_{\gamma-;\varpi}^{\beta}\mathcal{F}(\varkappa,\kappa_3)\right] \qquad (4.8)\\
&-\frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)}\left[\mathcal{J}_{\varkappa+;\rho}^{\alpha}\mathcal{F}(\kappa_2,\gamma)+\mathcal{J}_{\varkappa-;\rho}^{\alpha}\mathcal{F}(\kappa_1,\gamma)\right]+\frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\\
&\times\left[\mathcal{J}_{\varkappa+,\gamma+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}_{\varkappa+,\gamma-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{J}_{\varkappa-,\gamma+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}_{\varkappa-,\gamma-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_1,\kappa_3)\right]\Bigg|\\
\leq\;& \frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}
\end{aligned}$$

$$\times\left[\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\bigvee_{\varkappa}^{\tau}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\,d\varsigma d\tau\right.$$
$$+\int_{\varkappa}^{\kappa_2}\int_{\kappa_1}^{\gamma}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}\bigvee_{\varkappa}^{\tau}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\,d\varsigma d\tau$$
$$+\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\bigvee_{\tau}^{\varkappa}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\,d\varsigma d\tau$$
$$+\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}\bigvee_{\tau}^{\varkappa}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\,d\varsigma d\tau$$
$$\leq\frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left[[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}\bigvee_{\varkappa}^{\kappa_2}\bigvee_{\gamma}^{\kappa_4}(\mathcal{F})\right.$$
$$+[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}\bigvee_{\varkappa}^{\kappa_2}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F})+[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\gamma}^{\kappa_4}(\mathcal{F})$$
$$\left.+[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F})\right]$$
$$\leq\frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left[\frac{1}{2}(\rho(\kappa_2)-\rho(\kappa_1))+\left|\rho(\varkappa)-\frac{\rho(\kappa_1)+\rho(\kappa_2)}{2}\right|\right]^{\alpha}$$
$$\times\left[\frac{1}{2}(\varpi(\kappa_4)-\varpi(\kappa_3))+\left|\varpi(\gamma)-\frac{\varpi(\kappa_3)+\varpi(\kappa_4)}{2}\right|\right]^{\beta}\bigvee_{\kappa_1}^{\kappa_2}\bigvee_{\kappa_3}^{\kappa_2}(\mathcal{F})$$

for all $(\varkappa,\gamma)\in\Delta$.

Proof . By using modulus and the inequalities (4.3)-(4.6) in Lemma 3.2, we obtain

$$\left|\mathcal{F}(\varkappa,\gamma)-\frac{1}{\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\left[\mathcal{J}_{\gamma+;\varpi}^{\beta}\mathcal{F}(\varkappa,\kappa_4)+\mathcal{J}_{\gamma-;\varpi}^{\beta}\mathcal{F}(\varkappa,\kappa_3)\right]\right.$$
$$-\frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)}\left[\mathcal{J}_{\varkappa+;\rho}^{\alpha}\mathcal{F}(\kappa_2,\gamma)+\mathcal{J}_{\varkappa-;\rho}^{\alpha}\mathcal{F}(\kappa_1,\gamma)\right]$$
$$+\frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}$$
$$\left.\times\left[\mathcal{J}_{\varkappa+,\gamma+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}_{\varkappa+,\gamma-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{J}_{\varkappa-,\gamma+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}_{\varkappa-,\gamma-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_1,\kappa_3)\right]\right|$$
$$\leq\frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}$$
$$\times\left[\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\bigvee_{\varkappa}^{\tau}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\,d\varsigma d\tau\right.$$
$$+\int_{\varkappa}^{\kappa_2}\int_{\kappa_1}^{\gamma}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}\bigvee_{\varkappa}^{\tau}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\,d\varsigma d\tau$$
$$+\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\bigvee_{\tau}^{\varkappa}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\,d\varsigma$$

$$+\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}\bigvee_{\tau}^{\varkappa}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\,d\varsigma d\tau\Bigg]$$

$$\leq \frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg[[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}\bigvee_{\varkappa}^{\kappa_2}\bigvee_{\gamma}^{\kappa_4}(\mathcal{F})$$

$$+[\rho(\kappa_2)-\rho(\varkappa)]^{\alpha}[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}\bigvee_{\varkappa}^{\kappa_2}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F})+[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}[\varpi(\kappa_4)-\varpi(\gamma)]^{\beta}\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\gamma}^{\kappa_4}(\mathcal{F})$$

$$+[\rho(\varkappa)-\rho(\kappa_1)]^{\alpha}[\varpi(\gamma)-\varpi(\kappa_3)]^{\beta}\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F})\Bigg].$$

This proves the first and second inequality in (4.8).

The last inequality in (4.8) is proved above. □

Theorem 4.3. Suppose that the assumptions of Theorem 4.1, then we have the following inequalities

$$\Bigg|\frac{\mathcal{F}(\kappa_1,\kappa_3)+\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{F}(\kappa_2,\kappa_4)}{4}\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4) \tag{4.9}$$

$$-\frac{\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4)}{4}\left[\mathcal{J}_{\kappa_1+;\rho}^{\alpha}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}_{\kappa_1+;\rho}^{\alpha}\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{J}_{\kappa_2-;\rho}^{\alpha}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}_{\kappa_2-;\rho}^{\alpha}\mathcal{F}(\kappa_1,\kappa_3)\right]$$

$$-\frac{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2)}{4}\left[\mathcal{J}_{\kappa_3+;\varpi}^{\beta}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}_{\kappa_4-;\varpi}^{\beta}\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{J}_{\kappa_3+;\varpi}^{\beta}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}_{\kappa_4-;\varpi}^{\beta}\mathcal{F}(\kappa_1,\kappa_3)\right]$$

$$+\frac{1}{4}\Big[\mathcal{J}_{\kappa_1+,\kappa_3+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}_{\kappa_1+,\kappa_4-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_2,\kappa_3)$$

$$+\mathcal{J}_{\kappa_2-,\kappa_3+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}_{\kappa_2-,\kappa_4-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_1,\kappa_3)\Big]\Bigg|$$

$$\leq \frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\bigvee_{\tau}^{\kappa_2}\bigvee_{\varsigma}^{\kappa_4}(\mathcal{F})d\varsigma d\tau$$

$$+\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}\bigvee_{\tau}^{\kappa_2}\bigvee_{\kappa_3}^{\varsigma}(\mathcal{F})d\varsigma d\tau$$

$$+\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\bigvee_{\kappa_1}^{\tau}\bigvee_{\varsigma}^{\kappa_4}(\mathcal{F})d\varsigma d\tau$$

$$+\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}\bigvee_{\kappa_1}^{\tau}\bigvee_{\kappa_3}^{\varsigma}(\mathcal{F})d\varsigma d\tau\Bigg]$$

$$\leq \frac{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{\kappa_1}^{\kappa_2}\bigvee_{\kappa_3}^{\kappa_4}(\mathcal{F})$$

for all $(\varkappa,\gamma)\in\Delta$.

Proof . By Lemma 3.3, we get

$$\Bigg|\frac{\mathcal{F}(\kappa_1,\kappa_3)+\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{F}(\kappa_2,\kappa_4)}{4}\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4) \tag{4.10}$$

$$-\frac{\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4)}{4}\left[\mathcal{J}_{\kappa_1+;\rho}^{\alpha}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}_{\kappa_1+;\rho}^{\alpha}\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{J}_{\kappa_2-;\rho}^{\alpha}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}_{\kappa_2-;\rho}^{\alpha}\mathcal{F}(\kappa_1,\kappa_3)\right]$$

$$-\frac{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2)}{4}\left[\mathcal{J}_{\kappa_3+;\varpi}^{\beta}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}_{\kappa_4-;\varpi}^{\beta}\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{J}_{\kappa_3+;\varpi}^{\beta}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}_{\kappa_4-;\varpi}^{\beta}\mathcal{F}(\kappa_1,\kappa_3)\right]$$

$$
\begin{aligned}
&+\frac{1}{4}\Big[\mathcal{J}^{\alpha,\beta}_{\kappa_1+,\kappa_3+;\rho,\varpi}\mathcal{F}(\kappa_2,\kappa_4)+\mathcal{J}^{\alpha,\beta}_{\kappa_1+,\kappa_4-;\rho,\varpi}\mathcal{F}(\kappa_2,\kappa_3)\\
&+\mathcal{J}^{\alpha,\beta}_{\kappa_2-,\kappa_3+;\rho,\varpi}\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{J}^{\alpha,\beta}_{\kappa_2-,\kappa_4-;\rho,\varpi}\mathcal{F}(\kappa_1,\kappa_3)\Big]\Big|\\
\leq\ & \frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\left[\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\right.\\
&\times|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_4)-\mathcal{F}(\kappa_2,\varsigma)+\mathcal{F}(\kappa_2,\kappa_4)|\,d\varsigma d\tau\\
&+\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_3)-\mathcal{F}(\kappa_2,\varsigma)+\mathcal{F}(\kappa_2,\kappa_3)|\,d\varsigma d\tau\\
&+\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_4)-\mathcal{F}(\kappa_1,\varsigma)+\mathcal{F}(\kappa_1,\kappa_4)|\,d\varsigma d\tau\\
&\left.+\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_3)-\mathcal{F}(\kappa_1,\varsigma)+\mathcal{F}(\kappa_1,\kappa_3)|\,d\varsigma d\tau\right].
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_4)-\mathcal{F}(\kappa_2,\varsigma)+\mathcal{F}(\kappa_2,\kappa_4)|\,d\varsigma d\tau\\
\leq\ &\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\bigvee_{\tau}^{\kappa_2}\bigvee_{\varsigma}^{\kappa_4}(\mathcal{F})d\varsigma d\tau\\
\leq\ &\bigvee_{\kappa_1}^{\kappa_2}\bigvee_{\kappa_3}^{\kappa_4}(\mathcal{F})\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}d\varsigma d\tau \qquad (4.11)\\
=\ &\frac{[\rho(\kappa_2)-\rho(\kappa_1)]^{\alpha}}{\alpha}\frac{[\varpi(\kappa_4)-\varpi(\kappa_3)]^{\beta}}{\beta}\bigvee_{\kappa_1}^{\kappa_2}\bigvee_{\kappa_3}^{\kappa_4}(\mathcal{F}).
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_3)-\mathcal{F}(\kappa_2,\varsigma)+\mathcal{F}(\kappa_2,\kappa_3)|\,d\varsigma d\tau\\
\leq\ &\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\kappa_2)-\rho(\tau)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}}\bigvee_{\tau}^{\kappa_2}\bigvee_{\kappa_3}^{\varsigma}(\mathcal{F})d\varsigma d\tau \qquad (4.12)\\
\leq\ &\frac{[\rho(\kappa_2)-\rho(\kappa_1)]^{\alpha}}{\alpha}\frac{[\varpi(\kappa_4)-\varpi(\kappa_3)]^{\beta}}{\beta}\bigvee_{\kappa_1}^{\kappa_2}\bigvee_{\kappa_3}^{\kappa_4}(\mathcal{F}),
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_4)-\mathcal{F}(\kappa_1,\varsigma)+\mathcal{F}(\kappa_1,\kappa_4)|\,d\varsigma d\tau\\
\leq\ &\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}}\frac{\varpi'(\varsigma)}{[\varpi(\kappa_4)-\varpi(\varsigma)]^{1-\beta}}\bigvee_{\kappa_1}^{\tau}\bigvee_{\varsigma}^{\kappa_4}(\mathcal{F})d\varsigma d\tau \qquad (4.13)\\
\leq\ &\frac{[\rho(\kappa_2)-\rho(\kappa_1)]^{\alpha}}{\alpha}\frac{[\varpi(\kappa_4)-\varpi(\kappa_3)]^{\beta}}{\beta}\bigvee_{\kappa_1}^{\kappa_2}\bigvee_{\kappa_3}^{\kappa_4}(\mathcal{F}),
\end{aligned}
$$

and

$$\begin{aligned}
&\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} \frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}} \frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}} \left|\mathcal{F}(\tau,\varsigma)-\mathcal{F}(\tau,\kappa_3)-\mathcal{F}(\kappa_1,\varsigma)+\mathcal{F}(\kappa_1,\kappa_3)\right| d\varsigma d\tau \\
\leq\ & \int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} \frac{\rho'(\tau)}{[\rho(\tau)-\rho(\kappa_1)]^{1-\alpha}} \frac{\varpi'(\varsigma)}{[\varpi(\varsigma)-\varpi(\kappa_3)]^{1-\beta}} \bigvee_{\kappa_1}^{\tau}\bigvee_{\kappa_3}^{\varsigma}(\mathcal{F}) d\varsigma d\tau \\
\leq\ & \frac{[\rho(\kappa_2)-\rho(\kappa_1)]^{\alpha}}{\alpha} \frac{[\varpi(\kappa_4)-\varpi(\kappa_3)]^{\beta}}{\beta} \bigvee_{\kappa_1}^{\kappa_2}\bigvee_{\kappa_3}^{\kappa_4}(\mathcal{F}).
\end{aligned} \tag{4.14}$$

If we sybstitute the inequalities (4.11)-(4.14) in (4.10), then we obtain the requires inequality (4.9). □

Corollary 4.4. If we take $\varkappa = \frac{\kappa_1+\kappa_2}{2}$ and $\gamma = \frac{\kappa_3+\kappa_4}{2}$ in Theorem 4.1-Theorem 4.3, then we obtain some midpoint type inequalities, but the details are not presented here.

Remark 4.5. If we choose $\rho(\tau) = \tau$, $\tau \in [\kappa_1,\kappa_2]$ and $\varpi(\varsigma) = \varsigma$, $\varsigma \in [\kappa_3,\kappa_4]$ in Theorem 4.1-Theorem 4.3, then we obtain some Ostrowski type inequalities for Riemann-Liouville fractional integrals which were proved by Erden et al. in [17].

5 Some Inequalities For Hadamard Fractional Integrals

By choosing the $\rho(\tau) = \ln\tau$, $\tau \in [\kappa_1,\kappa_2]$ and $\varpi(\varsigma) = \ln\varsigma$, $\varsigma \in [\kappa_3,\kappa_4]$ in Theorem 4.1-Theorem 4.3, we have the following theorems.

Theorem 5.1. If $\mathcal{F} : \Delta \to \mathbb{R}$ is of bounded variation on Δ, then we have the following inequalities for Hadamard fractional integrals

$$\begin{aligned}
&\Bigg|\mathcal{F}(\varkappa,\gamma) - \frac{1}{\mathcal{N}_{\ln}^{\beta}(\kappa_3,\kappa_4;\gamma)}\left[\mathbf{J}_{\kappa_3+;\varpi}^{\beta}\mathcal{F}(\varkappa,\gamma)+\mathbf{J}_{\kappa_4-;\varpi}^{\beta}\mathcal{F}(\varkappa,\gamma)\right] \\
&-\frac{1}{\mathcal{M}_{\ln}^{\alpha}(\kappa_1,\kappa_2;\varkappa)}\left[\mathbf{J}_{\kappa_1+;\rho}^{\alpha}\mathcal{F}(\varkappa,\gamma)+\mathbf{J}_{\kappa_2-;\rho}^{\alpha}\mathcal{F}(\varkappa,\gamma)\right] \\
&+\frac{1}{\mathcal{M}_{\ln}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\ln}^{\beta}(\kappa_3,\kappa_4;\gamma)} \\
&\times\left[\mathbf{J}_{\kappa_1+,\kappa_3+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma)+\mathbf{J}_{\kappa_1+,\kappa_4-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma)+\mathbf{J}_{\kappa_2-,\kappa_3+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma)+\mathbf{J}_{\kappa_2-,\kappa_4-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\varkappa,\gamma)\right]\Bigg| \\
\leq\ & \frac{1}{\mathcal{M}_{\ln}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\ln}^{\beta}(\kappa_3,\kappa_4;\gamma)}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[\int_{\kappa_1}^{\varkappa}\int_{\kappa_3}^{\gamma}\left(\ln\frac{\varkappa}{\tau}\right)^{\alpha-1}\left(\ln\frac{\gamma}{\varsigma}\right)^{\beta-1}\bigvee_{\tau}^{\varkappa}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\frac{d\varsigma d\tau}{\varsigma\tau} \\
&+\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}\left(\ln\frac{\varkappa}{\tau}\right)^{\alpha-1}\left(\ln\frac{\varsigma}{\gamma}\right)^{\beta-1}\bigvee_{\tau}^{\varkappa}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\frac{d\varsigma d\tau}{\varsigma\tau}+\int_{\varkappa}^{\kappa_2}\int_{\kappa_3}^{\gamma}\left(\ln\frac{\tau}{\varkappa}\right)^{\alpha-1}\left(\ln\frac{\gamma}{\varsigma}\right)^{\beta-1}\bigvee_{\varkappa}^{\tau}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\frac{d\varsigma d\tau}{\varsigma\tau} \\
&+\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}\left(\ln\frac{\tau}{\varkappa}\right)^{\alpha-1}\left(\ln\frac{\varsigma}{\gamma}\right)^{\beta-1}\bigvee_{\varkappa}^{\tau}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\frac{d\varsigma d\tau}{\varsigma\tau}\Bigg] \\
\leq\ & \frac{1}{\mathcal{M}_{\rho}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\varpi}^{\beta}(\kappa_3,\kappa_4;\gamma)}\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg[\left(\ln\frac{\varkappa}{\kappa_1}\right)^{\alpha}\left(\ln\frac{\gamma}{\kappa_3}\right)^{\beta}\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F}) \\
&+\left(\ln\frac{\varkappa}{\kappa_1}\right)^{\alpha}\left(\ln\frac{\kappa_4}{\gamma}\right)^{\beta}\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\gamma}^{\kappa_4}(\mathcal{F})+\left(\ln\frac{\kappa_2}{\varkappa}\right)^{\alpha}\left(\ln\frac{\gamma}{\kappa_3}\right)^{\beta}\bigvee_{\varkappa}^{\kappa_2}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F})
\end{aligned}$$

$$
\begin{aligned}
&+\left(\ln \frac{\kappa_2}{\varkappa}\right)^{\alpha}\left(\ln \frac{\kappa_4}{\gamma}\right)^{\beta} \bigvee_{\varkappa}^{\kappa_2} \bigvee_{\gamma}^{\kappa_4}(\mathcal{F})\Bigg]\\
\leq\ & \frac{1}{\mathcal{M}_{\ln}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\ln}^{\beta}(\kappa_3,\kappa_4;\gamma)} \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\\
&\times\left[\frac{1}{2}\left(\ln \frac{\kappa_2}{\kappa_1}\right)+\left|\ln \frac{\varkappa}{\sqrt{\kappa_1\kappa_2}}\right|\right]^{\alpha}\left[\frac{1}{2}\left(\ln \frac{\kappa_4}{\kappa_3}\right)+\left|\ln \frac{\gamma}{\sqrt{\kappa_3\kappa_4}}\right|\right]^{\beta} \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_3}^{\kappa_2}(\mathcal{F})
\end{aligned}
$$

for all $(\varkappa, \gamma) \in \Delta$.

Theorem 5.2. If $\mathcal{F} : \Delta \to \mathbb{R}$ is of bounded variation on Δ, then we have the following inequalities for Hadamard fractional integrals

$$
\begin{aligned}
&\Bigg|\mathcal{F}(\varkappa,\gamma)-\frac{1}{\mathcal{N}_{\ln}^{\beta}(\kappa_3,\kappa_4;\gamma)}\left[\mathbf{J}_{\gamma+;\varpi}^{\beta}\mathcal{F}(\varkappa,\kappa_4)+\mathbf{J}_{\gamma-;\varpi}^{\beta}\mathcal{F}(\varkappa,\kappa_3)\right]\\
&-\frac{1}{\mathcal{M}_{\ln}^{\alpha}(\kappa_1,\kappa_2;\varkappa)}\left[\mathbf{J}_{\varkappa+;\rho}^{\alpha}\mathcal{F}(\kappa_2,\gamma)+\mathbf{J}_{\varkappa-;\rho}^{\alpha}\mathcal{F}(\kappa_1,\gamma)\right]\\
&+\frac{1}{\mathcal{M}_{\ln}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\ln}^{\beta}(\kappa_3,\kappa_4;\gamma)}\\
&\left[\mathbf{J}_{\varkappa+,\gamma+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_2,\kappa_4)+\mathbf{J}_{\varkappa+,\gamma-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_2,\kappa_3)+\mathbf{J}_{\varkappa-,\gamma+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_1,\kappa_4)+\mathbf{J}_{\varkappa-,\gamma-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_1,\kappa_3)\right]\Bigg|\\
\leq\ & \frac{1}{\mathcal{M}_{\ln}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\ln}^{\beta}(\kappa_3,\kappa_4;\gamma)} \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\left[\int_{\varkappa}^{\kappa_2}\int_{\gamma}^{\kappa_4}\left(\ln\frac{\kappa_2}{\tau}\right)^{\alpha-1}\left(\ln\frac{\kappa_4}{\varsigma}\right)^{\beta-1}\bigvee_{\varkappa}^{\tau}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\frac{d\varsigma d\tau}{\varsigma\tau}\right.\\
&+\int_{\varkappa}^{\kappa_2}\int_{\kappa_1}^{\gamma}\left(\ln\frac{\kappa_2}{\tau}\right)^{\alpha-1}\left(\ln\frac{\varsigma}{\kappa_3}\right)^{\beta-1}\bigvee_{\varkappa}^{\tau}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\frac{d\varsigma d\tau}{\varsigma\tau}\\
&+\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}\left(\ln\frac{\tau}{\kappa_1}\right)^{\alpha-1}\left(\ln\frac{\kappa_4}{\varsigma}\right)^{\beta-1}\bigvee_{\tau}^{\varkappa}\bigvee_{\gamma}^{\varsigma}(\mathcal{F})\frac{d\varsigma d\tau}{\varsigma\tau}\\
&\left.+\int_{\kappa_1}^{\varkappa}\int_{\gamma}^{\kappa_4}\left(\ln\frac{\tau}{\kappa_1}\right)^{\alpha-1}\left(\ln\frac{\varsigma}{\kappa_3}\right)^{\beta-1}\bigvee_{\tau}^{\varkappa}\bigvee_{\varsigma}^{\gamma}(\mathcal{F})\frac{d\varsigma d\tau}{\varsigma\tau}\right]\\
\leq\ & \frac{1}{\mathcal{M}_{\ln}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\ln}^{\beta}(\kappa_3,\kappa_4;\gamma)} \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left[\left(\ln\frac{\kappa_2}{\varkappa}\right)^{\alpha}\left(\ln\frac{\kappa_4}{\gamma}\right)^{\beta}\bigvee_{\varkappa}^{\kappa_2}\bigvee_{\gamma}^{\kappa_4}(\mathcal{F})\right.\\
&+\left(\ln\frac{\kappa_2}{\varkappa}\right)^{\alpha}\left(\ln\frac{\gamma}{\kappa_3}\right)^{\beta}\bigvee_{\varkappa}^{\kappa_2}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F})+\left(\ln\frac{\varkappa}{\kappa_1}\right)^{\alpha}\left(\ln\frac{\kappa_4}{\gamma}\right)^{\beta}\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\gamma}^{\kappa_4}(\mathcal{F})\\
&\left.+\left(\ln\frac{\kappa_2}{\varkappa}\right)^{\alpha}\left(\ln\frac{\kappa_4}{\gamma}\right)^{\beta}\bigvee_{\kappa_1}^{\varkappa}\bigvee_{\kappa_3}^{\gamma}(\mathcal{F})\right]\\
\leq\ & \frac{1}{\mathcal{M}_{\ln}^{\alpha}(\kappa_1,\kappa_2;\varkappa)\mathcal{N}_{\ln}^{\beta}(\kappa_3,\kappa_4;\gamma)}\\
&\times\left[\frac{1}{2}\left(\ln \frac{\kappa_2}{\kappa_1}\right)+\left|\ln \frac{\varkappa}{\sqrt{\kappa_1\kappa_2}}\right|\right]^{\alpha}\left[\frac{1}{2}\left(\ln \frac{\kappa_4}{\kappa_3}\right)+\left|\ln \frac{\gamma}{\sqrt{\kappa_3\kappa_4}}\right|\right]^{\beta} \bigvee_{\kappa_1}^{\kappa_2} \bigvee_{\kappa_3}^{\kappa_2}(\mathcal{F})
\end{aligned}
$$

for all $(\varkappa, \gamma) \in \Delta$.

Theorem 5.3. If $\mathcal{F} : \Delta \to \mathbb{R}$ is of bounded variation on Δ, then we have the following inequalities for Hadamard fractional integrals

$$
\left|\frac{\mathcal{F}(\kappa_1,\kappa_3)+\mathcal{F}(\kappa_1,\kappa_4)+\mathcal{F}(\kappa_2,\kappa_3)+\mathcal{F}(\kappa_2,\kappa_4)}{4}\mathcal{M}_{\ln}^{\alpha}(\kappa_1,\kappa_2)\mathcal{N}_{\ln}^{\beta}(\kappa_3,\kappa_4)\right.
$$

$$
\begin{aligned}
& -\frac{\mathcal{N}_{\ln}^{\beta}(\kappa_3,\kappa_4)}{4}\left[\mathbf{J}_{\kappa_1+;\rho}^{\alpha}\mathcal{F}(\kappa_2,\kappa_4)+\mathbf{J}_{\kappa_1+;\rho}^{\alpha}\mathcal{F}(\kappa_2,\kappa_3)+\mathbf{J}_{\kappa_2-;\rho}^{\alpha}\mathcal{F}(\kappa_1,\kappa_4)+\mathbf{J}_{\kappa_2-;\rho}^{\alpha}\mathcal{F}(\kappa_1,\kappa_3)\right] \\
& -\frac{\mathcal{M}_{\ln}^{\alpha}(\kappa_1,\kappa_2)}{4}\left[\mathbf{J}_{\kappa_3+;\varpi}^{\beta}\mathcal{F}(\kappa_2,\kappa_4)+\mathbf{J}_{\kappa_4-;\varpi}^{\beta}\mathcal{F}(\kappa_2,\kappa_3)+\mathbf{J}_{\kappa_3+;\varpi}^{\beta}\mathcal{F}(\kappa_1,\kappa_4)+\mathbf{J}_{\kappa_4-;\varpi}^{\beta}\mathcal{F}(\kappa_1,\kappa_3)\right] \\
& +\frac{1}{4}\left[\mathbf{J}_{\kappa_1+,\kappa_3+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_2,\kappa_4)+\mathbf{J}_{\kappa_1+,\kappa_4-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_2,\kappa_3)\right. \\
& \left.\left.+\mathbf{J}_{\kappa_2-,\kappa_3+;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_1,\kappa_4)+\mathbf{J}_{\kappa_2-,\kappa_4-;\rho,\varpi}^{\alpha,\beta}\mathcal{F}(\kappa_1,\kappa_3)\right]\right| \\
\leq\ & \frac{1}{4\Gamma(\alpha)\Gamma(\beta)}\left[\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\left(\ln\frac{\kappa_2}{\tau}\right)^{\alpha}\left(\ln\frac{\kappa_4}{\varsigma}\right)^{\beta}\bigvee_{\tau}^{\kappa_2}\bigvee_{\varsigma}^{\kappa_4}(\mathcal{F})\frac{d\varsigma d\tau}{\varsigma\tau}\right. \\
& +\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\left(\ln\frac{\kappa_2}{\tau}\right)^{\alpha}\left(\ln\frac{\varsigma}{\kappa_3}\right)^{\beta}\bigvee_{\tau}^{\kappa_2}\bigvee_{\kappa_3}^{\varsigma}(\mathcal{F})\frac{d\varsigma d\tau}{\varsigma\tau} \\
& +\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\left(\ln\frac{\tau}{\kappa_1}\right)^{\alpha}\left(\ln\frac{\kappa_4}{\varsigma}\right)^{\beta}\bigvee_{\kappa_1}^{\tau}\bigvee_{\varsigma}^{\kappa_4}(\mathcal{F})\frac{d\varsigma d\tau}{\varsigma\tau} \\
& \left.+\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\left(\ln\frac{\tau}{\kappa_1}\right)^{\alpha}\left(\ln\frac{\varsigma}{\kappa_3}\right)^{\beta}\bigvee_{\kappa_1}^{\tau}\bigvee_{\kappa_3}^{\varsigma}(\mathcal{F})\frac{d\varsigma d\tau}{\varsigma\tau}\right] \\
\leq\ & \frac{\mathcal{M}_{\ln}^{\alpha}(\kappa_1,\kappa_2)\mathcal{N}_{\ln}^{\beta}(\kappa_3,\kappa_4)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\bigvee_{\kappa_1}^{\kappa_2}\bigvee_{\kappa_3}^{\kappa_4}(\mathcal{F})
\end{aligned}
$$

for all $(\varkappa,\gamma)\in\Delta$.

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