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New qualitative results to delay integro-differential equations

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Abstract

This work deals with various qualitative analyses of solutions of a certain delay integro-differential equation (DIDE). We prove here six new theorems including sufficient conditions, on uniformly stability (US), boundedness, asymptotically stability (AS), exponentially stability (ES), integrability and instability of solutions, respectively. By defining a suitable Lyapunov function (LF) and using the Razumikhin method (RM), the proofs of the theorems are provided. We gave two examples to demonstrate applications of the established new conditions.

Keywords: DIDE, Lyapunov–Razumikhin method, stability, instability, boundedness, integrability
2020 MSC: 34D05, 34K20, 45J05

1 Introduction

It is well known that mathematical models are commonly used in physical, biological and engineering sciences, etc., to understand and explain industrial processes and complex systems. For example, integro-differential equations (IDEs) can be found as mathematical models of phenomena in the life sciences, population dynamics, ecology problems, medicine, electricity, physics, artificial neural networks and engineering sciences, heat flow, electricity and so on (see, for example, [1]-[17],[19]-[44],[46],[47], [52], [55]-[57],[59],[60]). Motivated by numerous theoretical works and applications in applied sciences, various qualitative behaviors of IDEs and delay differential equations have been studied intensively during the past decades and many interesting results were obtained in the relevant literature (see, the papers or the books given in [1]-[60] and the references of these sources). It is also known from the relevant literature that the qualitative behaviors of solutions called as US, AS, ES, convergence, boundedness, integrability and instability of solutions are important qualitative concepts during the investigation of dynamics of the solutions of numerous kinds of DIDEs and delay differential equations. Indeed, the books of Burton [4] and Hale [18] can be considered as the references books for these and some others qualitative concepts .

When we look for the relevant literature on the fundamental properties of solutions of integro-differential equations with and without delays, it can be seen that, during the last decades, various fundamental properties of solutions of these equations have been investigated. For example, the fundamental properties of solutions such as the integrability, US, AS, ES, etc. of integro-differential equations of the form

$$x'(t) = A(t)x(t) + \int_0^t C(t,s)x(s)ds,$$

$$x'(t) = A(t)x(t) + \int_{t-\tau}^t C(t,s)x(s)ds,$$

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and related different linear and nonlinear modified forms have been discussed by numerous authors. Many interesting results have been obtained on the subject. This fact can be easily checked from the works in the references of this paper. .

In this work, motivated from the works on the qualitative properties of solutions DIDEs ([1],[2],[4]-[12],[15]-[17],[19]-[35],[36]-[44], [47], [52]-[58],[60]), we consider the following nonlinear DIDE:

$$x'(t) = -a(t)g(x) - r(x) + \int_{t-\tau}^{t} C(t,s,h(x(s)))f(s,x(s))ds, \tag{1.1}$$

where $t \in [\tau, \infty)$, $x \in \mathbb{R}$, $a(t) \in C(\mathbb{R}^+, (0,\infty))$ and $C(.) \in C([-\tau,\infty) \times [-\tau,\infty) \times \mathbb{R}, \mathbb{R}))$, $-\tau \le s \le t < \infty$, $f \in C([-\tau,\infty) \times \mathbb{R}, \mathbb{R}))$, $g, r \in C(\mathbb{R}, \mathbb{R})$, $g(0) = r(0) = 0$, $f(s,0) = 0$, $h(0) = 0$, $h(x) \neq 0$ when $x \neq 0$, $C(t,s,0) = 0$ and $\tau > 0$ is constant delay. Hence, DIDE (1.1) includes the zero solution.

Since the functions a, g, r, C, h and f are continuous, the continuity of these functions is a sufficient conditions for the existence of the solutions of DIDE (1.1). We also assume that the functions g, r, C, h and f satisfy the Lipschitz condition in x. This assumption is a sufficient condition for the uniqueness of the solutions of DIDE (1.1).

To the best of information, the mentioned qualitative concepts have not been discussed for DIDE (1.1) in the literature up to now. Our aim is to give new results for these qualitative concepts and to do contribution to the theory of DIDEs.

We suppose $x(t, t_0, \phi)$, $t \ge t_0$, is a solution of DIDE (1.1) on $[t_0 - \tau, \beta)$, $\beta > 0$, such that $x(t) = \phi(t)$ on $[t_0 - \tau, t_0]$ and $|\phi(t)| = \sup_{t \in [t_0 - \tau, t_0]} |\phi(t)|$, where $\phi \in C([t_0 - \tau, t_0], \mathbb{R}])$, ϕ is the initial function.

2 Stability and Integrability

In this section, we investigate the US, AS and integrability of DIDE (1.1).

For our main results, we will consider the following conditions.

A. Assumptions

(A1) Let g_0, r_0 and f_0 be positive constants such that

$$g(0) = 0, xg(x) > 0, x \neq 0,$$

$$\frac{|g(x)|}{|x|} \ge g_0, x \neq 0, x \in \mathbb{R},$$

$$r(0) = 0, xr(x) > 0, x \neq 0,$$

$$\frac{|r(x)|}{|x|} \ge r_0, x \neq 0, x \in \mathbb{R},$$

$$f(0) = 0, |f(s, x(s))| \le f_0|x(s)|, -\tau \le s \le t < \infty, x \in \mathbb{R}.$$

(A2) Let $C_0, K_0 \in \mathbb{R}$, $C_0, K_0 > 0$, such that

$$C(t,s,0) = 0, |C(t,s,h(x(s))| \le C_0|K(t,s)|$$

and

$$|K(t,s)| \le K_0, -\tau \le s \le t < \infty, x \in \mathbb{R}.$$

(A3) There exist positive constants g_0, r_0, f_0 from (A1), C_0, K_0 from (A2) and τ such that

$$a(t) > 0, \ g_0 a(t) + r_0 - \tau f_0 C_0 K_0 \ge 0.$$

(A4) There exist positive constants g_0, r_0, f_0 from (A1), C_0, K_0 from (A2), τ from (A3) and ς_0 such that

$$a(t) > 0, \ g_0 a(t) + r_0 - \tau f_0 C_0 K_0 \ge \varsigma_0.$$

(A5) There exist positive constants g_0 and r_0 from (A1) such that

$$g(0) = 0, xg(x) > 0, x \neq 0, x \in \mathbb{R},$$

$$\frac{|g(x)|}{|x|} \geq g_0, x \neq 0,$$

$$r(0) = 0, xr(x) < 0, x \neq 0, x \in \mathbb{R},$$

$$|r(x)| \geq r_0|x|,$$

$$f(0) = 0, |f(s, x(s))| \leq f_0|x(s)|, -\tau \leq s \leq t < \infty, x \in \mathbb{R}.$$

(A6) There exist positive constants g_0, r_0, f_0 from (A1), C_0, K_0 from (A2) and τ from (A3) such that

$$a(t) < 0, g_0 a(t) - r_0 + \tau f_0 C_0 K_0 < 0.$$

Firstly, we give a new US result for DIDE (1.1).

Theorem 1. Let assumptions (A1)-(A3) hold. Then, the null solution of DIDE (1.1) is uniformly stable.

Proof . Consider the LF

$$V(t, x) = |x|,$$

where x represents $x(t)$ and also without mention through the paper, when we need.

Clearly

$$V(t, 0) = 0, V(t, x) \geq d_1|x|, d_1 \in (0, 1].$$

Differentiating the LF V and using conditions (A1)-(A2), we obtain

$$\begin{aligned}
\frac{d}{dt}V(t,x) =& \frac{dx}{dt} sgnx(t+0) \\
=& - a(t)g(x)sgnx(t+0) - r(x)sgnx(t+0) \\
&+ sgnx(t+0)\int_{t-\tau}^{t} C(t, s, h(x(s)))f(s, x(s))ds \\
\leq& - a(t)|g(x)| - |r(x)| + \int_{t-\tau}^{t} |C(t, s, h(x(s)))||f(s, x(s))|ds \\
\leq& - a(t)g_0|x| - r_0|x| + C_0 f_0 \int_{t-\tau}^{t} |K(t, s)||x(s)|ds \\
\leq& - a(t)g_0|x| - r_0|x| + C_0 f_0 K_0 \int_{t-\tau}^{t} |x(s)|ds.
\end{aligned} \tag{2.1}$$

Consider integral term

$$C_0 f_0 K_0 \int_{t-\tau}^{t} |x(s)|ds,$$

which is taken from (2.1).

When we use the RM on the interval $-\tau \leq s \leq 0$, we have

$$\begin{aligned}
C_0 f_0 K_0 \int_{t-\tau}^{t} |x(s)|ds &= C_0 f_0 K_0 \int_{-\tau}^{0} |x(t+\xi)|ds \\
&< C_0 f_0 K_0 \int_{-\tau}^{0} |x(t)|ds \\
&= \tau C_0 f_0 K_0 |x(t)|.
\end{aligned}$$

Putting the last term in (2.1) and using condition (A3), we obtain

$$\begin{aligned}\frac{d}{dt}V(t,x) \leq & -a(t)g_0|x| - r_0|x| + C_0 f_0 K_0 \tau |x| \\ = & -(a(t)g_0 + r_0 - \tau C_0 f_0 K_0)|x| \leq 0.\end{aligned}$$

This result finishes the proof of Theorem 1. □

Secondly, we show the boundedness result for the solutions of DIDE (1.1), when $t \to \infty$.

Theorem 2. Let assumptions (A1)-(A3) hold. Then, the solutions of DIDE (1.1) are bounded at infinity.

Proof . From the previous theorem, we have

$$\frac{d}{dt}V(t,x) \leq 0.$$

From this result, it follows that

$$V(t,x) \leq V(t_0,\phi(t_0)), t \geq t_0.$$

Next, it is clear that

$$V(t_0,\phi(t_0)) = |\phi(t_0)| = V_0 > 0,$$

where V_0 is a positive constant provided that $\phi(t_0) \neq 0$.

Hence, we have

$$|x(t)| \leq V_0 \text{ as } t \to \infty.$$

This inequality completes the proof. □

Theorem 3. Let assumptions (A1),(A2) and (A4) hold. Then, the null solution of DIDE (1.1) is asymptotically stable.

Proof . It is obvious that

$$V(t,x) \leq d_2|x|, \ (x \neq 0), d_2 \geq 1.$$

By the assumptions (A1),(A2), (A4) and the way of Theorem 1, we obtain

$$\frac{d}{dt}V(t,x) \leq -\varsigma_0|x|, t \geq t_0.$$

This is end of the proof. □

Theorem 4. Let assumptions (A1),(A2) and (A4) hold. Then, the null solution DIDE (1.1) is exponentially stable.

Proof . From Theorem 3 , we have

$$\frac{d}{dt}V(t,x(t)) \leq -\varsigma_0|x(t)|, t \geq t_0.$$

Then, from the definition of Lyapunov function V, we derive

$$\frac{d}{dt}V(t,x(t)) \leq -\varsigma_0|x| = -\varsigma_0 V(t,x(t)).$$

Hence,

$$\frac{\frac{d}{dt}V(t,x(t))}{V(t,x(t))} \leq -\varsigma_0.$$

By some elementary calculations, we derive

$$V(t,x(t)) \leq V(t_0,\phi(t_0)) \exp\left(-\varsigma_0(t-t_0)\right), \ \phi(t_0) \neq 0.$$

As for the next step, we conclude that

$$|x(t)| \leq V_0 \exp\left(-\varsigma_0(t-t_0)\right).$$

The proof is done. □

Theorem 5. Let assumptions (A1), (A2) and (A4) hold. Then, the absolute value of solutions of DIDE (1.1) are integrable on $[0, \infty)$.

Proof . Via Theorem 4, it is known that

$$\frac{d}{dt}V(t,x(t)) \leq -\varsigma_0|x|,\ t \geq t_0.$$

An integration gives that

$$V(t,x(t)) - V(t_0,\phi(t_0)) \leq -\varsigma_0 \int_{t_0}^{t} |x(s)|ds.$$

Therefore, it is clear that

$$\varsigma_0 \int_{t_0}^{t} |x(s)|ds \leq V(t_0,\phi(t_0)) - V(t,x(t)) \leq V(t_0,\phi(t_0)) = V_0$$

Then, we can get

$$\int_{t_0}^{\infty} |x(s)|ds \leq \varsigma_0^{-1} V_0 < \infty.$$

This result completes the proof of Theorem 5. □

Theorem 6. Let assumptions (A2), (A5) and (A6) hold. Then, the null solution of DIDE (1.1) is unstable.

Proof . By the same way of Theorem 1, we can obtain that

$$\begin{aligned}
\frac{d}{dt}V(t,x) =& \frac{dx}{dt} sgnx(t+0) \\
=& -a(t)g(x)sgnx(t+0) - r(x)sgnx(t+0) \\
& + sgnx(t+0)\int_{t-\tau}^{t} C(t,s,h(x(s)))f(s,x(s))ds \\
\geq& -a(t)|g(x)| + |r(x)| - \int_{t-\tau}^{t} |C(t,s,h(x(s)))||f(s,x(s))|ds \\
\geq& -g_0 a(t)|x| + r_0|x| - f_0 C_0 \int_{t-\tau}^{t} |K(t,s)||x(s)|ds \\
\geq& -g_0 a(t)|x| + r_0|x| - f_0 C_0 K_0 \int_{t-\tau}^{t} |x(s)|ds.
\end{aligned}$$

By applying Razumikhin condition to the term $f_0 C_0 K_0 \int_{t-\tau}^{t} |x(s)|ds$, we derive that

$$\frac{d}{dt}V(t,x(t)) \geq (-g_0 a(t) + r_0 - \tau f_0 C_0 K_0)|x| > 0,\ (x \neq 0).$$

The last inequality is the end of the proof of Theorem 6. □

Example 2.1. We take into consideration a nonlinear DIDE given by

$$\begin{aligned}
\dot{x}(t) =& -\left(25 + \frac{3}{2+\exp(t)}\right)\left(3 + \frac{2}{1+\exp(x^2(t))}\right)x(t) \\
& -\left(1 + \frac{1}{1+\exp(x^2(t))}\right)x(t) + \int_{t-\frac{1}{5}}^{t} \frac{1}{1+\exp(t^2+s^2)} \times \frac{x(s)}{1+x^2(s)} \times \frac{x(s)}{1+s^2} ds.
\end{aligned} \tag{2.2}$$

Doing a comparison between the DIDE (1.1) and (2.2), one can obtain the below relations:

$$a(t) = 25 + \frac{3}{2+\exp(t)} > 0, \tau = \frac{1}{5} \text{ is the fixed delay,}$$

$$g(x) = \left(3 + \frac{2}{1+\exp(x^2)}\right) x, g(0) = 0,$$

$$|g(x)| = \left(3 + \frac{2}{1+\exp(x^2)}\right) |x| \geq 3\,|x|\,,\ g_0 = 3,$$

$$xg(x) = \left(3 + \frac{2}{1+\exp(x^2)}\right) x^2 > 0, x \neq 0,$$

$$r(x) = \left(1 + \frac{1}{1+\exp(x^2)}\right)x, r(0) = 0, xr(x) = \left(1 + \frac{1}{1+\exp(x^2)}\right)x^2 > 0, x \neq 0,$$

$$\frac{|r(x)|}{|x|} = 1 + \frac{1}{1+\exp(x^2)} \geq 1 = r_0,$$

$$C(t,s,h(x)) = \frac{1}{1+\exp(t^2+s^2)} \times \frac{x}{1+x^2},$$

$$h(x) = \frac{x}{1+x^2}, h(0) = 0, h(x) \neq 0 \text{ if } x \neq 0, C(t,s,0) = 0,$$

$$\begin{aligned} |C(t,s,h(x))| &= \frac{1}{1+\exp(t^2+s^2)} \times \frac{|x|}{1+x^2} \\ &\leq \frac{1}{1+\exp(t^2+s^2)} = C_0 K(t,s), \end{aligned}$$

where

$$\begin{aligned} K(t,s) &= \frac{1}{1+\exp(t^2+s^2)}, C_0 = 1, \\ &\leq \frac{1}{1+\exp(t^2+s^2)} \leq \frac{1}{2}, K_0 = \frac{1}{2}, \end{aligned}$$

$$f(s,x(s)) = \frac{x(s)}{1+s^2},\ f(s,0) = 0,$$

$$|f(t,x(s))| = \frac{|x(s)|}{1+s^2} \leq |x(s)|\,,\ f_0 = 1.$$

In view of the given relations, we derive

$$g_0 a(t) + r_0 - \tau f_0 C_0 K_0 = 75 + 1 + \frac{9}{2+\exp(t)} - \frac{1}{10} > 75 = \varsigma_0.$$

Hence, conditions (A1)-(A4) are satisfied. Hence, the zero solution of nonlinear DIDE (2.2) is uniformly stable, asymptotically stable, exponentially stable, the absolute values of the solutions of DIDE (2.2) are integrable and non-zero solutions of DIDE (2.2) are bounded at the infinity.

Example 2.2. We take into consideration a nonlinear DIDE given by

$$\begin{aligned} \dot{x}(t) &= \left(25 + \frac{3}{2+\exp(t)}\right)\left(3 + \frac{2}{1+\exp(x^2(t))}\right)x(t) \\ &+ \left(1 + \frac{1}{1+\exp(x^2(t))}\right)x(t) \\ &+ \int_{t-\frac{1}{5}}^{t} \frac{1}{1+\exp(t^2+s^2)} \times \frac{x(s)}{1+x^2(s)} \times \frac{x(s)}{1+s^2} ds. \end{aligned} \tag{2.3}$$

Doing a comparison between DIDEs (2.3) and (1.1), one can obtain the below relations :

$$a(t) = -25 - \frac{3}{2+\exp(t)} < 0, \tau = \frac{1}{5},$$

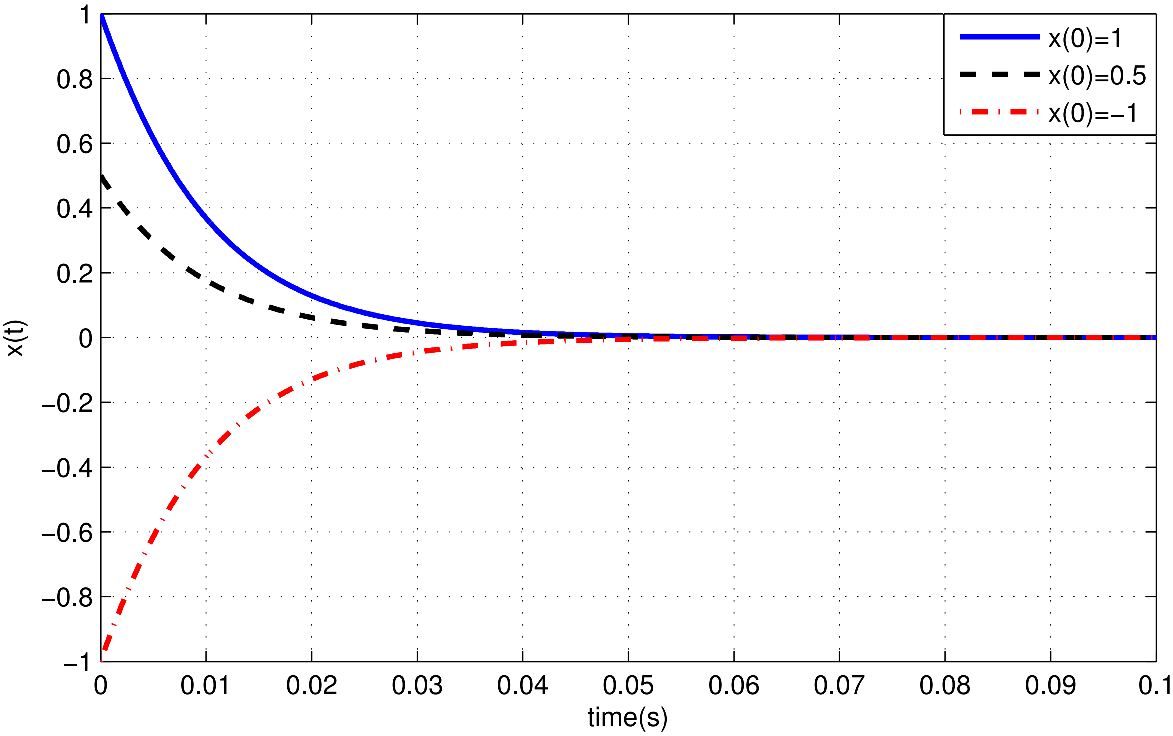

Figure 1: This figure shows the results of Theorem 1-5 for DIDE (2.2) and different initial values, when $\tau = \frac{1}{5}$.

$$a(t) \leq -25 < 0,$$

$$r(x) = -\left(1 + \frac{1}{1+\exp(x^2)}\right)x$$

$$xr(x) = -\left(1 + \frac{1}{1+\exp(x^2)}\right)x^2 < 0, x \neq 0,$$

$$\frac{|r(x)|}{|x|} = 1 + \frac{1}{1+\exp(x^2)} \geq 1 = r_0.$$

Next, the functions g, h, f, K and the kernel C are the same as in Example 2.1. Hence, the discussions related to these functions remain as the same in Example 2.1.

As for the next step, in view of the given relations, we derive

$$g_0a(t) - r_0 + \tau f_0 C_0 K_0 = -75 - \frac{9}{2+\exp(t)} - 1 + \frac{1}{10} = -75.9 - \frac{9}{2+\exp(t)} < 0.$$

Hence, conditions (A5) and (A6) Theorem 6 are satisfied hold. Therefore, the zero solution of nonlinear DIDE (2.3) is unstable.

3 Conclusion

This paper deals with behaviors of solutions of a DIDE. Six theorems, which have new sufficient conditions on the uniformly stability, asymptotic stability, exponential stability, instability of zero solution as well as boundedness and integrability of non-zero solutions of the considered DIDE, have been proved via the Lyapunov–Razumikhin method. Finally, two examples are given to show applications of the given theorems.

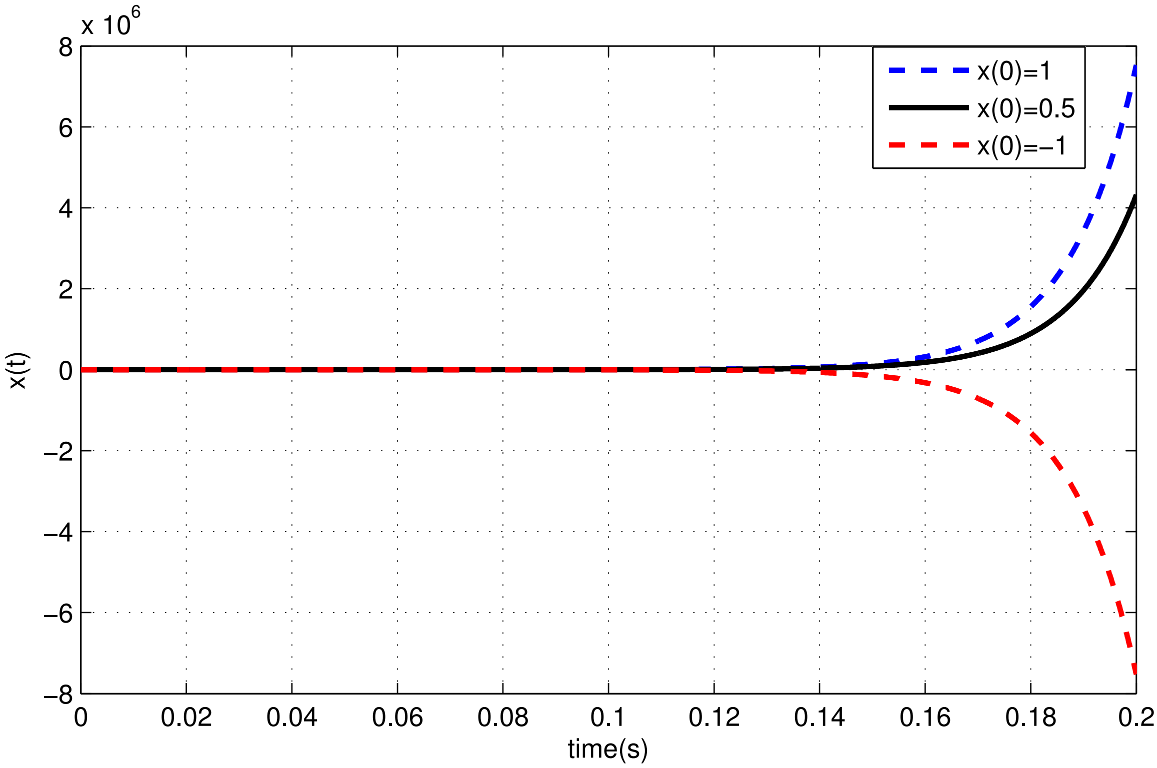

Figure 2: This figure shows the result of Theorem 6 for DIDE (2.3) different initial values, when $\tau = \frac{1}{5}$.

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