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On a topology induced by order convergence of monotone nets

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Abstract

In this paper, we will study on some topologies induced by order convergence in a Riesz space. We will investigate the relationships of them.

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1 Introduction

In general, by [4] we know that the order convergence is not topological in infinite dimensional Riesz spaces, researches are towards special cases or other definitions of convergence. For example, DeMarr proved in [5], that a locally convex space can be seen as an ordered vector space such that the τ-convergence of nets is equivalent to order convergence if and only if it is normable. Also, he proved in [6] that we can embed each locally convex space into an appropriate ordered vector space E such that its topological convergence and uo-convergence coincide. The authors of [10], characterized ordered normed spaces in which the order convergence of nets coincides with norm convergence. Also, they characterized ordered normed spaces in which the order convergence and norm convergence coincide. Chuchaev in [3] investigated ordered locally convex spaces, where the topological convergence agrees with order convergence of topologically eventually bounded nets.

In this paper, we study topology induced by monotone nets and investigate some of its properties. We also compare it with order topology defined on a Riesz space.

2 Preliminaries

Recall that a real vector space E (with elements x,y,...) is called an ordered vector space if E is partially ordered in such a manner that the vector space structure and order structure are compatible, that is to say, $x \leq y$ implies $x + z \leq y + z$ for every $z \in E$ and $x \geq y$ implies $\alpha x \geq \alpha y$ for every $\alpha \geq 0$ in $\mathbb{R}$. A Riesz space E is an order vector space in which $\sup\{x, y\}$ (it is customary to write sometimes $x \vee y$ instead of $\sup\{x, y\}$ and $x \wedge y$ instead of $\inf\{x, y\}$) exists for every $x, y \in E$. In a Riesz space E, two elements x and y are said to be disjoint (in symbols

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$x \perp y$) whenever $|x| \wedge |y| = 0$. If A is a nonempty subset of Riesz space E, then its disjoint complement A^d is defined by $A^d := \{x \in E : x \perp y \ for \ all \ y \in A\}$. Let E be a Riesz space, for each $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. A Riesz space is said to be Dedekind complete (resp. σ-Dedekind complete) if every order bounded above subset (resp. countable subset) has a supremum. A subset A of a Riesz space E is said to be solid if it follows from $|y| \leq |x|$ with $x \in A$ and $y \in E$ that $y \in A$. An order ideal of E is a solid subspace. A band of E is an order closed order ideal. A Banach lattice E is a Banach space $(E, \|.\|)$ such that E is a Riesz space and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A Banach lattice E has order continuous norm if $\|x_\alpha\| \to 0$ for every decreasing net $(x_\alpha)_\alpha$ with $\inf_\alpha x_\alpha = 0$. Recall that a net $(x_\alpha)_{\alpha \in \mathcal{A}}$ in a Riesz space E is *order convergent* to $x \in E$, denoted by $x_\alpha \xrightarrow{o} x$ whenever there exists another net $(y_\beta)_{\beta \in \mathcal{B}}$ in E such that $y_\beta \downarrow 0$ and that for every $\beta \in \mathcal{B}$, there exists $\alpha_0 \in \mathcal{A}$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_0$. If there exists a net $(y_\alpha)_{\alpha \in \mathcal{A}}$ (with the same index set) in a Riesz space E such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for each $\alpha \in \mathcal{A}$, then $x_\alpha \xrightarrow{o} x$. Conversely, if E is a Dedekind complete Riesz space and $(x_\alpha)_{\alpha \in \mathcal{A}}$ is order bounded, then $x_\alpha \xrightarrow{o} x$ in E implies that there exists a net $(y_\alpha)_{\alpha \in \mathcal{A}}$ (with the same index set) such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for each $\alpha \in \mathcal{A}$. For sequences in a Riesz space E, $x_n \xrightarrow{o} x$ if and only if there exists a sequence (y_n) such that $y_n \downarrow 0$ and $|x_n - x| \leq y_n$ for each $n \in \mathbb{N}$ (cf. [1, P.17 and P.18]). Recall that a Riesz subspace G of a Riesz space E is said to be order dense in E whenever for each $0 < x \in E$ there exists some $y \in G$ with $0 < y \leq x$. A net (x_α) in a Riesz space E is *unbounded order convergent* (or, *uo*-convergent for short) to $x \in E$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $u \in E^+$. We denote this convergence by $x_\alpha \xrightarrow{uo} x$ and write that x_α *uo*-convergent to x. This is an analogue of pointwise convergence in function spaces. Let $\mathbb{R}^A$ be the Riesz space of all real-valued functions on a non-empty set A, equipped with the pointwise order. It is easily seen that a net (x_α) in $\mathbb{R}^A$ *uo*-converges to $x \in \mathbb{R}^A$ if and only if it converges pointwise to x. For instance in c_0 and $\ell_p (1 \leq p \leq \infty)$, *uo*-convergence of nets is the same as coordinate-wise convergence. In Banach lattice E we write $x_\alpha \xrightarrow{un} x$ and say that (x_α) is *un*-convergent to x if $|x_\alpha - x| \wedge u \xrightarrow{\|\cdot\|} 0$ for every $u \in E^+$. A vector $x > 0$ in a Riesz space E is called an atom if $E_x = \{y \in E : \exists \lambda > 0, \ |y| \leq \lambda x\}$, the ideal generated by x, is one-dimensional if and only if $u, v \in [0, x]$ with $u \wedge v = 0$ implies $u = 0$ or $v = 0$. A Riesz space E is said to be atomic if the linear span of all atoms is order dense in E if and only if it is the band generated by its atoms. For example, c, c_0, $\ell_p (1 \leq p \leq \infty)$ are atomic Banach lattices and $C[0, 1]$, $L_1[0, 1]$ are atomless Banach lattices. Let E, F be Riesz spaces. An operator $T : E \to F$ is said to be order bounded if it maps each order bounded subset of E into order bounded subset of F. The collection of all order bounded operators from a Riesz space E into a Riesz space F will be denoted by $\mathcal{L}_b(E, F)$. The collection of all order bounded linear functionals on a Riesz space E will be denoted by $E^\sim$, that is $E^\sim = \mathcal{L}_b(E, \mathbb{R})$. A functional on a Riesz space is order continuous (resp. σ-order continuous) if it maps order null nets (resp. sequences) to order null nets (resp. sequences). The collection of all order continuous (resp. σ-order continuous) linear functionals on a Riesz space E will be denoted by $E_n^\sim$ (resp. $E_c^\sim$). For unexplained terminology and facts on Banach lattices and positive operators, we refer the reader to the excellent book of [2].

3 Order Topology

Definition 3.1. Let E be a Riesz space. A subset A of E is said to be quasi-order closed whenever for every $(x_\alpha) \subseteq A$ with $x_\alpha \uparrow x$ or $x_\alpha \downarrow x$ implies $x \in A$.

We observe that a solid subset $A \subseteq E$ is quasi-order closed if and only if A is order closed. If G is a quasi-order closed Riesz subspace of E, it is obvious that G is order dense in E. Clearly, if B is a band in Riesz space E, then it is quasi-order closed in E.

Remark 3.2. Let A be a quasi-order closed subset of E. If $x \in A$, then there is a net $(x_\alpha) \subseteq A$ with $x_\alpha \uparrow x$ or $x_\alpha \downarrow x$, so without loss of generality we assume that $x_\alpha \uparrow x$. Consider $|\lambda| \leq 1$. If $\lambda \geq 0$, then $\lambda x_\alpha \uparrow \lambda x$ and if $\lambda < 0$, then $\lambda x_\alpha \downarrow \lambda x$. Since A is quasi-order closed, we have $\lambda x \in A$. Hence A is a circled set.

Remark 3.3. Let G be a subspace of Riesz space E and $A \subseteq G \subseteq E$. If G is order dense in E, then A is quasi-order closed in G if and only if it is quasi-order closed in E.

Definition 3.4. $\theta \subseteq E$ is called order open if and only if $E \setminus \theta$ is quasi-order closed. Now, consider the following topologies:

1. First topology is called quasi-order topology which we define as follows.

$$\tau_o = \{\theta \subseteq E : \ E \setminus \theta \text{ is quasi-order closed}\}$$

   Clearly, τ_o is a topology for E.
2. Assume that τ_e be a topology for E with the following basis

$$\{(a, b) : \ a, b \in E \text{ and } a < b\}.$$

   We call this topology as order topology.

In the following proposition, we show that τ_o and τ_e are both linear topologies.

Theorem 3.5. Let E be a Dedekind complete Riesz space. Then

1. τ_o and τ_e both are linear topologies.
2. τ_e is a locally convex topology.

Proof .

1. Obviously, τ_e is a vector topology. We only show that τ_o is vector topology. First, we prove that the operation $x \to tx$ for each $t \in R$ is continuous. Let $\theta \subset E$ be an order open subset of E, then we must show that $t\theta$ is an order open subset of E for each $t \in R$. Since θ is order open, it follows that $\theta^c = F$ is quasi-order closed. Put $(t\theta)^c = G$ and $(x_\alpha) \subseteq G$ with $x_\alpha \uparrow x$. Then we have $x_\alpha \notin t\theta$ iff $t^{-1}x_\alpha \notin \theta$ iff $t^{-1}x_\alpha \in F$ for each α and since $t^{-1}x_\alpha \uparrow t^{-1}x$, follows that $t^{-1}x \in F$, implies that $x \in tF$. Then we have $t^{-1}x \in F$ iff $t^{-1}x \notin \theta$ iff $x \notin \theta$ iff $x \in G$, which follows that G is quasi order closed, and so $t\theta$ is an order open subset of E.
   Now we show that the operation $(x, y) \to x + y$ is continuous. Set θ_1 and θ_2 order open subsets of E, we show that $\theta_1 + \theta_2$ is an order open subset of E. Let $a \in \theta_1$. First we prove that $a + \theta_2$ is an order open subset of E. Put $\theta_2^c = F$ and $(a + \theta_2)^c = G$. We show that G is quasi-order closed. Let $(x_\alpha) \subseteq G$ and $x_\alpha \uparrow x$ in G. Then we have $x_\alpha \in G$ iff $x_\alpha \notin (a + \theta_2)$ iff $(x_\alpha - a) \notin \theta_2$. Since $(x_\alpha - a) \uparrow (x - a)$, follows that $(x - a) \in F$, and so $(x - a) \notin \theta_2$ iff $x \notin (a + \theta_2)$ iff $x \in G$. Thus G is quasi-order closed, and so $a + \theta_2$ is an order open subset of E. Now by $\theta_1 + \theta_2 = \bigcup_{a \in \theta_1}(a + \theta_2)$, the proof follows.
2. Let (a, b) belong to basis of τ_e, and let $0 \leq \lambda \leq 1$ and $x, y \in (a, b)$. It is clear that $a < \lambda x + (1 - \lambda)y < b$. Therefore, $\lambda x + (1 - \lambda)y \in (a, b)$. Hence τ_e is a locally convex topology.

□

Lemma 3.6. Let E be a Dedekind complete Riesz space and τ_o be the order topology for E. Then for each $c \in E$ and neighborhood U_c of c, there are $a, b \in E$ such that $c \in (a, b) \subset U_c$.

Proof . Let $c \in E$ and let U_c be an neighborhood of c in order topology. First we show that there is $a \in E$ such that $(a, c) \subset U_c$. By contradiction, let $(a, c) \cap U_c^c \neq \emptyset$. Then for each $a < c$ there is $c_a \in (a, c) \cap U_c^c$. It follows that

$$\sup\{c_a : \ c_a \in (a, c) \cap U_c^c\} = c.$$

For each $a < b < c$, we can set $c_a < c_b$. Hence for each $a < c$, there exists $c_{\alpha(a)} \in (a, c) \cap U_c^c$ with $c_{\alpha(a)} \uparrow c$. Therefore, $c \in U_c^c$, which is not possible. Thus, there is $a < x$ such that $(a, c) \subset U_c$. In the similar way there is a $c < b$ such that $(c, b) \subset U_c$ and proof follows. □

The preceding lemma shows that $\tau_o \subseteq \tau_e$, but as following example, in general two topologies not coincide.

Example 3.7. Consider $E = \ell^\infty$ and $e_1 = (1, 0, 0, 0...)$. Then $(-e_1, e_1)$ is a member of τ_e, but it is not belong to τ_o. Consider $x_n \in \ell^\infty$ which first n terms are zero and others are 1. Obviously, $x_n \downarrow 0$, but $x_n \notin (-e_1, e_1)$ for each n. This example shows that the sequence (x_n) is order convergent to zero, but is not topological convergence to zero. On the other hand, since $(-e_1, e_1) \notin \tau_o$, two topologies not coincide.

Theorem 3.8. Let E be a Dedekind complete Riesz space with topology τ_e and $(x_\alpha) \subset E$. If $x_\alpha \xrightarrow{\tau_e} x$ for some $x \in E$, then (x_α) is order convergence to x.

Proof . Assume that $a, b \in E$ with $x \in (a, b) \subseteq E$. Since $x_\alpha \xrightarrow{\tau_e} x$, there exists $\alpha_{(a,b)}$ such that $x_\alpha \in (a, b)$ for each $\alpha \geq \alpha_{(a,b)}$. Put $y_{\alpha_{(a,b)}} = |b - a|$. On the other hands, $(\alpha_{(a,b)})$ is a directed set with the following order relation

$$\alpha_{(a,b)} \leq \alpha_{(c,d)} \text{ iff } (c, d) \subseteq (a, b).$$

It follows that

$$|x_\alpha - x| \leq (x_\alpha \vee x) - (x_\alpha \wedge x) \leq b - a = y_{\alpha_{(a,b)}} \downarrow 0.$$

Thus $x_\alpha \xrightarrow{o} x$. □

Remark 3.9. By Example 3.7, the converse of Theorem 3.8 in general not holds.

Corollary 3.10. If E is a Dedekind complete Riesz space, then the following assertions are true.

1. If E is normed Riesz space with order continuous norm, then $x_\alpha \xrightarrow{\tau_e} 0$, implies $x_\alpha \xrightarrow{\|\cdot\|} 0$.
2. τ_e-limits are unique.
3. τ_e is a Hausdorff topology.

Proof .

1. Let E be a Dedekind complete normed Riesz space, $(x_\alpha) \subseteq E$ and $x_\alpha \xrightarrow{\tau_e} 0$. By Theorem 3.8, $x_\alpha \xrightarrow{o} 0$. So, by assumption $x_\alpha \xrightarrow{\|\cdot\|} 0$.
2. Let E be a Dedekind complete Riesz space, $(x_\alpha) \subseteq E$ that $x_\alpha \xrightarrow{\tau_e} x$ and $x_\alpha \xrightarrow{\tau_e} y$. By Theorem 3.8, $x_\alpha \xrightarrow{o} x$ and $x_\alpha \xrightarrow{o} y$. Since order convergence is unique, so $x = y$.
3. Since τ_e-limits are unique in Dedekind complete Riesz space E, therefore for each $x \in E$, $\{x\}$ is τ_e-closed. Hence τ_e is a vector topology on E, and (E, τ_e) is a topological vector space. By Theorem 1.12 of [9], τ_e is a Hausdorff topology in Dedekind complete Riesz space E.

□

Remark 3.11. Let E be a Dedekind complete Riesz space. τ_e is a vector Hausdorff topology on E which is locally convex.

Proposition 3.12. Let E be a Riesz space and τ_o be the order topology for E. If I is an ideal and quasi-order closed subset of E, then the following assertions are true.

1. $I^d = \{0\}$.
2. If E is a Dedekind complete Riesz space, then I is a band in E.

Proof .

1. Since I is a quasi-order closed subspace of E, it is order dense in E. By Theorem 1.36 of [2], $I^d = \{0\}$.
2. Let $(x_\alpha) \subseteq I$ and $x_\alpha \xrightarrow{o} x$, we show that $x \in I$. We know that, $\sup\{x_\alpha \wedge x\} = x$. Set $y_\beta = (\bigvee_{\alpha \leqslant \beta} x_\alpha) \wedge x$, then $y_\beta \uparrow x$. Since $(y_\beta) \subseteq I$ and I is quasi-order closed, hence $x \in I$ and the result follows.

□

It was observed in [7] that *un*-convergence is topological. Also, *uaw*-topology, τ_{uaw}, and *wun*-topology, τ_{wun}, are introduced in [11] and [8], respectively. If E is atomic with order continuous norm, it has been shown that τ_{un} and τ_{wun} are two locally convex topologies.

Remark 3.13. Let E be a Dedekind complete Riesz space, then

1. if E is Banach lattice with order continuous norm, τ_{un} is weaker than τ_e. Let $(x_\alpha) \subseteq E$ and $x_\alpha \xrightarrow{\tau_e} 0$. By Theorem 3.8, $x_\alpha \xrightarrow{o} 0$ and therefore $x_\alpha \xrightarrow{uo} 0$. Since E has order continuous norm, hence $x_\alpha \xrightarrow{un} 0$. It means that $\tau_{un} \subseteq \tau_e$. $(e_n) \subseteq \ell^1$ is *un*-null, while is not *o*-null. By Theorem 3.8, it is not τ_e-null. Therefore, $\tau_e \neq \tau_{un}$.

2. in E, τ_e and τ_o both are locally convex. Therefore, τ_w is weaker than τ_e and τ_o. Let E be a Banach lattice that E and the norm dual E^* have order continuous norms and (x_α) be a norm bounded net in E that is not w-null. By Theorem 6.4 of [7], it is not un-null. Obviously, (x_α) is not $\|.\|$-null. Since E has order continuous norm, then it is not o-null. By Theorem 3.8, it is not τ_e-null. It means that E and E^* have order continuous norms and $(x_\alpha) \subseteq E$ is norm bounded and τ_e-null, then is w-null.
3. if $(x_\alpha) \subseteq E$ and τ_e-null or τ_o-null. By 2, (x_α) is w-null. Therefore, it is uaw-null and wun-null. Hence τ_{uaw} and τ_{wun} are weaker than τ_e and τ_o. Note that $(e_n) \subseteq \ell^1$ is wun-null and uaw-null, while it is not τ_e-null. Therefore $\tau_e \neq \tau_{wun}$ and $\tau_e \neq \tau_{uaw}$. Similar to Part 2, if E and E^* have order continuous norms, $(x_\alpha) \subseteq E$ is norm bounded and τ_e-null, then it is wun-null and uaw-null.

Proposition 3.14. Let E be a Dedekind complete normed Riesz space, $(x_\alpha) \subseteq E$ and $x_\alpha \xrightarrow{\tau_e} x$, then $\|x\| \leq \liminf_\alpha \|x_\alpha\|$.

Proof . Let $(x_\alpha) \subseteq E$ where $x_\alpha \xrightarrow{\tau_e} x$. By Remark 3.13, $x_\alpha \xrightarrow{w} x$. By Exercise 4 of page 167 of [2], $\|x\| \leq \liminf_\alpha \|x_\alpha\|$. □

Remark 3.15. Note that if E is a Dedekind complete Banach lattice with order continuous norm, $(x_\alpha) \subseteq E$ and $x_\alpha \xrightarrow{\tau_e} x$, then by Theorem 3.8, $x_\alpha \xrightarrow{o} x$ and therefore $x_\alpha \xrightarrow{un} x$. Hence, $|x_\alpha| \wedge |x| \xrightarrow{\|\cdot\|} |x|$. It follows that $\|x\| \leq \liminf_\alpha \|x_\alpha\|$.

Let E be a Dedekind complete Riesz space. $A \subseteq E$ is said to be τ_e-bounded if to every τ_e-neighborhood V of 0 in E corresponds a number $s > 0$ such that $A \subseteq tV$ for every $t \geq s$. An operator $T : E \to F$ between two Dedekind complete Riesz spaces is said to be τ_e-τ_e-continuous, if $x_\alpha \xrightarrow{\tau_e} 0$ in E implies $T(x_\alpha) \xrightarrow{\tau_e} 0$ in F for each net $(x_\alpha) \subseteq E$.

It is clear that if $T : E \to F$ is τ_e-τ_e-continuous and A is a τ_e-bounded subset of E, then $T(A)$ is τ_e-bounded in F.

Proposition 3.16. Let E and F be two Dedekind complete normed Riesz spaces and $T : E \to F$ be τ_e-τ_e-continuous, then $|T|$ exists and is τ_e-τ_e-continuous.

Proof . By assumption, (E, τ_e) is locally convex-solid Riesz space. Let $A \subseteq E$ be order bounded set. By Theorem 3.47 of [2], A is τ_e-bounded. Since T is τ_e-τ_e-continuous, therefore $T(A)$ is τ_e-bounded. By Theorem 3.8, $T(A)$ is order bounded. It means that T is an order bounded operator. By Theorem 1.18 of [2], $|T|$ exists. Let $(x_\alpha) \subseteq E$ be τ_e-null. It is obvious that (x_α) is τ_e-bounded. By Theorem 3.8, it is order bounded. Therefore, $|T|(A)$ is order bounded. Now by Theorem 3.47 of [2], $|T|(A)$ is τ_e-bounded. Hence $|T|$ is τ_e-τ_e-continuous. □

Corollary 3.17. If E and F are two Dedekind complete normed Riesz spaces, then T is τ_e-τ_e-continuous if and only if T is order bounded.

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