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On the convergence of new algorithms procedures in Banach spaces

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Abstract

In this paper, a new algorithms type three-step via projection Jungck Suzuki generalized mappings are introduced, and the convergence of projection Jungck- Zenor algorithm and projection Jungck P-algorithm are proved. On the other hand, we proved that the projection Jungck-Zenor algorithm converges to a common fixed point faster than of projection Jungck P- algorithm in Banach spaces.

Keywords: projection Mappings, Jungck algorithms, Rate of convergence
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1 Introduction

The fixed point theory is a wonderful, and exciting topic for mathematics. In many fields such as general and algebraic topology, mathematical economics, approximation theory, game theory and etc. The scientist Luitzen Brouwer presented in 1912 [6] a theory named after him the Brouwer theory, which is considered one of the oldest theorems of the fixed point

Scientists also developed the concept of this theory until the polish mathematician S. Banach presented in [4] a basic result after this theory called the Banach contraction principle. This theory has a great benefit and is considered one of the most important results that were used in the analysis, as it played a major and important role in studying many problems in different fields, see [3, 13, 17, 16]. Finding fixed point values is not easy but rather complicated, therefore researchers have resorted to discovering algorithm methods to find their value. In 1953 Mann introduced a one-step algorithm called the Mann algorithm [11]. In [9] the scientist Ishikawa proposed a two-step Ishikawa algorithm. Noor in [12], presented a three-step algorithm defined as:
$y_0 \in Y$
$y_{n+1} = (1-\alpha_n) y_n + \alpha_n T(\mathrm{i}_n)$
$\mathrm{i}_n = (1-\beta_n) y_n + \beta_n T(\mathrm{j}_n)$
$\mathrm{j}_n = (1-\gamma_n) y_n + \gamma_n T(y_n)$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ lie in $[0,1]$. On the other hands, Sainuan in [14] defined a three-step algorithm as follows :
$h_{n+1} = (1-\alpha_n) To_n + \alpha_n Tl_n$

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$l_n = (1 - \beta_n) o_n + \beta_n T o_n$
$o_n = (1 - \gamma_n) h_n + \gamma_n T h_n$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ lie in $[0, 1]$.

Sainuan bears out the convergence theorem for the P - algorithm and also proved that this algorithm converges faster than S and Ishikawa algorithm.it was Jungck who first introduced the algorithms scheme, named after him Jungck iteration [10] to approximate common fixed points now known as Jungck contraction maps. Suzuki [18] likewise presented a type of map that fulfils condition (C). In addition, researchers Dhompongsa and Kaewcharoen developed this concept and got results that contribute to the fixed point theory that satisfies the condition (C), see [7, 8, 15].

Definition 1.1. [10] Any mappings $T, \mathcal{S} : C \to X$ are called Jungck contraction if $L \in (0, 1)$ such that

$$\|Tx - Ty\| \le L\|\mathcal{S}x - \mathcal{S}y\| \forall x, y \in C.$$

Definition 1.2. [1] Any mappings $T, \mathcal{S} : C \to X$ are called Jungck non-expansive if we have

$$\|Tx - Ty\| \le \|\mathcal{S}x - \mathcal{S}y\| \forall x, y \in C.$$

Definition 1.3. [5] Let $\{a_n\}$ and $\{d_n\}$ be sequences convergent to a and d, respectively. Then, $\{a_n\}$ converges faster than $\{d_n\}$ if $\lim_{n\to\infty} \left| \frac{a_n - a}{d_n - d} \right| = 0$.

Lemma 1.4. [2] Let $\{x_n\}$ and $\{y_n\}$ be two sequence in X such that $\lim_{n\to\infty} \sup x_n \le c, \lim_{n\to\infty} \sup y_n \le c$ and $\lim_{n\to\infty} \sup \|t_n x_n + (1 - t_n) y_n\| = c$, holds for some $c \ge 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$

2 Main Results

In this section, we define a new type of mapping, such as projection Jungck Suzuki generalized mapping and introduce iterative methods type three step. Also, we study the rate of convergence on these methods.

Definition 2.1. Let C be a non-empty subset of a normed space X. Then $T : C \to X$ and $\mathcal{P}_c : X \to C$ are called projection Jungck Suzuki Generalized mapping if
$\frac{1}{2} \|x - \mathcal{P}_c(x)\| \le \|\mathcal{S}x - \mathcal{S}y\|$ implies that $\|\mathcal{P}_c(x) - \mathcal{P}_c(y)\| \le L\|\mathcal{S}x - \mathcal{S}y\| + \phi (\|x - \mathcal{P}_c(x)\| + \|x - \mathcal{S}x\|)$, where $L \in (0, 1)$ and $\phi : R^+ \to R^+$such that $\phi(0) = 0$

Definition 2.2. Let $T, \mathcal{S} : C \to X$ and $\mathcal{P}_c : X \to C$.Then we define the following algorithms:

1. The projection Jungck P-algorithm
   For $h_0 \in C$ the sequence $\{h_n\}$ is defied by
   $\mathcal{S}h_{n+1} = (1 - \alpha_n) \mathcal{P}_c T o_n + \alpha_n \mathcal{P}_c T l_n$
   $\mathcal{S}l_n = (1 - \beta_n) \mathcal{S}\mathcal{P}_c (o_n) + \beta_n \mathcal{P}_c T o_n$
   $\mathcal{S}o_n = (1 - \gamma_n) \mathcal{S}\mathcal{P}_c (h_n) + \gamma_n \mathcal{P}_c T h_n$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ lies in $[0, 1]$
2. The projection Jungck Zenor-algorithm
   For $c_0 \in C$ the sequence $\{c_n\}$ is defied by
   $\mathcal{S}c_{n+1} = \mathcal{P}_c T d_n$
   $\mathcal{S}d_n = (1 - \alpha_n) \mathcal{P}_c T e_n + \alpha_n \mathcal{G} T e_n$
   $\mathcal{S}e_n = (1 - \beta_n) \mathcal{S}\mathcal{P}_c (c_n) + \beta_n T c_n$, where $\mathcal{G}$ is nonexpansive function and $\{\alpha_n\}, \{\beta_n\}$ lie in $[0, 1]$.

A mapping $\mathcal{S}\mathcal{P}_c$ is commute, i.e., $\mathcal{S}\mathcal{P}_c (x_n) = \mathcal{P}_c \mathcal{S} (x_n)$.

Lemma 2.3. Let C be a non-empty closed convex subset of a Banach space X. $T : C \to X$ be a projection Jungck Suzuki Generalized if $\{\mathcal{S}c_n\}$ Generated by projection Jungck Zenor algorithm such that
$0 \le \lambda \le \beta_n, \alpha_n \le 1$ for all $n \in \mathbb{N}$ then

1. $\lim_{n\to\infty} \|\mathcal{S}c_n - t\|$ exists for all $t \in F (\mathcal{P}_c, T)$
2. $\lim_{n\to\infty} \|\mathcal{P}_c T e_n - \mathcal{G}\mathcal{T}\rceil, \| =, 0$ and $\lim_{n\to\infty} \|\mathcal{P}_c \mathcal{S} (c_n) - \mathcal{S}c_n\| = 0$

Proof . Let $t \in \mathcal{F}(\mathcal{P}_c, T)$, where $\mathcal{F}(\mathcal{P}_c, T)$ is a family of common fixed point, then

$$\begin{aligned}
\|\mathcal{S}c_{n+1} - t\| &= \|\mathcal{P}_c T d_n - t\| \\
&\leq \|T d_n - t\| \\
&\leq L\|\mathcal{S}d_n - t\| + \phi\left(\|t - \mathcal{P}_c(t)\| + \|t - \mathcal{S}t\|\right) \\
&\leq L\|\mathcal{S}d_n - t\| \\
&\leq \|\mathcal{S}d_n - t\| \\
&= \|(1-\alpha_n)\mathcal{P}_c T e_n + \alpha_n \mathcal{G} T e_n - (1 - \alpha_n + \alpha_n)t\| \\
&= \|(1-\alpha_n)(\mathcal{P}_c T e_n - \mathcal{S}) + \alpha_n(\mathcal{G} T e_n - t)\| \\
&\leq [(1-\alpha_n)\|\mathcal{P}_c T e_n - t\| + \alpha_n\|\mathcal{G} T e_n - t\|] \\
&\leq [(1-\alpha_n)\|\mathcal{P}_c T e_n - t\| + \alpha_n\|T e_n - t\|] \\
&\leq [(1-\alpha_n)\|T e_n - t\| + \alpha_n\|T e_n - t\|] \\
&\leq \|T e_n - t\| \\
&\leq L\|\mathcal{S}e_n - t\| + \phi\left(\|t - \mathcal{P}_c(t)\| + \|t - \mathcal{S}t\|\right) \\
&\leq L\|\mathcal{S}e_n - t\| \\
&\leq \|\mathcal{S}e_n - t\| \qquad (2.1)
\end{aligned}$$

Now,

$$\begin{aligned}
\|\mathcal{S}e_n - t\| &= \|(1-\beta_n)\mathcal{S}\mathcal{P}_c(c_n) + \beta_n T c_n - (1 - \beta_n + \beta_n)t\| \\
&= \|(1-\beta_n)(\mathcal{S}\mathcal{P}_c(c_n) - t) + \beta_n(T c_n - t)\| \\
&\leq (1-\beta_n)\|\mathcal{S}\mathcal{P}_c(c_n) - t\| + \beta_n\|T c_n - t\| \\
&= (1-\beta_n)\|\mathcal{P}_c\mathcal{S}(c_n) - t\| + \beta_n\|T c_n - t\| \\
&\leq (1-\beta_n)\|\mathcal{S}c_n - t\| + \beta_n\|T c_n - t\| \\
&\leq (1-\beta_n)\|\mathcal{S}c_n - t\| + L\beta_n\|\mathcal{S}c_n - t\| + \phi\left(\|t - \mathcal{P}_c(t)\| + \|t - \mathcal{S}t\|\right) \\
&\leq (1-\beta_n)\|\mathcal{S}c_n - t\| + L\beta_n\|\mathcal{S}c_n - t\| \\
&\leq (1-\beta_n)\|\mathcal{S}c_n - t\| + \beta_n\|\mathcal{S}c_n - t\| \\
&\leq \|\mathcal{S}c_n - t\| \qquad (2.2)
\end{aligned}$$

Substitute Equation (2.2) in to Equation (2.1)

$$\begin{aligned}
&\|\mathcal{S}c_{n+1} - t\| \leq \|\mathcal{S}c_n - t\| \\
\text{So, } &\|\mathcal{S}c_{n+1} - t\| \leq \|\mathcal{S}c_n - t\| \Longrightarrow \{\mathcal{S}c_n\} \text{ is non-increasing} \qquad (2.3) \\
&\qquad\qquad\quad \leq \|\mathcal{S}c_{n-1} - t\|
\end{aligned}$$

$$\|\mathcal{S}c_{n+1} - t\| \leq \|\mathcal{S}c_0 - t\| \Longrightarrow \{\mathcal{S}c_n\} \text{ is bounded} \qquad (2.4)$$

From (2.3) and (2.4) we get

$$\lim_{n\to\infty}\|\mathcal{S}c_n - t\| \text{ is exists.}$$

Now to prove,

$$\lim_{n\to\infty}\|\mathcal{P}_c T e_n - \mathcal{G} T e_n\| = 0 \text{ and } \lim_{n\to\infty}\|\mathcal{P}_c\mathcal{S}(c_n) - \mathcal{S}c_n\| = 0$$

$$\text{Let, } c \in R;\ \lim_{n\to\infty}\|\mathcal{S}c_n - t\| = c \Longrightarrow \lim_{n\to\infty}\sup\|\mathcal{S}c_n - t\| \leq c \qquad (2.5)$$

Now, we apply the three condition To prove that, $\lim_{n\to\infty}\sup\|\mathcal{P}_cTe_n - t\| \le c$

$$\begin{aligned}
\lim_{n\to\infty}\sup\|\mathcal{P}_cTe_n - t\| &\le \lim_{n\to\infty}\sup\|Te_n - t\| \\
&\le \lim_{n\to\infty}\sup\left[L\|\mathcal{S}e_n - t\| + \phi\left(\|t-\mathcal{P}_c(t)\| + \|t-\mathcal{S}t\|\right)\right] \\
&\le \lim_{n\to\infty}\sup\left[L\|\mathcal{S}e_n - t\|\right] \\
&\le \lim_{n\to\infty}\sup\|\mathcal{S}e_n - t\| \\
&\le \lim_{n\to\infty}\sup\|(1-\beta_n)\mathcal{S}\mathcal{P}_c(c_n) + \beta_n Tc_n - (1-\beta_n+\beta_n)t\| \\
&\le \lim_{n\to\infty}\sup\|(1-\beta_n)(\mathcal{S}\mathcal{P}_c(c_n) - t) + \beta_n(Tc_n - t)\| \\
&\le \lim\sup\left[(1-\beta_n)\|\mathcal{S}\mathcal{P}_c(c_n) - t\| + \beta_n\|Tc_n - t\|\right] \\
&\le \lim_{n\to\infty}\sup\left[(1-\beta_n)\|\mathcal{P}_c\mathcal{S}(c_n) - t\| + \beta_n\|Tc_n - t\|\right] \\
&\le \lim_{n\to\infty}\sup\left[(1-\beta_n)\|\mathcal{S}c_n - t\| + \beta_n\|Tc_n - t\|\right] \\
&\le \lim_{n\to\infty}\sup\left[(1-\beta_n)\|\mathcal{S}c_n - t\| + L\beta_n\|\mathcal{S}c_n - t\| + \phi\left(\|t-\mathcal{P}_c(t)\| + \|t-\mathcal{S}t\|\right)\right] \\
&\le \lim_{n\to\infty}\sup\left[(1-\beta_n)\|\mathcal{S}c_n - t\| + L\beta_n\|\mathcal{S}c_n - t\|\right] \\
&\le \lim_{n\to\infty}\sup\left[(1-\beta_n)\|\mathcal{S}c_n - t\| + \beta_n\|\mathcal{S}c_n - t\|\right] \\
&\le \lim_{n\to\infty}\sup\|\mathcal{S}c_n - t\| \\
&\le c.
\end{aligned}$$

So,

$$\lim_{n\to\infty}\sup\|\mathcal{P}_cTe_n - t\| \le c. \tag{2.6}$$

To proof that, $\lim_{n\to\infty}\sup\|\mathcal{P}_c\mathcal{S}(c_n) - t\| \le c$, we have

$$\begin{aligned}
\lim_{n\to\infty}\sup\|\mathcal{P}_c\mathcal{S}(c_n) - t\| &\le \lim_{n\to\infty}\sup\|\mathcal{S}c_n - t\| \,| \\
&\le c.
\end{aligned}$$

So,

$$\lim_{n\to\infty}\operatorname{Sup}\|\mathcal{P}_c\mathcal{S}(c_n) - t\| \le c. \tag{2.7}$$

To prove that, $\lim_{n\to\infty}\sup\|\mathcal{G}Te_n - t\| \le c$,

$$\begin{aligned}
\lim_{n\to\infty}\sup\|\mathcal{G}Te_n - t\| &\le \lim_{n\to\infty}\sup\|Te_n - t\| \\
&\le \lim_{n\to\infty}\sup\left[L\|\mathcal{S}e_n - t\| + \phi\left(\|t-\mathcal{P}_c(t)\| + \|t-\mathcal{S}t\|\right)\right] \\
&\le \lim_{n\to\infty}\sup\left[L\|\mathcal{S}e_n - t\|\right] \\
&\le \lim_{n\to\infty}\sup\|\mathcal{S}e_n - t\| \\
&\le \lim_{n\to\infty}\sup\|(1-\beta_n)\mathcal{S}\mathcal{P}_c(c_n) + \beta_n Tc_n - (1-\beta_n+\beta_n)t\| \\
&\le \lim_{n\to\infty}\sup\|(1-\beta_n)(\mathcal{S}\mathcal{P}_c(c_n) - t) + \beta_n(Tc_n - t)\| \\
&\le \lim_{n\to\infty}\sup\left[(1-\beta_n)\|\mathcal{S}\mathcal{P}_c(c_n) - t\| + \beta_n\|Tc_n - t\|\right] \\
&\le \lim_{n\to\infty}\sup\left[(1-\beta_n)\|\mathcal{P}_c\mathcal{S}(c_n) - t\| + \beta_n\|Tc_n - t\|\right] \\
&\le \lim_{n\to\infty}\sup\left[(1-\beta_n)\|\mathcal{S}c_n - t\| + \beta_n\|Tc_n - t\|\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \lim_{n\to\infty} \sup \left[(1-\beta_n)\|\mathcal{S}c_n - t\| + L\beta_n \|\mathcal{S}c_n - t\| + \phi\left(\|t - \mathcal{P}_c(t)\| + \|t - \mathcal{S}t\|\right)\right]\\
&\leq \lim_{n\to\infty} \sup \left[(1-\beta_n)\|\mathcal{S}c_n - t\| + L\beta_n \|\mathcal{S}c_n - t\|\right]\\
&\leq \lim_{n\to\infty} \sup \left[(1-\beta_n)\|\mathcal{S}c_n - t\| + \beta_n \|\mathcal{S}c_n - t\|\right]\\
&\leq \lim_{n\to\infty} \sup \|\mathcal{S}c_n - t\|\\
&\leq c.
\end{aligned}$$

So,

$$\lim_{n\to\infty} \sup \|\mathcal{G}Te_n - t\| \leq c. \tag{2.8}$$

Now,

$$\begin{aligned}
c &= \lim_{n\to\infty} \sup \|\mathcal{S}c_{n+1} - t\|\\
&= \lim_{n\to\infty} \sup \|\mathcal{P}_c Td_n - t\|\\
&\leq \lim_{n\to\infty} \sup \|Td_n - t\|\\
&\leq \lim_{n\to\infty} \sup \left[L\|\mathcal{S}d_n - t\| + \phi\left(\|t - \mathcal{P}_c(t)\| + \|t - \mathcal{S}t\|\right)\right]\\
&\leq \lim_{n\to\infty} \sup \left[L\|\mathcal{S}d_n - t\|\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \lim_{n\to\infty} \sup \|\mathcal{S}d_n - t\|\\
&= \lim_{n\to\infty} \sup \|(1-\alpha_n)\mathcal{P}_c Te_n + \alpha_n \mathcal{G}Te_n - (1-\alpha_n+\alpha_n)t\|\\
&= \lim_{n\to\infty} \sup \|(1-\alpha_n)(\mathcal{P}_c Te_n - t) + \alpha_n(\mathcal{G}Te_n - t)\| \qquad (2.9)\\
&\leq \lim_{n\to\infty} \sup \left[(1-\alpha_n)\|\mathcal{P}_c Te_n - t\| + \alpha_n\|\mathcal{G}Te_n - t\|\right]\\
&\leq \lim_{n\to\infty} \sup \left[(1-\alpha_n)\|Te_n - t\| + \alpha_n\|Te_n - t\|\right]\\
&\leq \lim_{n\to\infty} \sup \|Te_n - t\|\\
&\leq \lim_{n\to\infty} \sup \left[L\|\mathcal{S}e_n - t\| + \phi\left(\|t - \mathcal{P}_c(t)\| + \|t - \mathcal{S}t\|\right)\right]\\
&\leq \lim_{n\to\infty} \sup \left[L\|\mathcal{S}e_n - t\|\right]\\
&\leq \lim_{n\to\infty} \sup \|\mathcal{S}e_n - t\|\\
&= \lim_{n\to\infty} \sup \|(1-\beta_n)\mathcal{S}\mathcal{P}_c(c_n) + \beta_n Tc_n - (1-\beta_n+\beta_n)t\|\\
&= \lim_{n\to\infty} \sup \|(1-\beta_n)(\mathcal{S}\mathcal{P}_c(c_n) - t) + \beta_n(Tc_n - t)\|\\
&\leq \lim_{n\to\infty} \sup \left[(1-\beta_n)\|\mathcal{S}\mathcal{P}_c(c_n) - t\| + \beta_n\|Tc_n - t\|\right]\\
&= \lim_{n\to\infty} \sup \left[(1-\beta_n)\|\mathcal{P}_c\mathcal{S}(c_n) - t\| + \beta_n\|Tc_n - t\|\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \lim_{n\to\infty} \sup \left[(1-\beta_n)\|\mathcal{P}_c\delta(c_n) - t\| + L\beta_n\|\mathcal{S}c_n - t\|\right]\\
&\leq \lim_{n\to\infty} \sup \left[(1-\beta_n)\|\mathcal{P}_c\delta(c_n) - t\| + \beta_n\|\delta c_n - t\|\right] \qquad (2.10)\\
&\leq \lim_{n\to\infty} \sup \left[(1-\beta_n)\|\mathcal{S}c_n - t\| + \beta_n\|\mathcal{S}c_n - t\|\right]\\
&= \lim_{n\to\infty} \sup \|\delta c_n - t\|\\
&= c
\end{aligned}$$

So, $\lim_{n\to\infty} \sup \|(1-\alpha_n)(\mathcal{P}_c Te_n - t) + \alpha_n(\mathcal{G}Te_n - t)\| = c$. From (2.6), (2.8), (2.9) and by using lemma (1.4) we get

$$\lim_{n\to\infty} \|\mathcal{P}_c Te_n - \mathcal{G}Te_n\| = 0.$$

Also, $\lim_{n\to\infty}\sup\left[(1-\beta_n)\left\|\mathcal{P}_c\mathcal{S}\left(c_n\right)-t\right\|+\beta_n\left\|\mathcal{S}c_n-t\right\|\right]=c$. From (2.5), (2.7), (2.10) and by using lemma (1.4) we get

$$\lim_{n\to\infty}\left\|\mathcal{P}_c\mathcal{S}\left(c_n\right)-\mathcal{S}c_n\right\|=0$$

□

Lemma 2.4. Let $T:C\to X$ be a projection Jungck Suzuki generalized. If $\{\mathcal{S}h_n\}$ generated by projection Jungck $-P$ algorithm such that $0\le\lambda\le\beta_n,\alpha_n\le 1$, for all $n\in\mathbb{N}$, then

1. $\lim_{n\to\infty}\left\|\mathcal{S}h_n-t\right\|$ exists for all $t\in\mathcal{F}\left(\mathcal{P}_c,T\right)$
2. $\lim_{n\to\infty}\left\|\mathcal{P}_cTo_n-\mathcal{P}_cTl_n\right\|=0$ and $\lim_{n\to\infty}\left\|\mathcal{P}_c\mathcal{S}h_n-\mathcal{S}h_n\right\|=0$

Theorem 2.5. Let X be a normed space, T be a projection Jungck Suzuki generalized mapping and $\mathcal{F}\left(T,\mathcal{S},\mathcal{P}_c\right)\neq\phi$. Then the projection Jungck Zenor algorithm converge faster than projection Jungck P-algorithm.

Proof . For Projection Jungck- Zenor algorithm

$$\begin{aligned}
\left\|\mathcal{S}c_{n+1}-t\right\| &= \left\|\mathcal{P}_cTd_n-t\right\|\\
&\le \left\|Td_n-t\right\|\\
&\le L\left\|\mathcal{S}d_n-t\right\|+\phi\left(\left\|t-\mathcal{P}_c(t)\right\|+\left\|t-\mathcal{S}t\right\|\right)\\
&\le L\left\|\mathcal{S}d_n-t\right\|\\
&= L\left\|(1-\alpha_n)\,\mathcal{P}_cTe_n+\alpha_n\mathcal{G}Te_n-(1-\alpha_n+\alpha_n)\,t\right\|\\
&= L\left\|(1-\alpha_n)\left(\mathcal{P}_cTe_n-t\right)+\alpha_n\left(\mathcal{G}Te_n-t\right)\right\|\\
&\le L\left[(1-\alpha_n)\left\|\mathcal{P}_cTe_n-t\right\|+\alpha_n\|\mathcal{G}Te_n-t\|\right]\\
&\le L\left[(1-\alpha_n)\left\|\mathcal{P}_cTe_n-t\right\|+\alpha_n\left\|Te_n-t\right\|\right]\\
&\le L\left[(1-\alpha_n)\left\|Te_n-t\right\|+\alpha_n\left\|Te_n-t\right\|\right]\\
&\le L\left[L\left(1-\alpha_n\right)\left\|Se_n-t\right\|+\phi\left(\left\|t-\mathcal{P}_c(t)\right\|+\left\|t-\mathcal{S}t\right\|\right)+L\alpha_n\left\|Se_n-t\right\|\right.\\
&\qquad\left.+\phi\left(\left\|t-\mathcal{P}_c(t)\right\|+\left\|t-\mathcal{S}t\right\|\right)\right]\\
&\le L\left[L\left(1-\alpha_n\right)\left\|Se_n-t\right\|+L\alpha_n\left\|Se_n-t\right\|\right]\\
&\le L\left[L-L\alpha_n+L\alpha_n\right]\left\|Se_n-t\right\|\\
&\le L^2\left\|\mathcal{S}e_n-t\right\|. \qquad (2.11)
\end{aligned}$$

Now

$$\begin{aligned}
&\left\|\mathcal{S}e_n-t\right\|=\left\|(1-\beta_n)\,\mathcal{S}\mathcal{P}_c\left(c_n\right)+\beta_nTc_n-(1-\beta_n+\beta_n)\,t\right\|\\
&=\left\|(1-\beta_n)\left(\mathcal{S}\mathcal{P}_c\left(c_n\right)-t\right)+\beta_n\left(Tc_n-t\right)\right\|\\
&\le(1-\beta_n)\left\|\mathcal{S}\mathcal{P}_c\left(c_n\right)-t\right\|+\beta_n\left\|Tc_n-t\right\|\\
&=(1-\beta_n)\left\|\mathcal{P}_c\mathcal{S}\left(c_n\right)-t\right\|+\beta_n\left\|Tc_n-t\right\|\\
&\le(1-\beta_n)\left\|\mathcal{S}c_n-t\right\|+\beta_n\left\|Tc_n-t\right\|\\
&\le(1-\beta_n)\left\|\mathcal{S}c_n-p\right\|+L\beta_n\left\|\mathcal{S}c_n-t\right\|+\phi\left(\left\|t-\mathcal{P}_c(t)\right\|+\left\|t-\mathcal{S}t\right\|\right)\\
&\le(1-\beta_n)\left\|\mathcal{S}c_n-t\right\|+L\beta_n\left\|\mathcal{S}c_n-t\right\|\\
&\le\left(1-\beta_n(1-L)\right)\left\|\mathcal{S}c_n-t\right\|\\
&\le\left(1-\lambda(1-L)\right)\left\|\mathcal{S}c_n-t\right\|. \qquad (2.12)
\end{aligned}$$

Substitute Equation (2.12) in to Equation (2.11)

$$\begin{aligned}
\left\|\mathcal{S}c_{n+1}-t\right\| &\le L^2[1-\lambda(1-L)]\left\|\mathcal{S}c_n-t\right\|\\
&\le L^{2n}[1-\lambda(1-L)]^n\left\|\mathcal{S}c_0-t\right\|\\
\text{Put,}\mathcal{P}\cdot\mathcal{J}\cdot\mathcal{Z}\cdot I &= L^{2n}[1-\lambda(1-L)]^n\left\|\mathcal{S}c_0-t\right\|.
\end{aligned}$$

For Projection Jungck-P algorithm

$$\begin{aligned}\|Sh_{n+1}-t\| &= \|(1-\alpha_n)\mathcal{P}_cTo_n+\alpha_n\mathcal{P}_cTl_n-(1-\alpha_n+\alpha_n)t\| \\ &= \|(1-\alpha_n)(\mathcal{P}_cTo_n-t)+\alpha_n(\mathcal{P}_cTl_n-t)\|\end{aligned}$$

$$\begin{aligned}&\leq (1-\alpha_n)\|\mathcal{P}_cTo_n-t\|+\alpha_n\|\mathcal{P}_cTl_n-t\| \\ &\leq (1-\alpha_n)\|To_n-t\|+\alpha_n\|Tl_n-t\| \\ &\leq L(1-\alpha_n)\|\mathcal{S}o_n-t\|+\phi(\|t-\mathcal{P}_c(t)\|+\|t-\mathcal{S}t\|)+L\alpha_n\|\mathcal{S}l_n-t\| \\ &\quad +\phi(\|t-\mathcal{P}_c(t)\|+\|t-\mathcal{S}t\|) \\ &\leq L(1-\alpha_n)\|\mathcal{S}o_n-t\|+L\alpha_n\|\mathcal{S}l_n-t\|. \end{aligned} \tag{2.13}$$

Now

$$\begin{aligned}\|\mathcal{S}l_n-t\| &= \|(1-\beta_n)\mathcal{S}\mathcal{P}_c(o_n)+\beta_n\mathcal{P}_cTo_n-(1-\beta_n+\beta_n)t\| \\ &= \|(1-\beta_n)(\mathcal{S}\mathcal{P}_c(o_n)-t)+\beta_n(\mathcal{P}_cTo_n-t)\| \\ &\leq (1-\beta_n)\|\mathcal{S}\mathcal{P}_c(o_n)-t\|+\beta_n\|\mathcal{P}_cTo_n-t\| \\ &= (1-\beta_n)\|\mathcal{P}_c\mathcal{S}(o_n)-t\|+\beta_n\|\mathcal{P}_cTo_n-t\| \\ &\leq (1-\beta_n)\|\mathcal{S}o_n-t\|+\beta_n\|To_n-t\| \\ &\leq (1-\beta_n)\|\mathcal{S}o_n-t\|+L\beta_n\|\mathcal{S}o_n-t\|+\phi(\|t-\mathcal{P}_c(t)\|+\|t-\mathcal{S}t\|) \\ &\leq (1-\beta_n)\|\mathcal{S}o_n-t\|+L\beta_n\|\mathcal{S}o_n-t\| \\ &\leq (1-\beta_n(1-L))\|\mathcal{S}o_n-t\| \end{aligned} \tag{2.14}$$

$$\begin{aligned}\|\mathcal{S}o_n-t\| &= \|(1-\gamma_n)\mathcal{S}\mathcal{P}_c(h_n)+\gamma_n\mathcal{P}_cTh_n-(1-\gamma_n+\gamma_n)t\| \\ &= \|(1-\gamma_n)(\mathcal{S}\mathcal{P}_c(h_n)-t)+\gamma_n(\mathcal{P}_cTh_n-t)\| \\ &\leq (1-\gamma_n)\|\mathcal{S}\mathcal{P}_c(h_n)-t\|+\gamma_n\|\mathcal{P}_cTh_n-t\| \\ &= (1-\gamma_n)\|\mathcal{P}_c\mathcal{S}(h_n)-t\|+\gamma_n\|\mathcal{P}_cTh_n-t\| \\ &\leq (1-\gamma_n)\|\mathcal{S}h_n-t\|+\gamma_n\|Th_n-t\| \\ &\leq (1-\gamma_n)\|\mathcal{S}h_n-t\|+L\gamma_n\|\mathcal{S}h_n-t\|+\phi(\|t-\mathcal{P}_c(t)\|+\|t-\mathcal{S}t\|) \\ &\leq (1-\gamma_n)\|\mathcal{S}h_n-t\|+L\gamma_n\|\mathcal{S}h_n-t\| \\ &\leq (1-\gamma_n(1-L))\|\mathcal{S}h_n-t\| \\ &\leq (1-\lambda(1-L))\|\mathcal{S}h_n-t\|. \end{aligned} \tag{2.15}$$

Substitute Equation (2.15) in to Equation(2.14)

$$\begin{aligned}\|\mathcal{S}l_n-t\| &= (1-\beta_n(1-L))\left[(1-\lambda(1-L))\|\mathcal{S}h_n-t\|\right] \\ &= \left[(1-\beta_n(1-L))(1-\lambda(1-L))\right]\|\mathcal{S}h_n-t\| \\ &\leq \left[(1-\lambda(1-L))(1-\lambda(1-L))\right]\|\mathcal{S}h_n-t\| \\ &\leq [1-\lambda(1-L)]^2\|\mathcal{S}h_n-t\|. \end{aligned} \tag{2.16}$$

Substitute Equation (2.16) &(2.15) in to Equation (2.13)

$$\begin{aligned}&\|Sh_{n+1}-t\| = L(1-\alpha_n)\left[(1-\lambda(1-L))\|\mathcal{S}h_n-t\|\right]+L\alpha_n\left([1-\lambda(1-L)]^2\|\mathcal{S}_n-t\|\right) \\ &= \left[L(1-\alpha_n)-L\lambda\left(1-\alpha_n(1-L)+L\alpha_n\left[1-2\lambda(1-L)+\lambda^2(1-L)\right]\right)\|\mathcal{S}h_n-t\|\right] \\ &\leq \left[L(1-\lambda)-L\lambda\left(1-\lambda(1-L)+L\lambda\left[1-2\lambda(1-L)+\lambda^2(1-L)\right]\right)\|\mathcal{S}h_n-t\|\right] \\ &\leq \left[L-L\lambda-L\lambda+L\lambda^2(1-L)+L\lambda-2L\lambda^2(1-L)+L\lambda^3(1-L)\right]\|\mathcal{S}h_n-t\| \\ &\leq \left[L-L\lambda-L\lambda^2(1-L)+L\lambda^3(1-L)\right]\|\mathcal{S}h_n-t\| \\ &\leq L\left[(1-\lambda)-\lambda^2(1-L)+\lambda^3(1-L)\right]\|\mathcal{S}h_n-t\| \\ &\leq L^n\left[(1-\lambda)-\lambda^2(1-L)+\lambda^3(1-L)\right]^n\|\mathcal{S}h_0-t\| \end{aligned}$$

$$\text{Put, } \mathcal{P}\cdot\mathcal{J}\cdot\mathcal{P}\cdot I = L^n\left[(1-\lambda)-\lambda^2(1-L)+\lambda^3(1-L)\right]^n\|\mathcal{S}h_0-t\|.$$

Now, because

$$\frac{\text{P.J.Z.I}}{\text{P.J.J.I}} = \frac{L^{2n}[1-\lambda(1-L)]^n\|\mathcal{S}c_0-t\|}{L^n\left[(1-\lambda)-\lambda^2(1-L)+\lambda^3(1-L)\right]^n\|\mathcal{S}h_n-t\|} \quad \text{as} \quad n\to\infty,$$

therefore, projection Jungck Zenor-algorithm converge faster than projection Jungck P-algorithm. □

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