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Conditional reciprocal continuity and a common fixed point in a b-metric space

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Abstract

A unique common fixed point is obtained for compatible and non-compatible self-maps on a b-metric space, through the notion of conditional reciprocal continuity, due to Pant and Bist.

Keywords: b-metric space, conditionally reciprocally continuous maps, compatible maps, common fixed point.
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1 Introduction

Let X be a non-empty set and $\rho : X \times X \to \mathbb{R}$ be such that

$(m1)$ $\rho(x, y) \geq 0$ for all $x, y \in X$

$(m2)$ $\rho(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$

$(m3)$ $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$

$(m4)$ $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$ for all $x, y, z \in X$.

Then the pair (X, ρ) denotes a metric space with metric ρ. Let $X = \mathbb{R}$. Then the metric $\rho(x, y) = |x - y|$ for all $x, y \in X$ is called the *usual metric* and it gives the *distance* between the points x and y on the number line $\mathbb{R}^1$. Let $X = \mathbb{R} \times \mathbb{R}$ and $\rho(x, y) = |x - y|$ for all $x, y \in X$. Condition (m4) says that the length of one side in a triangle with vertices x, y and z never exceeds the sum of the lengths of other sides in it. Hence it is referred to as the *triangle inequality* of the metric ρ. The notion of metric space was due to Frechet in 1926.

In the last few decades, fixed point theorems were developed in a metric space, normed linear space, topological space etc., while the conditions imposed on the underlying mappings are usually metrical or compact type conditions. Further, new ambient algebraic structures were formulated to improve the results. One such was a b-metric, introduced by Bakhtin [4], by generalizing the triangle inequality (m4).

Definition 1.1. Let $s \geq 1$, X be a nonempty set and $\rho_s : X \times X \to [0, \infty)$ be such that

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(b1) $\rho_s(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$

(b2) $\rho_s(x,y) = \rho_s(y,x)$ for all $x, y \in X$

(b3) $\rho_s(x,y) \leq s[\rho_s(x,z) + \rho_s(y,z)]$ for all $x, y, z \in X$.

Then ρ_s is a b-metric on X, and (X, ρ_s) denotes a b-metric space.

Metric space is a particular case of a b-metric space, when $s = 1$. However, a b-metric space is not necessarily a metric space. For instance, consider the pair (X, ρ_s), where $X = \mathbb{R}$ and $\rho_s(x,y) = |x-y|^2$ for all $x, y \in \mathbb{R}$. Then the conditions (b1) and (b2) are obvious. Further,

$$\rho_s(x,y) = |x-y|^2 = |x-z+z-y|^2 \leq 2\left(|x-z|^2 + |z-y|^2\right) = 2[\rho_s(x,z) + \rho_s(y,z)]$$

for all $x, y \in X$. Thus $(X = \mathbb{R}, \rho_s)$ is a b-metric space with $b = 2$. Since $\rho_s(1,3) + \rho_s(1,0) = 4 + 1 = 5$ and $\rho_s(0,3) = 9$, (m3) fails to hold good, showing that ρ_s is not a metric. Thus a b-metric space is not a metric space. In view of the convexity of $f(x) = x^p$, where $x > 0$ and $1 < p < \infty$, it follows that $(\mathbb{R}, |x-y|^p)$ is a b-metric space, which is not a metric space. In other words, the class of b-metric spaces contains that of metric spaces.

It is well-known that unlike the set $\mathbb{R}$ of real numbers, the set $\mathbb{C}$ of all complex numbers does not have the ordering property. Azam et al. In [3] introduced the notion of a complex-valued metric space in terms of a partial ordering on $\mathbb{C}$, which was further generalized to a complex-valued b-metric space in [9]. For some of its applications, one may refer to [2] [8] and mebetal2.

Definition 1.2. A b-ball in a b-metric space (X, ρ_s) is defined by

$$B_{\rho_s}(x,r) = \{y \in X : \rho_s(x,y) < r\}.$$

The family of all b-balls forms a base topology, called the *b-metric topology* $\tau(\rho_s)$ on X.

Definition 1.3. Let (X, ρ_s) be a b-metric space with parameter s. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be

(a) b-convergent, with limit $p \in X$, if it converges to p in the b-metric topology $\tau(\rho_s)$

(b) b-Cauchy, if $\lim_{n,m\to\infty} \rho_s(x_n, x_m) = 0$.

Like in a metric space, every b-convergent sequence has a unique limit, and is necessarily b-Cauchy.

Definition 1.4. A b-metric space X is said to be b-complete, if every b-Cauchy sequence in X is b-convergent in it.

Since a b-metric is not jointly continuous in general in its coordinate variables x and y, though a metric d is known to be continuous (See Example 2.13, [20]), we use the following results from [17]:

Lemma 1.1. Let (X, ρ_s) be a b-metric space with parameter s. Suppose that $\{x_n\}_{n=1}^{\infty}$ is b-convergent with limit x and $\{y_n\}_{n=1}^{\infty}$ is b-convergent with limit y in X. Then

$$\frac{1}{s^2}\rho_s(x,y) \leq \liminf_{n\to\infty} \rho_s(x_n, y_n) \leq \limsup_{n\to\infty} \rho_s(x_n, y_n) \leq s^2 \rho_s(x,y). \tag{1.1}$$

In particular, $x = y$, then $\lim_{n\to\infty} \rho_s(x_n, y_n) = 0$. Further, for each $z \in X$, we have

$$\frac{1}{s}\rho_s(x,z) \leq \liminf_{n\to\infty} \rho_s(x_n, z) \leq \limsup_{n\to\infty} \rho_s(x_n, z) \leq s\rho_s(x,z). \tag{1.2}$$

Lemma 1.2. Let (X, ρ_s) be a b-metric space with parameter s. Suppose that there exist sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \rho_s(x_n, y_n) = 0$, whenever $\lim_{n\to\infty} x_n = t$ for some $t \in X$, then $\lim_{n\to\infty} y_n = t$.

2 Conditional Reciprocal Continuity in b-Metric Spaces

Self-maps f and r on a metric space (X,ρ) are known to be commuting, if $frx = rfx$ for all $x \in X$. As a weaker form of it, Sessa [18] introduced weakly commuting maps f and r on X with the choice $\rho(frx, rfx) \le \rho(fx, rx)$ for all $x \in X$. weakly commuting maps were generalized as R-weakly commuting maps by Pant [10], which satisfy the condition:

$$\rho(frx, rfx) \le R\rho(fx, rx) \text{ for all } x \in X \text{ for some } R > 0. \tag{2.1}$$

Writing $R = 1$ in (2.1), we get weakly commuting pair (f, r). Splitting the condition (2.1), Pathak et al. [14] defined R-weakly commuting of types (A_g) and (A_f). In fact, self-maps f and r on X are said to be R-weakly commuting of type (A_g), if

$$\rho(frx, rrx) \le R\rho(fx, rx) \text{ for all } x \in X \text{ for some } R > 0. \tag{2.2}$$

Interchanging the roles of f and r in (2.2), we get R-weakly commuting of type (A_f). In a comparative study of various weaker forms of commuting maps, Singh and Tomar [19] remarked that R-weak commutativity is independent of these two types. Prior to these notions, Gerald Jungck [6] introduced compatible maps as a generalization for weakly commuting maps as follows:

Definition 2.1. Self-maps f and r on X are said to be compatible, if

$$\lim_{n\to\infty} \rho(frx_n, rfx_n) = 0 \tag{2.3}$$

whenever there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} rx_n = z \quad \text{for some} \quad z \in X. \tag{2.4}$$

It was observed from [15] that a pair (f, r) of self-maps can be weakly commuting, but there may not be any sequence $\{x_n\}_{n=1}^{\infty}$ with the choice (2.4). Such maps are *vacuously* compatible. While, self-maps f and r are non-compatible, if there is a sequence $\{x_n\}_{n=1}^{\infty}$ with (2.4) but $\lim_{n\to\infty} \rho(frx_n, rfx_n) \neq 0$ or $+\infty$. In the study of common fixed points for non-compatible and discontinuous maps, the notions of reciprocal continuity, weak reciprocal continuity and conditional reciprocal continuity were introduced as follows:

Definition 2.2 (Pant et al., [11]). Self-maps f and r on X are reciprocally continuous at $z \in X$, if for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with the choice (2.4), we have

$$\lim_{n\to\infty} frx_n = fz \text{ and } \lim_{n\to\infty} rfx_n = rz, \tag{2.5}$$

where f and r are reciprocally continuous (on X) if and only if they are reciprocally continuous at each $z \in X$.

Example 2.1. Consider $X = \mathbb{R}$ with the usual metric $\rho_u(x,y) = |x - y|$ for all $x, y \in X$. Define $f, r : X \to X$ by $fx = x/2$ and $rx = fx + 1$ for all $x \in X$. Since $\rho(fx, rx) = 1$ for all x, we see that there exists no sequence $\{x_n\}_{n=1}^{\infty}$ satisfying (2.4). Thus the condition reciprocal continuity is vacuously satisfied. We call such maps *vacuously* reciprocally continuous.

Definition 2.3 (*Pant et al.*, [13]). Self-maps f and r on X are said to be weakly reciprocally continuous at $z \in X$, if for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with the choice (2.4), we have

$$\lim_{n\to\infty} frx_n = fz \text{ or } \lim_{n\to\infty} rfx_n = rz. \tag{2.6}$$

It is obvious that reciprocal continuity implies weak reciprocal continuity.

Definition 2.4 (*Pant and Bist*, [12]). Self-maps f and r on X are said to be conditionally reciprocally continuous at $z \in X$, if for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with the choice (2.4), there corresponds a sequence $\{y_n\}_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} fy_n = \lim_{n\to\infty} ry_n = z \text{ for some } z \in X \text{ implies } \lim_{n\to\infty} fry_n = fz \text{ and } \lim_{n\to\infty} rfy_n = rz. \tag{2.7}$$

Example 4 and Example 5 of [12] suggests that the notions of nonvacuous weak reciprocal continuity and nonvacuous conditional reciprocal continuity are independent of each other.

Definition 2.5 (*Jungck*, **[7]**)**.** A point $x \in X$ is called a coincidence point for self-maps f and r if $fx = rx = y$. Self-maps which commute at their coincidence points are called weakly compatible.

Weakly compatible maps are also called coincidentally commuting maps [5].

Remark 2.1. Compatible maps do commute at their coincident points, and hence are weakly compatible.

Splitting the condition (2.3) in different ways, Pathak and Khan [16] introduced different types of compatible maps:

Definition 2.6. Self-maps f and r on X are said to be f-compatible, if

$$\lim_{n\to\infty} \rho(frx_n, rrx_n) = 0 \tag{2.8}$$

whenever there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with the choice (2.4). Whereas, self-maps f and r on X are r-compatible, if

$$\lim_{n\to\infty} \rho(ffx_n, rfx_n) = 0 \tag{2.9}$$

whenever there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ with the choice (2.4).

Remark 2.2. It was observed that each of these two types is independent of the compatibility. However, f-compatibility, r-compatibility and the compatibility are equivalent, whenever f and r are continuous.

Remark 2.3. It may be noted that non-vacuously compatible, compatible maps of all types and non-compatible maps are included in the wider class of self-maps $\{f, g\}$ satisfying the property (EA) [1], in which (2.4) holds good for some $\{x_n\}_{n=1}^{\infty} \subset X$.

Remark 2.4. Compatibility and all its types, and R-weak commutativity and its types imply the weak compatibility [19]. Since two self-maps fail to be weakly compatible, only if they have a coincidence point at which they do not commute, weak compatibility is the minimal condition for the maps to have a common fixed point. Further, weak compatibility and property (EA) are weaker than the compatibility and all its types. However, both these notations are independent of each other [14].

3 Common Fixed Points in b-Metric Spaces

We now prove the following common fixed point theorem for conditionally reciprocally continuous pairs of maps in a b-metric space:

Theorem 3.1. Let f, g, r, h be self-maps on a complete b-metric space (X, ρ_s) with $s \geq 1$, satisfying the inclusions:

$$f(X) \subset h(X),\ g(X) \subset r(X), \tag{3.1}$$

and the inequality

$$\rho_s(fx, gy) \leq \frac{q}{s^4} \max\left\{\rho_s(rx, hy), \rho_s(fx, rx), \rho_s(gy, hy), \frac{1}{2}[\rho_s(rx, gy) + \rho_s(fx, hy)]\right\} \text{ for all } x, y \in X, \tag{3.2}$$

where $0 < q < 1$. Suppose that $\{f, r\}$ and $\{g, h\}$ are conditionally reciprocally continuous pairs and one of the following conditions holds good:

(*a*) $\{f, r\}$ and $\{g, h\}$ are compatible,

(*b*) $\{f, r\}$ is r-compatible and $\{g, h\}$ is h-compatible,

(*c*) $\{f, r\}$ is f-compatible and $\{g, h\}$ is g-compatible.

Then f, g, r and h have a unique common fixed point.

Proof . Let $x_0 \in X$ be arbitrary. By virtue of the inclusions (3.1), we choose points x_1, x_2, ... in X such that

$$y_{2n-1} = fx_{2n-1} = hx_{2n},\ y_{2n} = gx_{2n} = rx_{2n+1} \text{ for } n \geq 1. \tag{3.3}$$

We first establish that $\{y_n\}_{n=1}^{\infty}$ is b-Cauchy sequence in X.

Writing $x = x_{2n-1}$ and $y = x_{2n}$ in (3.2) and using (3.3) and (b4),

$$\begin{aligned} \rho_s(y_{2n-1}, y_{2n}) &= \rho_s(fx_{2n-1}, gx_{2n}) \\ &\leq \frac{q}{s^4} \max\big\{\rho_s(rx_{2n-1}, hx_{2n}), \rho_s(fx_{2n-1}, rx_{2n-1}), \rho_s(gx_{2n}, hx_{2n}), \\ &\quad \frac{1}{2}[\rho_s(rx_{2n-1}, gx_{2n}) + \rho_s(fx_{2n-1}, hx_{2n})]\big\} \\ &= \frac{q}{s^4} \max\big\{\rho_s(y_{2n-2}, y_{2n-1}), \rho_s(y_{2n-1}, y_{2n-2}), \rho_s(y_{2n}, y_{2n-1}), \\ &\quad \frac{1}{2}[\rho_s(y_{2n-2}, y_{2n}) + \rho_s(y_{2n-1}, y_{2n-1})]\big\} \\ &= \frac{q}{s^4} \max\big\{\rho_s(y_{2n-2}, y_{2n-1}), \rho_s(y_{2n-1}, y_{2n}), \frac{1}{2}[\rho_s(y_{2n-2}, y_{2n})]\big\} \\ &\leq \frac{q}{s^4} \max\big\{\rho_s(y_{2n-2}, y_{2n-1}), \rho_s(y_{2n-1}, y_{2n}), \frac{s}{2}[\rho_s(y_{2n-2}, y_{2n-1}) + \rho_s(y_{2n-1}, y_{2n})]\big\}. \end{aligned} \tag{3.4}$$

If $\rho_s(y_{2n-1}, y_{2n}) > \rho_s(y_{2n-2}, y_{2n-1})$ for some $n = m$, then (3.4) would imply that

$$0 < \rho_s(y_{2m-1}, y_{2m}) \leq \frac{q}{s^2}\rho_s(y_{2m-1}, y_{2m}) < \rho_s(y_{2m-1}, y_{2m}), \tag{3.5}$$

which is a contradiction. Thus $\rho_s(y_{2n-1}, y_{2n}) \leq \rho_s(y_{2n-2}, y_{2n-1})$ for all n so that (3.4) gives

$$\rho_s(y_{2n-1}, y_{2n}) \leq \frac{q}{s^3}\rho_s(y_{2n-2}, y_{2n-1}) \text{ for all } n. \tag{3.6}$$

Similarly, it follows that

$$\rho_s(y_{2n-2}, y_{2n-1}) \leq \frac{q}{s^3}\rho_s(y_{2n-3}, y_{2n-2}) \text{ for all } n. \tag{3.7}$$

Combining (3.6) and (3.7),

$$\rho_s(y_{n-1}, y_n) \leq k\rho_s(y_{n-2}, y_{n-1}) \text{ for all } n \geq 3,$$

where $k = q/s^3$. By induction,

$$\rho_s(y_{n-1}, y_n) \leq k^{n-2}\rho_s(y_1, y_2) \text{ for all } n \geq 3. \tag{3.8}$$

Therefore, for all $n > m$, repeatedly using (3.8), we have

$$\begin{aligned} \rho_s(y_m, y_n) &\leq s\Big[\rho_s(y_m, y_{m+1}) + \rho_s(y_{m+1}, y_n)\Big] \\ &\leq s\rho_s(y_m, y_{m+1}) + s^2\Big[\rho_s(y_{m+1}, y_{m+2}) + \rho_s(y_{m+2}, y_n)\Big] \\ &\ \vdots \\ &\leq \underbrace{s\rho_s(y_m, y_{m+1}) + s^2\rho_s(y_{m+1}, y_{m+2}) + \cdots + s^{n-m}\rho_s(y_{n-1}, y_n)}_{n-m \text{ terms}} \\ &\leq \big[sk^{m-1} + s^2k^m + \cdots + s^{n-m}k^{n-2}\big]\rho_s(y_1, y_2) \\ &= sk^{m-1}\big(1 + sk + \cdots s^{n-m-1}k^{n-m-1}\big)\rho_s(y_1, y_2) \\ &\leq \frac{sk^{m-1}}{1 - sk} \cdot \rho_s(y_1, y_2) \text{ for all } n. \end{aligned}$$

As $m, n \to \infty$, this implies that $\rho_s(y_m, y_n) \to 0$. Thus $\{y_n\}_{n=1}^{\infty}$ is b-Cauchy sequence in X. Since X is b-complete, there exists a point $z \in X$ such that

$$\lim_{n\to\infty} fx_{2n-1} = \lim_{n\to\infty} rx_{2n} = \lim_{n\to\infty} gx_{2n} = \lim_{n\to\infty} hx_{2n+1} = z. \tag{3.9}$$

Since, $\{f, r\}$ is a conditionally reciprocally continuous pair, there exists $\{a_n\}_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} fa_n = \lim_{n\to\infty} ra_n = u \text{ for some } u \in X \text{ implies } \lim_{n\to\infty} fra_n = fu,\ \lim_{n\to\infty} rfa_n = ru. \tag{3.10}$$

Also, since $\{g, h\}$ is a conditionally reciprocally continuous pair, there exists $\{b_n\}_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} gb_n = \lim_{n\to\infty} hb_n = v \text{ for some } v \in X \text{ implies } \lim_{n\to\infty} ghb_n = gv \text{ and } \lim_{n\to\infty} hgb_n = hv. \tag{3.11}$$

Case (a): Suppose that (f, r) and (g, h) are compatible. Then

$$\lim_{n\to\infty} \rho_s(rfa_n, fra_n) = 0 \text{ and } \lim_{n\to\infty} \rho_s(hgb_n, ghb_n) = 0.$$

In view of Lemma 1.2, (3.10) and (3.11) imply that

$$ru = \lim_{n\to\infty} rfa_n = fu \text{ and } hv = \lim_{n\to\infty} ghb_n = gv. \tag{3.12}$$

Now, writing $x = u$, $y = v$ in (3.2), and using (3.12), we see that

$$\begin{aligned}\rho_s(fu, gv) &\le \frac{q}{s^4} \max\left\{\rho_s(ru, hv), \rho_s(fu, ru), \rho_s(gv, hv), \frac{1}{2}[\rho_s(ru, gv) + \rho_s(fu, hv)]\right\} \\ &\le \frac{q}{s^4} \max\left\{\rho_s(fu, gv), 0, 0, \frac{1}{2}[\rho_s(fu, gv) + \rho_s(fu, gv)]\right\} = \frac{q}{s^4}\rho_s(fu, gv)\end{aligned}$$

so that $\rho_s(fu, gv) = 0$ or $fu = gv$. Thus

$$fu = ru = gv = hv = p. \tag{3.13}$$

In view of Remark 2.1, (3.13) implies that

$$fp = rp \text{ and } gp = hp. \tag{3.14}$$

Writing $x = y = p$ in (3.2), and then using (3.13) and (3.14), we see that

$$\begin{aligned}\rho_s(fp, gp) &\le \frac{q}{s^4} \max\left\{\rho_s(rp, hp), \rho_s(fp, rp), \rho_s(gp, hp), \frac{1}{2}[\rho_s(rp, gp) + \rho_s(fp, hp)]\right\} \\ &= \frac{q}{s^4}\rho_s(fp, gp).\end{aligned}$$

In other words, p is a common coincidence point of f, g, r and h. Finally, writing $x = u$, $y = p$ in (3.2), and using this,

$$\begin{aligned}\rho_s(p, gp) = \rho_s(fu, gp) &\le \frac{q}{s^4} \max\left\{\rho_s(ru, hp), \rho_s(fu, ru), \rho_s(gp, hp), \frac{1}{2}[\rho_s(ru, gp) + \rho_s(fu, hp)]\right\} \\ &= \frac{q}{s^4} \max\left\{\rho_s(p, gp), 0, 0, \frac{1}{2}[\rho_s(p, gp) + \rho_s(p, gp)]\right\} = \rho_s(p, gp).\end{aligned}$$

Thus $\rho_s(p, gp) = 0$ or $p = gp$. In other words, p is a common fixed point of f, g, r and h.

Case (b): Suppose that $(\{f, r\}$ is r-compatible and $\{g, h\}$ is h-compatible. Then from (3.10) and (3.11),

$$\lim_{n\to\infty} ffa_n = \lim_{n\to\infty} rfa_n = ru \text{ and } \lim_{n\to\infty} ggb_n = \lim_{n\to\infty} hgb_n = hv. \tag{3.15}$$

Now, writing $x = fa_n$, $y = gb_n$ in (3.2), and using (3.15), we see that

$$\rho_s(ffa_n, ggb_n) \le \frac{q}{s^4} \max\left\{\rho_s(rfa_n, hgb_n), \rho_s(ffa_n, rfa_n), \rho_s(ggb_n, hgb_n), \frac{1}{2}[\rho_s(rfa_n, ggb_n) + \rho_s(ffa_n, hgb_n)]\right\}$$

Employing the limit superior as $n \to \infty$ and using Lemma 1.1, this gives

$$\begin{aligned}\frac{1}{s^2}\rho_s(ru,hv) &\le \limsup_{n\to\infty}\rho_s(ffa_n,ggb_n)\\ &\le \limsup_{n\to\infty}\frac{q}{s^4}\max\left\{\rho_s(rfa_n,hgb_n),\rho_s(ffa_n,rfa_n),\rho_s(ggb_n,hgb_n),\frac{1}{2}[\rho_s(rfa_n,ggb_n)+\rho_s(ffa_n,hgb_n)]\right\}\\ &\le \frac{q}{s^4}\cdot\max\left\{s^2\rho_s(ru,hv),0,0,\frac{s^2}{2}[\rho_s(ru,hv)+\rho_s(ru,hv)]\right\} = \frac{q}{s^2}\cdot\rho_s(ru,hv).\end{aligned}$$

If $\rho_s(ru,hv)>0$, this would imply a contradiction that

$$0<\frac{1}{s^2}\rho_s(ru,hv)\le\frac{q}{s^2}\cdot\rho_s(ru,hv)<\frac{1}{s^2}\rho_s(ru,hv).$$

Therefore, $\rho_s(ru,hv)=0$ or $ru=hv$.

Now, writing $x=u$, $y=gb_n$ in (3.2), and using (3.15), we see that

$$\rho_s(fu,ggb_n)\le\frac{q}{s^4}\max\left\{\rho_s(ru,hgb_n),\rho_s(fu,ru),\rho_s(ggb_n,hgb_n),\frac{\rho_s(ru,ggb_n)+\rho_s(fu,hgb_n)}{2}\right\}$$

Again applying the limit superior as $n\to\infty$ in this and using Lemma 1.1, this gives

$$\begin{aligned}\frac{1}{s^2}\rho_s(fu,hv) &\le \limsup_{n\to\infty}\rho_s(fu,ggb_n)\\ &\le \frac{q}{s^4}\max\left\{s^2\rho_s(fu,hv),s^2\rho_s(fu,ru),0,0,\frac{s^2}{2}[\rho_s(fu,hv)+\rho_s(fu,hv)]\right\}\\ &= \frac{q}{s^2}\cdot\rho_s(fu,hv)\end{aligned}$$

so that $\rho_s(fu,hv)=0$ or $fu=ru=hv$.

Whereas, writing $x=fa_n$, $y=v$ in (3.2), and using (3.15), we see that

$$\rho_s(ffa_n,gv)\le\frac{q}{s^4}\max\left\{\rho_s(rfa_n,hv),\rho_s(ffa_n,rfa_n),\rho_s(gv,hv),\frac{1}{2}[\rho_s(rfa_n,gv)+\rho_s(ffa_n,hv)]\right\}.$$

In view of Lemma 1.1, as $n\to\infty$, this gives

$$\frac{1}{s^2}\rho_s(ru,gv)\le\frac{q}{s^4}\max\left\{s^2\rho_s(ru,hv),0,s^2\rho_s(gv,hv),\frac{s^2}{2}[\rho_s(ru,gv)+\rho_s(ru,hv)]\right\}=\frac{s^2}{2}\cdot\rho_s(ru,gv)$$

so that $\rho_s(ru,gv)=0$ or $ru=gv$. In other words, $fu=ru=gv=hv=\xi$, say. Since r-compatible and h-compatible pairs commute at their coincidence points, we see that

$$f\xi=r\xi \text{ and } g\xi=h\xi. \tag{3.16}$$

Putting $x=y=\xi$ in (3.2), using (3.16) and then simplifying, we obtain that

$$f\xi=r\xi=g\xi=h\xi. \tag{3.17}$$

Finally, writing $x=w$, $y=\xi$ in (3.2), and using this,

$$\begin{aligned}\rho_s(p,g\xi)=\rho_s(fw,g\xi) &\le \frac{q}{s^4}\max\left\{\rho_s(rw,h\xi),\rho_s(fw,rw),\rho_s(g\xi,h\xi),\frac{1}{2}[\rho_s(rw,g\xi)+\rho_s(fw,h\xi)]\right\}\\ &= \frac{q}{s^4}\max\left\{\rho_s(\xi,g\xi),0,0,\frac{1}{2}[\rho_s(\xi,g\xi)+\rho_s(\xi,g\xi)]\right\}\\ &= \rho_s(\xi,g\xi).\end{aligned}$$

Thus $\rho_s(\xi, g\xi) = 0$ or $\xi = g\xi$. In other words, ξ is a common fixed point of f, g, r and h.

Case (c): Interchanging the roles of f and r, and g and h in case (b), a common fixed point can be obtained. The uniqueness of the common fixed point follows easily from (3.2) and the choice of k. $\square$

The following theorem was proved in [17]:

Theorem 3.2. Let (X, ρ_s) be a complete b-metric space, $s \geq 1$, and f, g, r, $h : X \to X$ satisfy the inclusions (3.1), the inequality (3.2), and the condition (a) of Theorem 3.1. If r and h are continuous, then f, g, r and h have a unique common fixed point.

Note that a unique common fixed point can be obtained by Theorem 3.1, even if neither r nor h is continuous. Where as, Theorem 3.2 employed the continuity of both r and h for finding a common fixed point.

Remark 3.1. In addition to the continuity of r and h, if f and g are also continuous Theorem 3.2, then in view of Remark 2.2, the compatibility of $\{f, r\}$ coincides with their f-compatibility and the compatibility of $\{g, h\}$ coincides with their h-compatibility. Therefore, it suffices to use only one of (a), (b) and (c) to obtain the unique common fixed point in Theorem 3.2.

Setting $f = g$, $r = h$ and $s = 1$ in Theorem 3.2, we get

Corollary 3.1 (Theorem 1, [12]). Let f and r be self-maps on a complete metric space (X, ρ), satisfying the inclusion:

$$f(X) \subset r(X), \tag{3.18}$$

and the inequality

$$\rho_s(fx, fy) \leq \frac{q}{s^4} \max\left\{\rho_s(rx, ry), \rho_s(fx, rx), \rho_s(fy, ry), \frac{1}{2}[\rho_s(rx, fy) + \rho_s(fx, ry)]\right\} \text{ for all } x, y \in X, \tag{3.19}$$

where $0 < q < 1$. Suppose that $\{f, r\}$ is a conditionally reciprocally continuous pair and one of the following conditions holds good:

(a) $\{f, r\}$ is compatible,

(b) $\{f, r\}$ is f-compatible,

(c) $\{f, r\}$ is r-compatible.

Then f and r have a unique common fixed point.

Corollary 3.2 (Theorem 1, [12]). Let f and r be self-maps on a complete metric space (X, ρ), satisfying the inclusion (3.18) and the inequality

$$\rho_s(fx, fy) \leq q\rho_s(rx, ry) \text{ for all } x, y \in X, \tag{3.20}$$

where $0 < q < 1$. Suppose that $\{f, r\}$ is a conditionally reciprocally continuous pair and one of the following conditions holds good:

(a) $\{f, r\}$ is compatible,

(b) $\{f, r\}$ is f-compatible,

(c) $\{f, r\}$ is r-compatible.

Then f and r have a unique common fixed point.

It may be noted that the right hand side of (3.20) is less than or equal to that of (3.19), and hence (3.19) holds good whenever (3.19) holds. Thus Corollary 3.2 follows from Corollary 3.1.

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