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The arrow edge domination in graphs

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Abstract

The idea of this paper is to study the arrow edge domination. The arrow edge dominating set D_e of a graph G is an arrow edge dominating set if every edge from D dominates exactly one edge from $V-D$ and is adjacent to two or more edges from D. The arrow edge domination number $\gamma_{\text{are}}(G)$ is the minimum cardinality of all arrow edge dominating sets in G. Several properties and bounds are introduced here. Our results are applied in some graphs such that the path graph, cycle graph, complete graph, wheel graph, complete bipartite graph, Barbell graph, helm graph, big helm graph, complement path graph, complement cycle graph, the complement of complete graph and complement of complete bipartite graph. An important fact given here is if G has no arrow vertex dominating set, then G may have an arrow edge dominating set and an example is given.

Keywords: Arrow edge domination, edge domination, dominating set, path graph, cycle graph
2020 MSC: 05C69

1 Introduction

Graph theory is an essential branch of mathematics. It is a mathematical representation of any network or subject. It describes the relationship between lines (edges) and points or nodes (vertices). A graph consists of some vertices and edges. The length of the edges and position of the vertices do not matter. Graph theory has several topics such as labeling, coloring, and domination. There are more connections between graph theory and other branches of mathematics such as topology, algebra, fuzzy, and probability. Also, with other sciences such as computer, engineering, physics, chemistry, and biology. The graph is required in our real life, such as in streets, planning of cities, residential houses, and service institutions. Let $G=(V,E)$ be a finite graph, simple and undirected. Where $V(G)$ be a set of vertices and $E(G)$ be a set of edges in G. The degree of any vertex v in G is the number of edges that incident on it and denoted by $\deg(v)$. If $\deg(v)=0$, then v is said isolated vertex and if $\deg(v)=1$, then v is said pendant vertex. A support vertex is a vertex that adjacent with pendant vertex. $\Delta(G)$ is the maximum degree over all vertices in G and $\delta(G)$ is the minimum degree over them in G. The open neighborhood of a vertex v is the set of neighbors of v in a graph G expressed as $N(v)=N_G(v)=\{u : uv \in E(G)\}$. The closed neighborhood of v is the set $N[v]=N_G[v]=N(v)\cup\{v\}$. Where two vertices u and v of G are adjacent if there is an edge $e=uv\in E(G)$. Link two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, such that construct a new graph having the vertex set $V_1\cup V_2$ and the edge set $E_1\cup E_2\cup\{v_1v_2 : \forall v_1\in V_1, v_2\in V_2\}$. This linking is called join operation and denoted by G_1+G_2. For more basic concepts in graph theory see [18]. For basic concepts in domination of graphs see [19-21]. There are more types of domination models discussed more problems and gave wide results and applications, see [1]-[17], [21]-[26]. In 2021,

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the authors introduced new type of domination said arrow domination in graphs [27]. They studied it on the vertices of the graph and proved more results and applications. In 2021 , they joined some other types of domination models with arrow model to introduce several new types of it by adding new conditions on the arrow dominating set or on its complement set or on both sets. Such models as co-independent arrow domination, restrained arrow domination, connected arrow domination, and complementary tree arrow domination [28]. A second new model of domination called the arrow edge domination is introduced here. Some bounds are given and properties are proved, then the arrow edge domination discussed for some stander graphs.

2 Arrow edge dominating set

The definition of arrow edge dominating set is introduced. Some essential properties are proved.

Definition 2.1. Let $G(V,E)$ be a finite, nontrivial, simple and undirected graph. A sub set $D_e \subseteq E$ is an arrow edge dominating set if every $e \in D_e$ dominates exactly one edge and adjacent with two or more edges from D_e.

Definition 2.2. A set D_e from a graph G is said to be an arrow edge dominating set if it has no proper subset as arrow edge dominating set.

Definition 2.3. The smallest arrow edge dominating set is called minimum arrow edge dominating set.

Definition 2.4. The cardinality of the minimum arrow edge dominating set is known as arrow edge domination number of G and denoted by $\gamma_{\text{are}}(G)$. Such set is referred as γ_{are} - set.

Observation 2.5. For any graph $G(V,E)$ with an arrow edge dominating set D_e and an arrow edge domination number $\gamma_{\text{are}}(G)$, we have :

1- $|N(e) \cap (E - D_e)| = 1 \forall e \in D_e$.

2- $|N(e) \cap D_e| \geq 2 \forall e \in D_e$.

3- $N(e) \cap D_e \neq \emptyset \forall e \in E - D_e$.

4- $\gamma_{\text{are}}(G) \geq 3$.

5- $m \geq 4$, where m is the size of a graph.

Remark 2.6. Let G be a disconnected graph with a component isomorphic to K_2 or K_3, then G has no arrow edge dominating set.

Theorem 2.7. For any graph $G(V,E)$ having arrow edge domination number $\gamma_{\text{are}}(G)$, then $\left\lceil \frac{m}{2} \right\rceil \leq \gamma_{\text{are}}(G) \leq m-1$.

Proof . Let D_e be a γ_{are} − set of G and the edges $e_i, e_j \in D_e, e_i \neq e_j$. Then, there are two cases to prove the lower bound as follows:

Case 1: If $N(e_i) \cap N(e_j) \cap N(E - D_e) = \emptyset$, then every edge in $E - D_e$ is dominated by exactly one edge in D_e, so $\gamma_{\text{are}}(G) = \frac{m}{2}$.

Case 2: If $N(e_i) \cap N(e_j) \cap N(E - D_e) \neq \emptyset$, then there are one or more edges from $E - D_e$ which is dominated by more than one edge from D_e. Hence, $\gamma_{\text{are}}(G) \geq \left\lceil \frac{m}{2} \right\rceil$

The upper bound proved depending on the fact $E - D_e \neq \emptyset$, where it must be having one edge at least. Thus, $\gamma_{\text{are}}(G) \leq m - 1$. □

Theorem 2.8. Every an arrow edge dominating set is a minimal arrow edge dominating set .

Proof . Let D_e be any arrow edge dominating set in a graph G. Suppose that D_e is not minimal arrow edge dominating set, then there exist at least one edge $e \in D_e$ such that $D_e - e$ is an arrow edge dominating set.We discuss two cases as follows :

Case 1 : Suppose that there is one edge $e_1 \in E - D_e$ dominated by an edge e. Then , if e_1 is dominated by only e, then $D_e - e$ is not dominates e_1. Thus, $D_e - e$ is not arrow edge dominating set.

Case 2 : If there is one or more edges in $D_e - e$ dominate $e_1 \in E - D_e$ that is dominated by the edge e. Then, we discuss which edges are dominate. Let $D_e - e$ dominates the edge e by at least two edges say e_2 and e_3, since e is adjacent with two edges from D_e at least by definition of arrow edge domination, the edges e_1 and e_2 dominate two edges in $E - (D_e - e)$, which is a contradiction. Therefore, $D_e - e$ is not arrow edge dominating set. Then, in each above cases $D_e - e$ is not arrow edge dominating set. Hence, D_e is a minimal arrow edge dominating set. □

3 Applying the arrow edge domination in some graphs

The arrow edge domination number is proved for standard graphs such as path, cycle, complete, bipartite, star graph, complete bipartite, helm, big helm, Barbell graphs and its complements.

Proposition 3.1. The path P_n and cycle C_n graphs have no arrow edge dominating sets.

Proof . Since every edge in P_n and C_n have only one or two neighborhoods which contradict the definition of arrow edge dominating set. □

Proposition 3.2. A complete graph $K_n(n \geq 4)$ has no arrow edge dominating set.

Proof . Suppose that K_n has arrow edge dominating set D_e. If D_e contains all edges of K_n unless one, then there is one edge or more in D_e doesn't dominate this edge. If there are two edges e_1, e_2 in E don't belongs to D_e, then two or more edges of D_e will dominate e_1 and e_2. Hence, D_e is not arrow edge dominating set in K_n. So that if $E - D_e$ has three or more edges. Therefore, K_n has no arrow edge domination. For example see Fig. 1.1. □

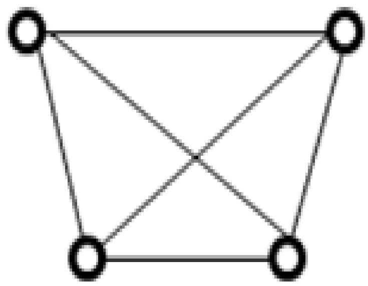


Figure 1: Complete Graph K_n

Proposition 3.3. A wheel graph $W_n(n \geq 3)$ has no arrow edge dominating set.

Proof . If we take all edges from $C_n \in D_e$, then every edge in D_e dominates two edges from $E - D_e$. If we take all edges that joined K_1 with the cycle graph, then every edge is dominates two edges from the cycle. If we take some edges from the cycle C_n and some of the edges that joined K_1 with the cycle in D_e, then there are some edges dominate two or more edges. In all above cases we get a contradiction. Thus, the wheel graph has no arrow edge dominating set . For example see Fig. 1.2. □

Theorem 3.4. A complete bipartite graph $K_{n,m}$ has arrow edge dominating set such that

$$\gamma_{\text{are}}\ (K_{n,m}) = \begin{cases} m-1 & \text{if } n = 1 \\ m(n-1) & \text{if } n \geq 2 \end{cases}$$

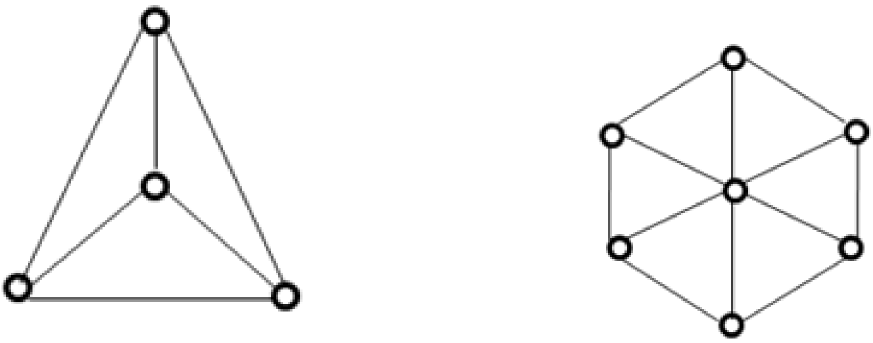


Figure 2: A wheel Graph W_n has no Arrow Edge Dominating Set

Proof . There are two cases as follows:

Case 1: Since all the edges in $K_{1,m}$ are adjacent by common vertex , then one edge must be outside the dominating set and the remaining edges belong to the dominating set. Thus, $|D_e| = m - 1 = \gamma_{\text{are}}\ (K_{1,m})$. If we take the arrow dominating set D'_e of order $m - 2$, then there are exactly two edges in $E - D'_e$. These edges are adjacent with all edges in D'_e. So, every edge in D'_e dominates two edges which is contradiction. Hence, D_e is a minimum arrow edge dominating set and $\gamma_{\text{are}}\ (K_{1,m}) = m - 1$.

Case 2: Let M_1 and M_2 be the two disjoint sets of $K_{n,m}$ such that $|M_1| = n$ and $|M_2| = m$. Suppose that D_e consists of all edges that joined $n-1$ vertices from M_1 with all vertices from M_2. Then, $E-D_e$ contains all edges that joint one vertex in M_1 with all vertices of M_2. Since every edge in D_e dominate exactly one edge from $E - D_e$ and adjacent with two or more edges from D_e. Thus, D_e is the arrow edge dominating set . Hence, $\gamma_{are}\,(K_{n,m}) = m(n-1)$. In similar technique of case 1 , we can prove D_e is the minimum arrow edge dominating set. See for example Fig. 1.3. □

Figure 3: Minimum Arrow Edge Domination in Complete Bipartite Graph $K_{n,m}$

Proposition 3.5. A Barbell graph $B_{n,n}(n \geq 3)$ has no arrow edge dominating set.

Proof . Since $B_{n,n}$ consists of two copies of K_n joined by an edge and since K_n has no arrow edge dominating set according to Proposition 3.2, $B_{n,n}$ has no arrow edge dominating set. □

Proposition 3.6. The helm graph $H_n(n \geq 3)$ has no arrow edge dominating set.

Proof . Since every edge in any edge dominating set dominates two or more edges from $E - D_e$ or has no neighbourhoods from D_e. □

Proposition 3.7. A big helm $\mathcal{H}_n$ graph has no arrow edge dominating set .

Proof . Similar to the proof of Proposition 3.6. □

Proposition 3.8. The complement path $\overline{P_n}$ graph has no arrow edge dominating set.

Proof . If $n \leq 4$, it is clear that $\overline{P_n}$ has no $\gamma_{\text{are}}-$ set where its edges less than four according to Observation 2.5(5). If $n = 5$, then $\overline{P_5}$ has no $\gamma_{\text{are}}-$ set because there are two edges in any edge dominating set D_e which are dominate two edges or adjacent with only one edge . If $n \geq 6$, then every dominating set D_e has either an edge that dominates two or more edges or an edge doesn't dominate any edge. Hence, there is no arrow edge dominating set in $\overline{P_n}$. For example see Fig. 1.4. □

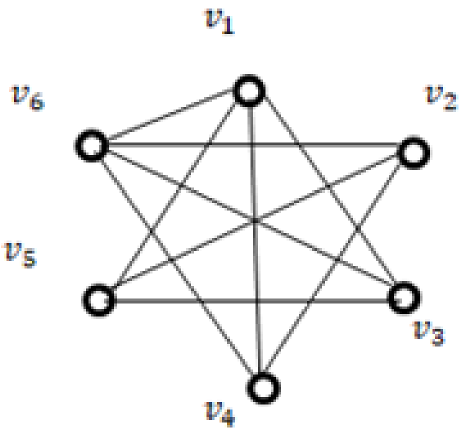
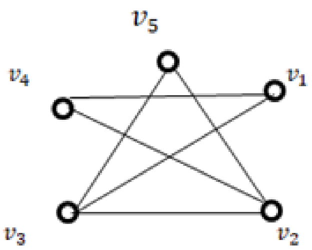


Figure 4: A complement path graph $\overline{P_6}$

Proposition 3.9. The complement cycle graph $\overline{C_n}$ has no arrow edge dominating set.

Proof . If $n = 3$ it is clear that $\overline{C_n}$ has no $\gamma_{\text{are}}-$ set because it is null graph. If $n = 4$, then $\overline{C_4}$ has only two edges and since $\gamma_{\text{are}}(G) \geq 3$, $\overline{C_4}$ has no $\gamma_{\text{are}}-$ set. If $n = 5$ every edge in any edge dominating set D_e dominates two edges or adjacent with only one edge . If $n \geq 6$, then $\overline{C_n}$ has no arrow edge dominating set in similar reason of Proposition 3.8. For example see Fig. 1.5. □



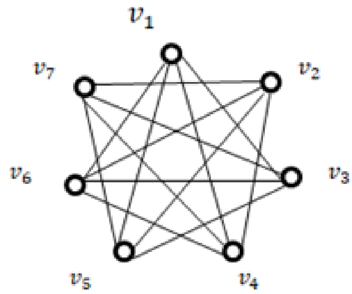


Figure 5: A Complement Cycle Graph C_n

Proposition 3.10. The complement of a complete graph $\overline{K_n}$ has no arrow edge dominating set.

Proof . Since $\overline{K_n}$ is a null graph, then $\overline{K_n}$ has no any edge, $\overline{K_n}$ has no arrow edge dominating set . □

Proposition 3.11. The complement of a complete bipartite graph $\overline{K_{n,m}}$ has no arrow dominating set ..

Proof . Since $\overline{K_{n,m}} = K_n \cup K_m$ is the union of two complete graphs and since a complete graph has no arrow edge dominating set according to Proposition 3.2, $\overline{K_{n,m}}$ has no arrow edge dominating set. □

Remark 3.12. If G has no arrow vertex dominating set, then G may be has arrow edge dominating set.

As an example, $S_n (n \geq 4)$ graph has no arrow vertex dominating set but it has arrow edge dominating set. For example, see Fig. 1.6.

(a) $\gamma_{ar}(S_4) = 3$

(b) S_4 without pitchfork vertex domination

Figure 6: The Star Graph S_4

4 Conclusion

A new model of edges domination called "arrow edge domination" is introduced in this paper. Several properties are given and some bounds of the arrow edge domination number are discussed with respect to the size of graph. This model of domination are applied on several known graphs. Some graphs haven't this type of domination, while other graphs have it. Such graphs that are discussed here, path graph, cycle graph, complete graph, wheel graph, complete bipartite graph, Barbell graph, helm graph, big helm graph, complement path graph, complement cycle graph, complement of complete graph and complement of complete bipartite graph. An important fact given here, if G has no arrow vertex dominating set, then G may be has arrow edge dominating set and an example is putted.

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