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Numerous new Jacobi elliptic function solutions for fractional space-time perturbed Gerdjikov-Ivanov equation with conformable derivative

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Abstract

In this article, Numerous new Jacobi elliptic function solutions for perturbed Gerdjikov-Ivanov equation with space-time conformable fractional derivative have been extracted using the new extended auxiliary equation method. Solitary and periodic solutions are retrieved from the Jacobi elliptic function solutions. These solutions' existence is likewise guaranteed by the constraint criteria. In order to better understand the physical processes, we also show some graphical representations of the solutions.

Keywords: Perturbed Gerdjikov-Ivanov equation, the Conformable Derivative, the new extended auxiliary equation method, Jacobi elliptic function solutions
2020 MSC: 39A14,35Q55,26A33, 35C05, 35C08, 35C09

1 Introduction

One of the most difficult tasks in mathematical physics is finding exact analytic solutions to partial differential equations (PDE). In general, it is impossible to find an accurate solution to many types of such equations. The employment of numerical and approximation approaches is unavoidable in these situations. However, accurate (PDE) solver approaches are always favored because they offer the answer immediately and without any limits on how it can be used. The goal of this paper is to look at the fractional nonlinear Schrödinger equation (FNLSE) which has a quintic nonlinearity, namely the perturbed Gerdjikov-Ivanov equation with space-time conformable fractional derivative [1] from an exact approach perspective. In nonlinear fiber optics, this equation is very important. It is also having a wide range of uses in photonic crystal fibers. The model has grabbed the attention of numerous scholars in recent years due to its great significance. The model with cubic non-linearity has been subjected to a range of powerful approaches, including, the semi-inverse variational principle [2], the sine-Gordon equation approach [3], the extended trial equation method [4], the $\exp(\phi(\zeta))$-expansion and the Kudryashov methods [5], the $\left(\frac{G'}{G^2}\right)$-expansion method [6]. The goal of this study was to use a well-known method [7] known as a new extended auxiliary equation method to find a large number of new Jacobi elliptic function solutions to the following nonlinear Schrödinger equation.

$$i\mathcal{D}_t^{\alpha}u + a\mathcal{D}_x^{2\alpha}u + b|u|^4u = i\left[cu^2\mathcal{D}_x^{\alpha}\bar{u} + \lambda_1\mathcal{D}_x^{\alpha}u + \lambda_2\mathcal{D}_x^{\alpha}(|u|^2u) + \theta\mathcal{D}_x^{\alpha}(|u|^2)u\right], \tag{1.1}$$

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Here $q(x,t)$ denoted the macroscopic complex-valued wave profile. In this equation, $D_t^{\alpha}u$ represents linear temporal evolution, $D_x^{2\alpha}u$ represents group velocity dispersion (GVD), and $|u|^4u$ represents quintic non-linearity. The parameters a, b are the coefficients of these quantities, respectively. Furthermore, the non-linear dispersion coefficient denoted by c. Last but not least, the constants λ_1, λ_2 and θ are known parameters related to perturbative effects.

The following is a breakdown of the structure of this paper: The new extended auxiliary equation method is described in Section 2. We use this method to solve the perturbed Gerdjikov-Ivanov equation with space-time conformable fractional derivative in Section 3. We ran numerical simulations of the given results in section 4. Conclusions are reached in Section 5.

2 Description of new extended auxiliary equation method

Assume that we are given nonlinear partial differential equation of the form;

$$W(u, u_x, u_t, u_{xx}, u_{xt}, ...) = 0, \tag{2.1}$$

where W is a polynomial function. The main steps for solving equations (1.1) using the new extended auxiliary equation method [7] are summarized as follows:

Step 1: We use the wave transformation

$$\begin{aligned} u(x,t) &= \psi(\zeta(x,t))e^{i\eta(x,t)}, \\ \zeta(x,t) &= (\frac{1}{\alpha})x^{\alpha} - (\frac{\nu}{\alpha})t^{\alpha}, \\ \eta(x,t) &= (\frac{-k}{\alpha})x^{\alpha} + (\frac{w}{\alpha})t^{\alpha}. \end{aligned} \tag{2.2}$$

Step 2: Substituting (2.2) into (2.1) yields an ordinary differential equation in ζ of the form;

$$Q(U, U^{'}(\zeta), , U^{''}(\zeta), U^{'''}(\zeta), ...) = 0, \tag{2.3}$$

where Q is a general polynomial.

Step 3: Assume that (2.3) has the formal solution;

$$U(\zeta) = \sum_{i=0}^{2N} A_i F^i(\zeta), \tag{2.4}$$

where A_i are constants to be determined, such that $A_{2N} \neq 0$, and $F(\zeta)$ satisfies the first order ODE:

$$\left[F^{'}(\zeta)\right]^2 = c_0 + c_2F^2 + c_4F^4 + c_6F^6, \tag{2.5}$$

where c_i,$(i = 0, 2, 4, 6)$ are arbitrary constants to be determined.

Step 4: Determining the positive integer N by balancing the highest order derivatives and the nonlinear terms in equation (2.3).

Step 5: Substituting (2.4) along with (2.5) into (2.3) and collecting all the coefficients of $F^j(F^{'})^l$, $(j = 0, 1, 2, ...)$ and $(l = 0, 1)$ then setting each coefficient to zero, a set of algebraic equations is obtained for c_i, $(i = 0, 2, 4, 6)$, A_j, $(j = 0, 1, 2, ..., 2N)$, k and μ. Solving the system we find c_i, $(i = 0, 2, 4, 6)$, A_j, $(j = 0, 1, 2, ..., 2N)$, k and μ.

Step 6: It is well known [7, 9] that Equation (2.5) has the following solutions:

$$F(\zeta) = \frac{1}{2}\left[\frac{-c_4}{c_6}\left(1 \pm f_i(\zeta)\right)\right]^{\frac{1}{2}}, \tag{2.6}$$

where the function f_i, $(i = 1, 2, 3, ..., 12)$ can be expressed through the Jacobi elliptic function $sn(\zeta, m)$, $cn(\zeta, m)$, $dn(\zeta, m)$, where $0 < m < 1$ is the modulus of the Jacobi elliptic functions. When $m \to 0$ or $m \to 1$, the Jacobi elliptic function degenerate to hyperbolic function and trigonometric functions, respectively.

The functions $f_i(\zeta)$, $(i = 1, 2, 3, ..., 12)$ given by (2.6) have 12 forms as follows:

If $c_0 = \frac{c_4^3(m^2-1)}{32c_6^2m^2}$, $c_2 = \frac{c_4^2(5m^2-1)}{16c_6m^2}$, $c_6 > 0$, then

$$f_1(\zeta) = sn(\rho\zeta), \qquad f_2 = \frac{1}{msn(\rho\zeta)}, \qquad \rho = \frac{c_4}{2m}\sqrt{\frac{1}{c_6}} \tag{2.7}$$

If $c_0 = \frac{c_4^3(1-m^2)}{32c_6^2}$, $c_2 = \frac{c_4^2(5-m^2)}{16c_6}$, $c_6 > 0$, then

$$f_3(\zeta) = msn(\rho\zeta), \qquad f_4 = \frac{1}{sn(\rho\zeta)}, \qquad \rho = \frac{c_4}{2}\sqrt{\frac{1}{c_6}} \tag{2.8}$$

If $c_0 = \frac{c_4^3}{32c_6^2m^2}$, $c_2 = \frac{c_4^2(4m^2+1)}{16c_6m^2}$, $c_6 < 0$, then

$$f_5(\zeta) = cn(\rho\zeta), \qquad f_6 = \frac{sn(\rho\zeta)}{dn(\rho\zeta)}\sqrt{1-m^2}, \qquad \rho = \frac{c_4}{2m}\sqrt{\frac{-1}{c_6}} \tag{2.9}$$

If $c_0 = \frac{c_4^3m^2}{32c_6^2(m^2-1)}$, $c_2 = \frac{c_4^2(5m^2-4)}{16c_6(m^2-1)}$, $c_6 < 0$, then

$$f_7(\zeta) = \frac{dn(\rho\zeta)}{\sqrt{1-m^2}}, \qquad f_8 = \frac{1}{dn(\rho\zeta)}, \qquad \rho = \frac{c_4}{2}\sqrt{\frac{-1}{c_6(1-m^2)}} \tag{2.10}$$

If $c_0 = \frac{c_4^3}{32c_6^2(1-m^2)}$, $c_2 = \frac{c_4^2(4m^2-5)}{16c_6(m^2-1)}$, $c_6 > 0$, then

$$f_9(\zeta) = \frac{1}{cn(\rho\zeta)}, \qquad f_{10} = \frac{dn(\rho\zeta)}{\sqrt{1-m^2}sn(\rho\zeta)}, \qquad \rho = \frac{c_4}{2}\sqrt{\frac{1}{c_6(1-m^2)}} \tag{2.11}$$

If $c_0 = \frac{m^2c_4^3}{32c_6^2}$, $c_2 = \frac{c_4^2(m^2+4)}{16c_6}$, $c_6 < 0$, then

$$f_{11}(\zeta) = dn(\rho\zeta), \qquad f_{12} = \frac{\sqrt{1-m^2}}{dn(\rho\zeta)}, \qquad \rho = \frac{c_4}{2}\sqrt{\frac{-1}{c_6}} \tag{2.12}$$

Step 7: Substituting c_i, $(i = 0, 2, 4, 6)$, A_j, $(j = 0, 1, 2, ..., 2N)$, k, μ and (2.7)-(2.12) into (2.4), a numerous new kinds of Jacobi elliptic function solutions of equations (1.1) will be obtained.

3 The Conformable Derivative

Let $\alpha \in (0, 1]$, $\mathbb{R}^+ = [0, \infty)$ and given a continuous function $u : \mathbb{R}^+ \to \mathbb{R}$.

Definition 3.1 ([1, 8]). For any $\alpha \in (0, 1]$ the conformable derivative $\mathcal{D}_t^\alpha$ of the function u of order $0 < \alpha \leq 1$ is defined by

$$\mathcal{D}_t^\alpha u(t) = \lim_{\epsilon\to 0}\frac{u(t+\epsilon t^{1-q}) - u(t)}{\epsilon}.$$

Proposition 3.2 ([8]). Let $\alpha \in (0, 1]$ and $u(t), y(t)$ be two α-differentiable functions at a point $t > 0$. Then:

(a) $\mathcal{D}_t^\alpha(a\,u + b\,y) = a\,\mathcal{D}_t^\alpha(u) + b\,\mathcal{D}_t^\alpha(y)$ for all $a, b \in \mathbb{R}$.

(b) $\mathcal{D}_t^\alpha(t^p) = p\,t^{p-\alpha}$ for any $p \in \mathbb{R}$.

(c) $\mathcal{D}_t^\alpha(u\,y) = u\,\mathcal{D}_t^\alpha(y) + y\,\mathcal{D}_t^\alpha(u)$.

(d) $\mathcal{D}_t^\alpha\left(\frac{u}{y}\right) = \frac{y\,\mathcal{D}_t^\alpha(u) - u\,\mathcal{D}_t^\alpha(y)}{y^2}$.

(e) $\mathcal{D}_t^\alpha(u) = 0$ for any $u = \lambda$, where λ is an arbitrary constant.

(f) $\mathcal{D}_t^\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

(g) $\mathcal{D}_t^\alpha(f \circ g)(t) = t^{1-\alpha}f'(t)g'(t)$.

4 Applications

Substituting (2.2) into (1.1) and by separating the real and the imaginary parts, we get:

$$\textbf{The Real Part:}\quad a\psi^{''}-(w+ak^2+\lambda_1 k)\psi+(c-\lambda_2)k\psi^3+b\psi^5=0. \tag{4.1}$$

$$\textbf{The Imaginary Part:}\quad \nu+\lambda_1+2ak+(c+3\lambda_2+2\theta)\psi^2=0. \tag{4.2}$$

From (4.2) we have $\nu=-(\lambda_1+2ak), \quad \theta=-\frac{1}{2}(c+3\lambda_2).$

From (4.1) balancing $\psi^{''}$ and ψ^5 gives $N=\frac{1}{2}$

Setting $\psi(\zeta)=\sqrt{\phi(\zeta)}$ gives

$$a(2\phi\phi^{''}-(\phi^{'})^2)-4(w+ak^2+\lambda_1 k)\phi^2+4(c-\lambda_2)k\phi^3+4b\phi^4=0. \tag{4.3}$$

Balancing $\phi\phi^{''}$ with ϕ^4 gives $N=1$. Substituting with $N=1$ in (2.5) we get

$$\phi=\sum_{i=0}^{2}A_iF^i(\zeta)=A_0+A_1f+A_2f^2, \tag{4.4}$$

The analytical solutions for Equation (1.1) will be derived as a result of substituting (4.4) into (4.3) and following the steps given in the technique we obtain.

Case 1:

$A_0=0, \quad A_1=0, \quad A_2=\pm\frac{\sqrt{3}\ a\ k\,(\lambda_2-c)\sqrt{-\frac{c_6}{a\,b}}}{k\,(\lambda_2-c)}$, provide $(\lambda_2-c)\,k\neq 0, \quad a\,b\neq 0$,
$c_2=-\frac{2(w+a\,k^2+\lambda_1\,k)}{a}$, provide $a\neq 0$, $c_4=\frac{\sqrt{3}\,k\,(\lambda_2-c)\sqrt{-\frac{c_6}{a\,b}}}{2}$, $c_0=c_0, \quad c_6=c_6$,
and we have $\zeta(x,t)=(\frac{1}{\alpha})x^{\alpha}+(\frac{(\lambda_1+2ak)}{\alpha})t^{\alpha}$

Substituting into (4.4) along with (2.6), we obtain the following solutions of equation (4.3):

$$\psi(\zeta)=\frac{1}{2}\left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2-c)\sqrt{-\frac{c_6}{a\,b}}\right)}{2\,c_6}\left(1\pm f_i(\zeta)\right)\right]^{\frac{1}{2}}. \tag{4.5}$$

Now, we obtain the following Jacobi elliptic function solutions for equation (1.1):

From (2.7),(4.5) and (2.2), we obtain

$$u_1=\left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2-c)\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1\pm sn\left(\frac{\sqrt{3}\,k\,(\lambda_2-c)\sqrt{-\frac{1}{a\,b}}}{4m}\zeta\right)\right)\right]^{\frac{1}{2}}e^{i\eta(x,t)}, \tag{4.6}$$

$$u_2=\left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2-c)\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1\pm\frac{1}{msn\left(\frac{\sqrt{3}\,k\,(\lambda_2-c)\sqrt{-\frac{1}{a\,b}}}{4m}\zeta\right)}\right)\right]^{\frac{1}{2}}e^{i\eta(x,t)}, \tag{4.7}$$

provided that $a\,b<0$.

If $m\to 1$, then $sn(\zeta)\to\tanh(\zeta)$, and hence equation (1.1) has the hyperbolic function solutions

$$u_1=\left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2-c)\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1\pm\tanh\left(\frac{\sqrt{3}\,k\,(\lambda_2-c)\sqrt{-\frac{1}{a\,b}}}{4m}\zeta\right)\right)\right]^{\frac{1}{2}}e^{i\eta(x,t)}, \tag{4.8}$$

$$u_2 = \left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1 \pm \coth\left(\frac{\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{1}{a\,b}}}{4m}\zeta\right)\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.9}$$

where $a\,b < 0$.

From (2.8),(4.5) and (2.2), we obtain

$$u_3 = \left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1 \pm msn\left(\frac{\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{1}{a\,b}}}{4}\zeta\right)\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.10}$$

$$u_4 = \left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1 \pm \frac{1}{sn\left(\frac{\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{1}{a\,b}}}{4}\zeta\right)}\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.11}$$

provided that $a\,b < 0$.

If $m \to 0$, then $sn(\zeta) \to \sin(\zeta)$, and hence equation (1.1) has the following solutions

$$u_3 = \left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.12}$$

and the periodic solution

$$u_4 = \left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1 \pm \csc\left(\frac{\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{1}{a\,b}}}{4}\zeta\right)\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.13}$$

where $a\,b < 0$.

From (2.9),(4.5) and (2.2), we obtain

$$u_5 = \left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1 \pm cn\left(\frac{\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{\frac{1}{a\,b}}}{4m}\zeta\right)\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.14}$$

$$u_6 = \left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1 \pm \frac{\sqrt{1-m^2}\,sn\left(\frac{\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{\frac{1}{a\,b}}}{4m}\zeta\right)}{dn\left(\frac{\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{\frac{1}{a\,b}}}{4m}\zeta\right)}\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.15}$$

provided that $a\,b > 0$.

If $m \to 1$, then $cn(\zeta) \to sech(\zeta)$, and hence equation (1.1) has the hyperbolic solutions

$$u_5 = \left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1 \pm sech\left(\frac{\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{\frac{1}{a\,b}}}{4m}\zeta\right)\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.16}$$

and the solution

$$u_6 = \left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.17}$$

From (2.10),(4.5) and (2.2), we obtain

$$u_7=\left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1\pm\frac{dn\left(\frac{\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{\frac{1}{a\,b(1-m^2)}}}{4}\zeta\right)}{\sqrt{1-m^2}}\right)\right]^{\frac{1}{2}}e^{i\eta(x,t)}, \tag{4.18}$$

$$u_8=\left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1\pm\frac{1}{dn\left(\frac{\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{\frac{1}{a\,b(1-m^2)}}}{4}\zeta\right)}\right)\right]^{\frac{1}{2}}e^{i\eta(x,t)}, \tag{4.19}$$

provided that $a\,b>0$.

If $m\to 0$, then $dn(\zeta)\to 1$, and hence equation (1.1) has the solutions

$$u_7=u_8=\left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{4\,c_6}\right]^{\frac{1}{2}}e^{i\eta(x,t)}, \tag{4.20}$$

From (2.11),(4.5) and (2.2), we obtain

$$u_9=\left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1\pm\frac{1}{cn\left(\frac{\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{1}{a\,b(1-m^2)}}}{4}\zeta\right)}\right)\right]^{\frac{1}{2}}e^{i\eta(x,t)}, \tag{4.21}$$

$$u_{10}=\left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1\pm\frac{dn\left(\frac{\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{1}{a\,b(1-m^2)}}}{4}\zeta\right)}{\sqrt{1-m^2}\,sn\left(\frac{\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{1}{a\,b(1-m^2)}}}{4}\zeta\right)}\right)\right]^{\frac{1}{2}}e^{i\eta(x,t)}, \tag{4.22}$$

provided that $a\,b>0$.

If $m\to 0$, then $dn(\zeta)\to 1$, $cn(\zeta)\to\cos(\zeta)$, $sn(\zeta)\to\sin(\zeta)$ and hence equation (1.1) has the periodic solutions

$$u_9=\left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1\pm\sec\left(\frac{\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{1}{a\,b(1-m^2)}}}{4}\zeta\right)\right)\right]^{\frac{1}{2}}e^{i\eta(x,t)}, \tag{4.23}$$

$$u_{10}=\left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1\pm\csc\left(\frac{\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{1}{a\,b(1-m^2)}}}{4}\zeta\right)\right)\right]^{\frac{1}{2}}e^{i\eta(x,t)}, \tag{4.24}$$

From (2.12),(4.5) and (2.2), we obtain

$$u_{11}=\left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1\pm dn\left(\frac{\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{\frac{1}{a\,b}}}{4}\zeta\right)\right)\right]^{\frac{1}{2}}e^{i\eta(x,t)}, \tag{4.25}$$

$$u_{12}=\left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1\pm\frac{\sqrt{1-m^2}}{dn\left(\frac{\sqrt{3}\,k\,(\lambda_2-c)\,\sqrt{\frac{1}{a\,b}}}{4}\zeta\right)}\right)\right]^{\frac{1}{2}}e^{i\eta(x,t)}, \tag{4.26}$$

provided that $a\,b > 0$.

If $m \to 0$, then $dn(\zeta) \to 1$, and hence equation (1.1) has the solutions

$$u_{11} = u_{12} = \left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{4\,c_6}\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.27}$$

If $m \to 1$, then $dn(\zeta) \to sech(\zeta)$, and hence equation (1.1) has the hyperbolic function solutions

$$u_{11} = \left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\left(1 \pm sech\left(\frac{\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{\frac{1}{a\,b}}}{4}\zeta\right)\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.28}$$

$$u_{12} = \left[\frac{-\left(\sqrt{3}\,k\,(\lambda_2 - c)\,\sqrt{-\frac{c_6}{a\,b}}\right)}{8\,c_6}\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.29}$$

Case 2:

$A_0 = 0, \quad A_1 = 0, \quad A_2 = \frac{2ac_4}{(\lambda_2 - c)k}, \quad$ provide $\quad (\lambda_2 - c)\,k \neq 0,$
$c_2 = -\frac{2(w + a\,k^2 + \lambda_1\,k)}{a}, \quad$ provide $a \neq 0, \quad c_6 = -\frac{4abc_4^2}{3(\lambda_2 - c)^2 k^2}, \quad c_0 = c_0, \quad c_4 = c_4,$
and we have $\zeta(x,t) = (\frac{1}{\alpha})x^\alpha + (\frac{(\lambda_1 + 2ak)}{\alpha})t^\alpha$

Substituting into (4.4) along with (2.6), we obtain the following solutions of equation (4.3):

$$\psi(\zeta) = \frac{1}{2}\left[\frac{2ac_4}{(\lambda_2 - c)k}\left(1 \pm f_i(\zeta)\right)\right]^{\frac{1}{2}}. \tag{4.30}$$

Now, we obtain the following Jacobi elliptic function solutions for equation (1.1):

From (2.7),(4.30) and (2.2), we obtain

$$u_1 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\left(1 \pm sn\left(\sqrt{-\frac{3(\lambda_2 - c)^2 k^2}{16\,m^2 a\,b}}\zeta\right)\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.31}$$

$$u_2 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\left(1 \pm \frac{1}{msn\left(\sqrt{-\frac{3(\lambda_2 - c)^2 k^2}{16\,m^2 a\,b}}\zeta\right)}\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.32}$$

provided that $a\,b < 0$.

If $m \to 1$, then $sn(\zeta) \to \tanh(\zeta)$, and hence equation (1.1) has the hyperbolic function solutions

$$u_1 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\left(1 \pm \tanh\left(\sqrt{-\frac{3(\lambda_2 - c)^2 k^2}{16\,m^2 a\,b}}\zeta\right)\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.33}$$

$$u_2 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\left(1 \pm \coth\left(\sqrt{-\frac{3(\lambda_2 - c)^2 k^2}{16\,m^2 a\,b}}\zeta\right)\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.34}$$

where $a\,b < 0$.

From (2.8),(4.30) and (2.2), we obtain

$$u_3 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\left(1 \pm msn\left(\sqrt{-\frac{3(\lambda_2 - c)^2 k^2}{16\,a\,b}}\zeta\right)\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.35}$$

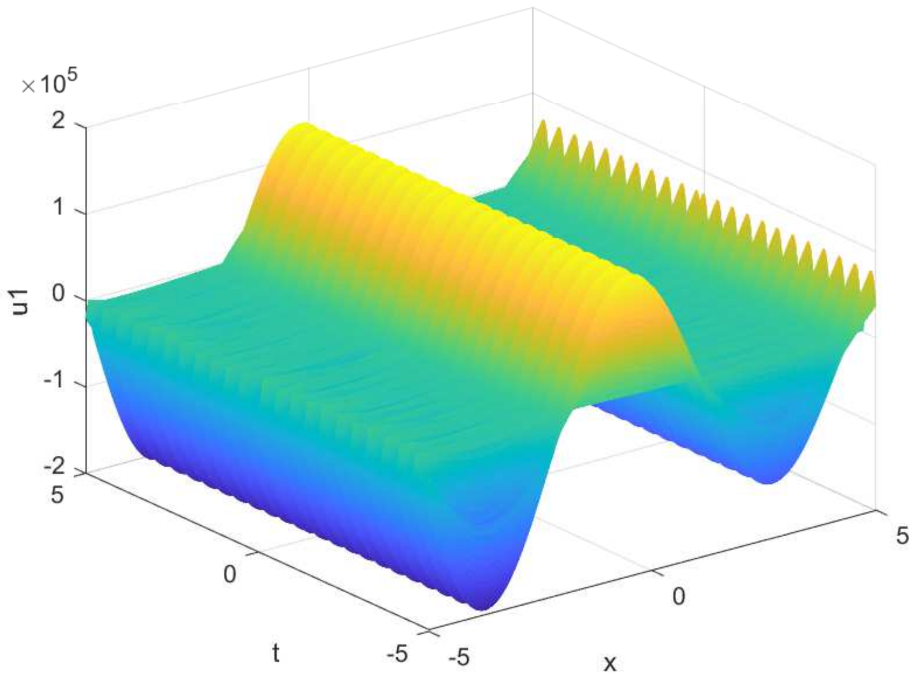

Figure 1: Plotting of the Real part of $|u_1(x,t)|$ Jacobi elliptic function solution (4.31) when $a = 15.2$, $b = -1$, $k = 1$, $\lambda_1 = 3$, $\lambda_2 = 3$, $c = 1$, $m = \frac{1}{\sqrt{5}}$, $\alpha = 0.97$.

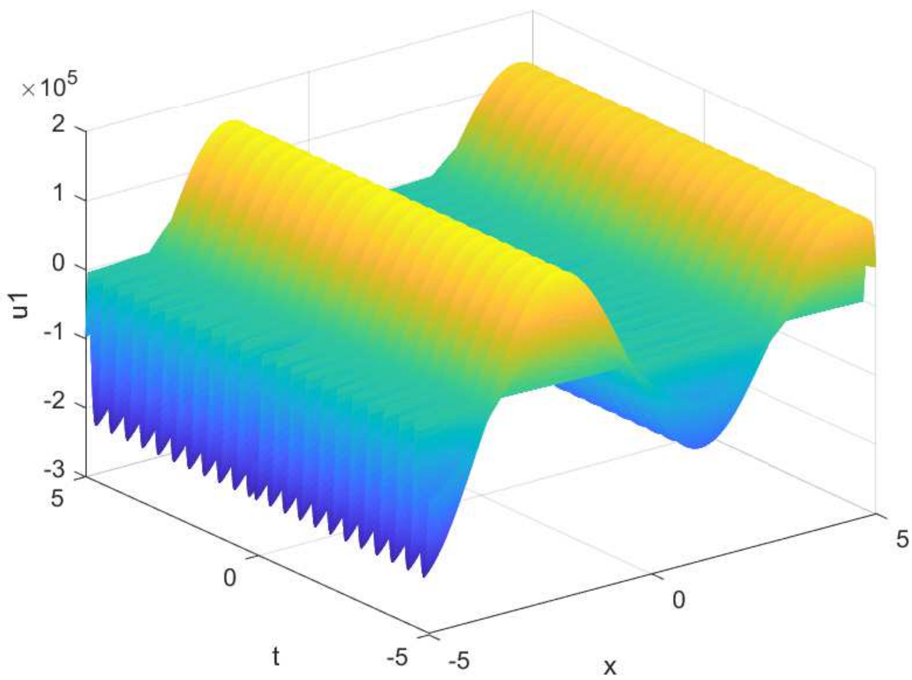

Figure 2: Plotting of the Imaginary part of $|u_1(x,t)|$ Jacobi elliptic function solution (4.31) when $a = 15.2$, $b = -1$, $k = 1$, $\lambda_1 = 3$, $\lambda_2 = 3$, $c = 1$, $m = \frac{1}{\sqrt{5}}$, $\alpha = 0.97$.

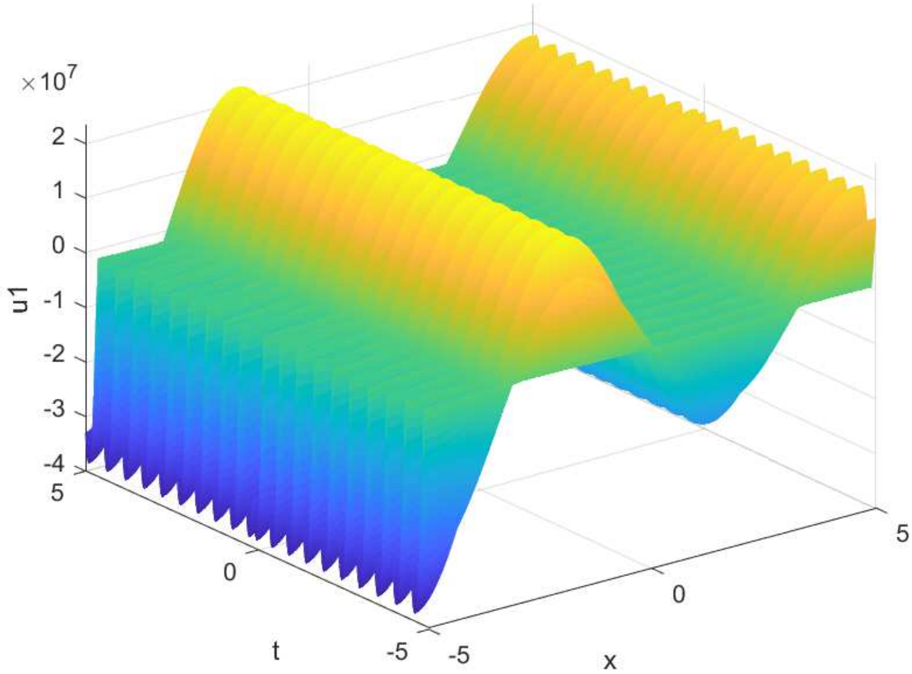

Figure 3: Plotting of the Real part of $|u_5(x,t)|$ Jacobi elliptic function solution (4.39) when $a = 15.2$, $b = -1$, $k = 1$, $\lambda_1 = 3$, $\lambda_2 = 3$, $c = 1$, $c_4 = 1$ $m = 0.5$, $\alpha = 0.95$.

$$u_4 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\left(1 \pm \frac{1}{sn\left(\sqrt{-\frac{3(\lambda_2-c)^2k^2}{16\,a\,b}}\zeta\right)}\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.36}$$

provided that $a\,b < 0$.

If $m \to 0$, then $sn(\zeta) \to \sin(\zeta)$, and hence equation (1.1) has the following solutions

$$u_3 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.37}$$

and the periodic solution

$$u_4 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\left(1 \pm \csc\left(\sqrt{-\frac{3(\lambda_2-c)^2k^2}{16\,a\,b}}\zeta\right)\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.38}$$

where $a\,b < 0$.

From (2.9),(4.30) and (2.2), we obtain

$$u_5 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\left(1 \pm cn\left(\sqrt{-\frac{3(\lambda_2-c)^2k^2}{16\,m^2\,a\,b}}\zeta\right)\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.39}$$

$$u_6 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\left(1 \pm \frac{\sqrt{1-m^2}\,sn\left(\sqrt{-\frac{3(\lambda_2-c)^2k^2}{16\,m^2\,a\,b}}\zeta\right)}{dn\left(\sqrt{-\frac{3(\lambda_2-c)^2k^2}{16\,m^2\,a\,b}}\zeta\right)}\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.40}$$

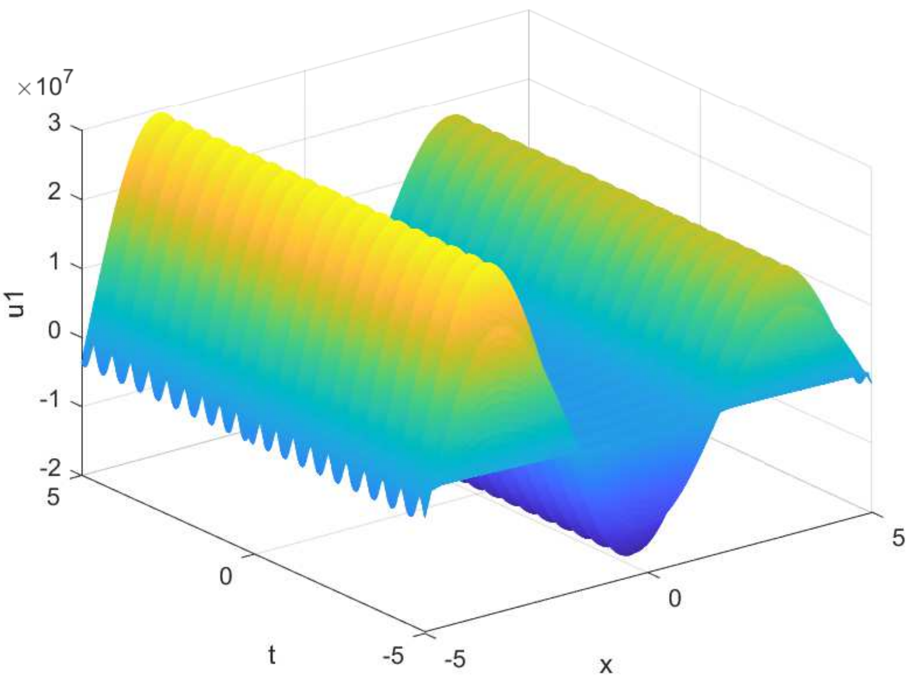

Figure 4: Plotting of the Imaginary part of $|u_5(x,t)|$ Jacobi elliptic function solution (4.39) when $a = 15.2$, $b = -1$, $k = 1$, $\lambda_1 = 3$, $\lambda_2 = 3$, $c = 1$, $c_4 = 1$ $m = 0.5$, $\alpha = 0.95$.

provided that $a\,b < 0$.

If $m \to 1$, then $cn(\zeta) \to sech(\zeta)$, and hence equation (1.1) has the hyperbolic solutions

$$u_5 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\left(1 \pm sech\left(\sqrt{-\frac{3(\lambda_2 - c)^2 k^2}{16\,m^2\,a\,b}}\zeta\right)\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.41}$$

and the solution

$$u_6 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.42}$$

From (2.10),(4.30) and (2.2), we obtain

$$u_7 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\left(1 \pm \frac{dn\left(\sqrt{\frac{3(\lambda_2-c)^2 k^2}{16\,(1-m^2)\,a\,b}}\zeta\right)}{\sqrt{1-m^2}}\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.43}$$

$$u_8 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\left(1 \pm \frac{1}{dn\left(\sqrt{\frac{3(\lambda_2-c)^2 k^2}{16\,(1-m^2)\,a\,b}}\zeta\right)}\right)\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.44}$$

provided that $a\,b > 0$.

If $m \to 0$, then $dn(\zeta) \to 1$, and hence equation (1.1) has the solutions

$$u_7 = u_8 = \left[\frac{ac_4}{2(\lambda_2 - c)k}\right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.45}$$

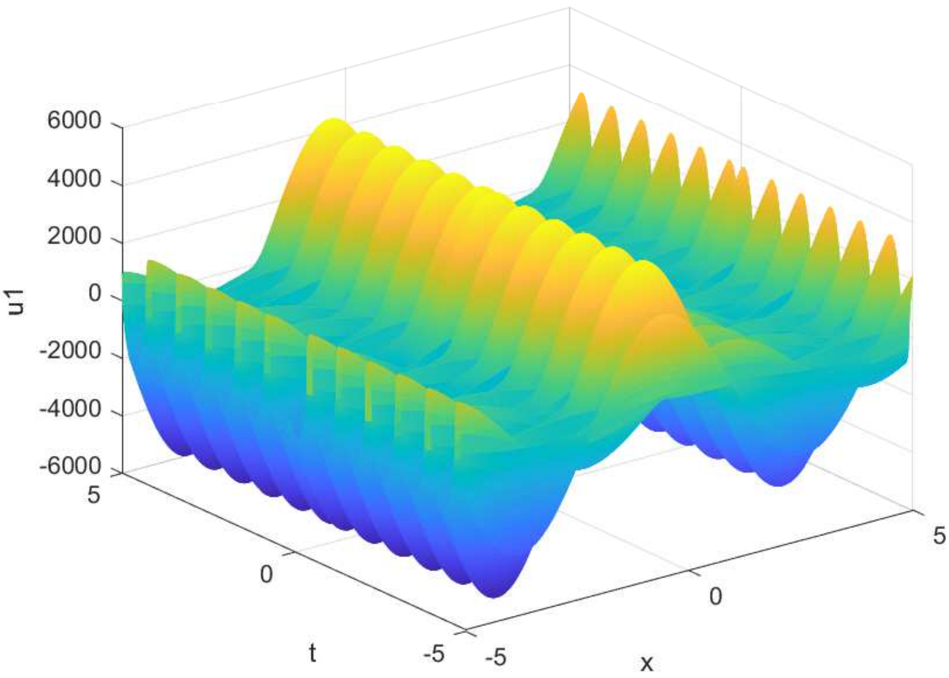

Figure 5: Plotting of the Real part of $|u_7(x,t)|$ Jacobi elliptic function solution (4.44) when $a = 15.2$, $b = 1$, $k = 1$, $\lambda_1 = 3$, $\lambda_2 = 3$, $c = 1$, $c_4 = 1$ $m = \frac{1}{\sqrt{3}}$, $\alpha = 0.98$.

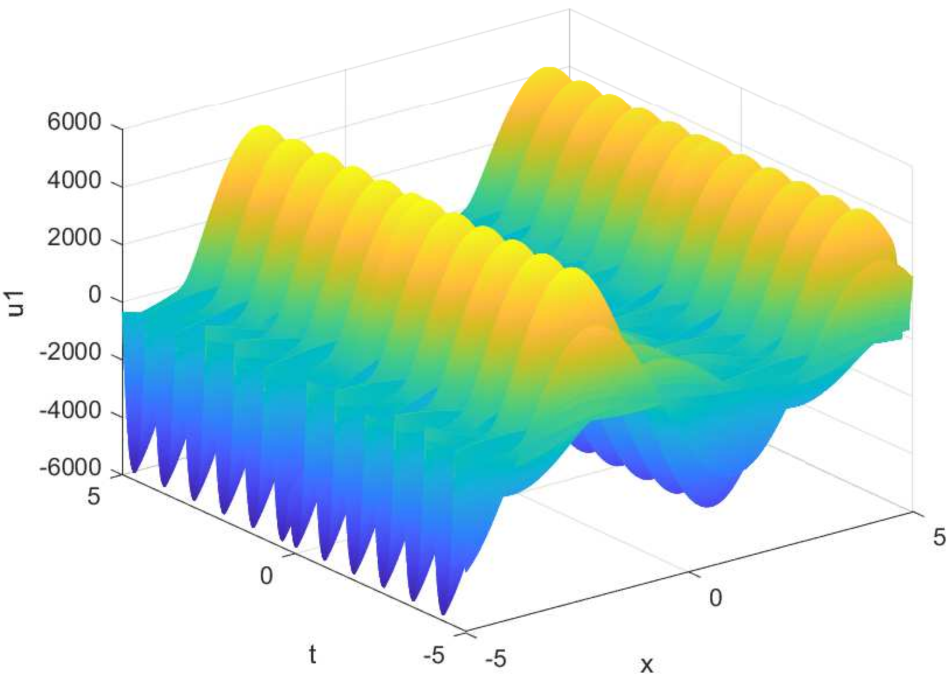

Figure 6: Plotting of the Imaginary part of $|u_7(x,t)|$ Jacobi elliptic function solution (4.44) when $a = 15.2$, $b = 1$, $k = 1$, $\lambda_1 = 3$, $\lambda_2 = 3$, $c = 1$, $c_4 = 1$ $m = \frac{1}{\sqrt{3}}$, $\alpha = 0.98$.

From (2.11),(4.30) and (2.2), we obtain

$$u_9 = \left[\frac{ac_4}{2(\lambda_2 - c)k} \left(1 \pm \frac{1}{cn\left(\sqrt{-\frac{3(\lambda_2-c)^2k^2}{16\,(1-m^2)\,a\,b}}\zeta\right)} \right) \right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.46}$$

$$u_{10} = \left[\frac{ac_4}{2(\lambda_2 - c)k} \left(1 \pm \frac{dn\left(\sqrt{-\frac{3(\lambda_2-c)^2k^2}{16\,(1-m^2)\,a\,b}}\zeta\right)}{\sqrt{1-m^2}\, sn\left(\sqrt{-\frac{3(\lambda_2-c)^2k^2}{16\,(1-m^2)\,a\,b}}\zeta\right)} \right) \right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.47}$$

provided that $a\,b < 0$.

If $m \to 0$, then $dn(\zeta) \to 1$, $cn(\zeta) \to \cos(\zeta)$, $sn(\zeta) \to \sin(\zeta)$ and hence equation (1.1) has the periodic solutions

$$u_9 = \left[\frac{ac_4}{2(\lambda_2 - c)k} \left(1 \pm \sec\left(\sqrt{\frac{3(\lambda_2-c)^2k^2}{16\,a\,b}}\zeta \right) \right) \right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.48}$$

$$u_{10} = \left[\frac{ac_4}{2(\lambda_2 - c)k} \left(1 \pm \csc\left(\sqrt{\frac{3(\lambda_2-c)^2k^2}{16\,a\,b}}\zeta \right) \right) \right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.49}$$

From (2.12),(4.30) and (2.2), we obtain

$$u_{11} = \left[\frac{ac_4}{2(\lambda_2 - c)k} \left(1 \pm dn\left(\sqrt{\frac{3(\lambda_2-c)^2k^2}{16\,a\,b}}\zeta \right) \right) \right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.50}$$

$$u_{12} = \left[\frac{ac_4}{2(\lambda_2 - c)k} \left(1 \pm \frac{\sqrt{1-m^2}}{dn\left(\sqrt{\frac{3(\lambda_2-c)^2k^2}{16\,a\,b}}\zeta\right)} \right) \right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.51}$$

provided that $a\,b > 0$.

If $m \to 0$, then $dn(\zeta) \to 1$, and hence equation (1.1) has the solutions

$$u_{11} = u_{12} = \left[\frac{ac_4}{2(\lambda_2 - c)k} \right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.52}$$

If $m \to 1$, then $dn(\zeta) \to sech(\zeta)$, and hence equation (1.1) has the hyperbolic function solutions

$$u_{11} = \left[\frac{ac_4}{2(\lambda_2 - c)k} \left(1 \pm sech\left(\sqrt{-\frac{3(\lambda_2-c)^2k^2}{16\,a\,b}}\zeta \right) \right) \right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.53}$$

$$u_{12} = \left[\frac{ac_4}{2(\lambda_2 - c)k} \right]^{\frac{1}{2}} e^{i\eta(x,t)}, \tag{4.54}$$

5 Graphs of some solutions

In this section, some graphs of the Jacobi elliptic function solutions of the given equation are presented based on the algorithm provided in subsection 2. Figures $1-3$ have been presented to highlight the dynamic characteristics of some selected analytical results acquired in section 4. Figure 1 illustrate the behavior of the real and the imaginary parts of the solution $|u_1(x,t)|$ defined in (4.31) for $a = 15.2$, $b = -1$, $k = 1$, $\lambda_1 = 3$, $\lambda_2 = 3$, $c = 1$, $m = \frac{1}{\sqrt{5}}$, $\alpha = 0.97$. Figure 2 shows the solution properties of $|u_5(x,t)|$ stated in (4.39), where the parameters are $a = 15.2$, $b = -1$, $k = 1$, $\lambda_1 = 3$, $\lambda_2 = 3$, $c = 1$, $c_4 = 1$ $m = 0.5$, $\alpha = 0.95$. Finally, Figure 3 represents the solution $|u_7(x,t)|$ as indicated in (4.44) for the provided values $a = 15.2$, $b = 1$, $k = 1$, $\lambda_1 = 3$, $\lambda_2 = 3$, $c = 1$, $c_4 = 1$, $m = \frac{1}{\sqrt{3}}$, $\alpha = 0.98$. According to the results of the numerical simulations, the solutions are periodic. Furthermore, a close examination of the structure of the derived solutions reveals that the appropriate conformable derivative parameter of α exists in the formula of all solutions.

6 Conclusion

This study looked at the perturbed Gerdjikov-Ivanov equation with a space-time conformable fractional derivative. The hyperbolic and the periodic solutions for the perturbed Gerdjikov-Ivanov equation with space-time conformable fractional derivative have been determined from Jacobi elliptic function solutions when the modulus is $m \to 1$ or $m \to 0$ using the new extended auxiliary equation approach. Furthermore, for a better understanding of the dynamical behavior of the solution of the investigated equation, we have given numerical simulations corresponding to the conformable fractional derivatives of order α. To the best of our knowledge, this is the first time to use the new extended auxiliary equation method to consider the perturbed Gerdjikov-Ivanov equation with space-time conformable fractional derivative.

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