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Estimation on initial coefficient bounds of generalized subclasses of bi-univalent functions

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Abstract

In the present investigation, we introduce the two subclasses $S_{\Sigma}^{\alpha}(\gamma,\rho,\lambda,\mu,\xi,\delta)$ and $S_{\Sigma}(\gamma,\rho,\lambda,\mu,\xi,\delta;\beta)$ of normalized analytic bi-univalent functions defined in the open unit disk and associated with the Ruscheweyh's operator. Further, we obtain bounds for the second and third Taylor-Maclaurin coefficients of the functions belong to these subclasses. We also provide relevant connections with earlier investigations of other researchers.

Keywords: Analytic function, Univalent function, Convolution, Coefficient bounds, Bi-univalent function
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1 Introduction and preliminaries

Let $\mathcal{A}$ be the class of analytic functions f defined in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ and normalized by conditions $f(0)=0$ and $f'(0)=1$. Hence, series expansion of $f\in\mathcal{A}$ is of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \quad (z\in\mathbb{U}). \tag{1.1}$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ containing univalent functions in $\mathbb{U}$ (for details, see [4]). A function $f\in\mathcal{S}$ is said to be starlike of order α $(0\le\alpha<1)$ if $Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha$. A function $f\in\mathcal{S}$ is said to be convex of order α $(0\le\alpha<1)$ if $Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha$. The classes of starlike functions of order α and convex functions of order α are denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively. By definition, it is clear that, $\mathcal{K}(\alpha)\subset\mathcal{S}^*(\alpha)$ and also we have, $f\in\mathcal{K}(\alpha)$ if and only if $zf'\in\mathcal{S}^*(\alpha)$.

Since each $f\in\mathcal{S}$ is univalent, they are invertible for some part of unit disk $\mathbb{U}$. In fact, the Koebe One Quarter Theorem [4] ensures that, f^{-1} exists at least on $\left\{z\in\mathbb{C}:|z|<\frac{1}{4}\right\}$ for each $f\in\mathcal{S}$. Thus every $f\in\mathcal{A}$ has an inverse f^{-1} which is defined by

$$f^{-1}(f(z))=z, \quad z\in\mathbb{U}$$

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and

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}$$

where,

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions. For brief history of the class Σ, see Srivastava et al.[16]. Here are some of the examples of functions in the class Σ

$$\frac{z}{1-z}, \quad -log(1-z), \quad \frac{1}{2}log\left(\frac{1+z}{1-z}\right).$$

Geometric behavior of any function can be analyzed by knowing coefficient bounds of that function. Hence many researchers obtained coefficient bounds for several interesting subclasses of Σ. This journey was started in 1967 by Lewin [7], who introduced the class Σ and showed that, $|a_2| < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that, $|a_2| < \sqrt{2}$. Later, Netanyahu [11] proved that, $max|a_2| = \frac{4}{3}$ if $f \in \Sigma$. The theory of bi-univalent functions has been revived in the year 2010 by the pioneering work of Srivastava et al.[16]. After that, many researchers viz. [1, 6, 8, 10, 12, 19] introduced various subclasses of Σ and found coefficient bounds for the functions in them.

Let $f, g \in \mathcal{A}$, given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad (z \in \mathbb{U}).$$

Then, the convolution (Hadamard product) of f and g is denoted by $f * g$ and is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

In 1975, Ruscheweyh [13] defined the operator D^λ involving convolution as follows.

Let $f \in \mathcal{A}$. The operator $D^\lambda : \mathcal{A} \to \mathcal{A}$ is defined as

$$D^\lambda(f(z)) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \quad (\lambda > -1)$$

where,

$$D^0(f(z)) = f(z), \quad D^1(f(z)) = zf'(z)$$

and

$$D^n(f(z)) = \frac{z(z^{n-1}f(z))^n}{n!}, \quad n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}.$$

Clearly, the operator D^λ satisfies the relation:

$$z(D^\lambda(f(z)))' = (1+\lambda)D^{\lambda+1}(f(z)) - \lambda D^\lambda(f(z)). \tag{1.3}$$

Using the operator D^λ, we define following two new subclasses of bi-univalent functions.

Definition 1.1. A function $f \in \Sigma$ given by (1.1) is said to be in the class $S_\Sigma^\alpha(\gamma, \rho, \lambda, \mu, \xi, \delta)$; $0 \leq \rho \leq 1, \lambda > -1, 0 \leq \mu \leq 1, 0 \leq \xi \leq 1, 0 \leq \delta \leq 1, 0 < \alpha \leq 1, \gamma \in \mathbb{C}^* = \mathbb{C} - \{0\}$, if it satisfies the following conditions:

$$\left|arg\left(1 + \frac{1}{\gamma}\left[\frac{(1-\rho)D^\lambda f(z) + (\rho - \mu - \mu\lambda)z(D^\lambda f(z))' + \mu(1+\lambda)z(D^{\lambda+1}f(z))'}{(1-\xi)z + \xi(1-\delta)D^\lambda f(z) + \xi\delta z(D^\lambda f(z))'} - 1\right]\right)\right| < \frac{\alpha\pi}{2}$$

and

$$\left|arg\left(1 + \frac{1}{\gamma}\left[\frac{(1-\rho)D^\lambda g(w) + (\rho - \mu - \mu\lambda)w(D^\lambda g(w))' + \mu(1+\lambda)w(D^{\lambda+1}g(w))'}{(1-\xi)w + \xi(1-\delta)D^\lambda g(w) + \xi\delta w(D^\lambda g(w))'} - 1\right]\right)\right| < \frac{\alpha\pi}{2}$$

for all $z, w \in \mathbb{U}$ and $g = f^{-1} \in \Sigma$ given by (1.2).

Definition 1.2. A function $f \in \Sigma$ given by (1.1) is said to be in the class $S_\Sigma(\gamma, \rho, \lambda, \mu, \xi, \delta; \beta)$; $0 \leq \rho \leq 1, \lambda > -1, 0 \leq \mu \leq 1, 0 \leq \xi \leq 1, 0 \leq \delta \leq 1, 0 \leq \beta < 1, \gamma \in \mathbb{C}^* = \mathbb{C} - \{0\}$, if it satisfies the following conditions:

$$Re\left(1 + \frac{1}{\gamma}\left[\frac{(1-\rho)D^\lambda f(z) + (\rho - \mu - \mu\lambda)z(D^\lambda f(z))' + \mu(1+\lambda)z(D^{\lambda+1}f(z))'}{(1-\xi)z + \xi(1-\delta)D^\lambda f(z) + \xi\delta z(D^\lambda f(z))'} - 1\right]\right) > \beta$$

and

$$Re\left(1 + \frac{1}{\gamma}\left[\frac{(1-\rho)D^\lambda g(w) + (\rho - \mu - \mu\lambda)w(D^\lambda g(w))' + \mu(1+\lambda)w(D^{\lambda+1}g(w))'}{(1-\xi)w + \xi(1-\delta)D^\lambda g(w) + \xi\delta w(D^\lambda g(w))'} - 1\right]\right) > \beta$$

for all $z, w \in \mathbb{U}$ and $g = f^{-1} \in \Sigma$ given by (1.2).

Remark 1.3. For particular values of $\gamma, \rho, \lambda, \mu, \xi$ and δ, we get following well known subclasses of the class of bi-univalent functions.

1. $S_\Sigma^\alpha(\gamma, \rho, 0, \mu, \xi, \delta)$ and $S_\Sigma(\gamma, \rho, 0, \mu, \xi, \delta; \beta)$ are the subclasses introduced by Saleh [14].
2. For $\lambda = 0$ and $\rho = 1$ in definitions 1.1 and 1.2, we get modified definitions of subclasses introduced by Srivastava et al. [15].
3. $S_\Sigma(1, \lambda, 0, \delta, 0, \gamma; \alpha) = \mathcal{N}_\Sigma(\alpha, \lambda, \delta)$ is the subclass introduced by Bulut [3].
4. $S_\Sigma^\alpha(1, 1, 0, 0, \lambda, 0) = S_\Sigma^{a,1,a}(\alpha, \lambda)$ and $S_\Sigma(1, 1, 0, 0, \lambda, 0; \beta) = \mathcal{M}_\Sigma^{a,1,a}(\beta, \lambda)$ are the subclasses introduced by Srivastava et al. [17].
5. $S_\Sigma^\alpha(1, 1, 0, 0, 1, \lambda) = \mathcal{G}_\Sigma(\alpha, \lambda)$ and $S_\Sigma(1, 1, 0, 0, 1, \lambda; \beta) = \mathcal{M}_\Sigma(\beta, \lambda)$ are subclasses introduced by Murugusundaramoorthy et al. [9].
6. $S_\Sigma^\alpha(1, 1, 0, \lambda, 1, \lambda) = \mathcal{B}_\Sigma(\alpha, \lambda)$ and $S_\Sigma(1, 1, 0, \lambda, 1, \lambda; \beta) = \mathcal{N}_\Sigma(\beta, \lambda)$ are subclasses introduced by Keerthi and Raja [18].
7. $S_\Sigma^\alpha(1, \lambda, 0, 0, 0, \gamma) = \mathcal{B}_\Sigma(\alpha, \lambda)$ and $S_\Sigma(1, \lambda, 0, 0, 0, \gamma; \beta) = \mathcal{B}_\Sigma(\beta, \lambda)$ are the subclasses introduced by Frasin and Aouf [6].
8. $S_\Sigma^\alpha(1, 1, 0, \beta, 0, \gamma) = \mathcal{H}_\Sigma(\alpha, \beta)$ and $S_\Sigma(1, 1, 0, \beta, 0, \gamma; \gamma) = \mathcal{H}_\Sigma(\gamma, \beta)$ are the subclasses introduced by Frasin [5].
9. $S_\Sigma^\alpha(1, 1, 0, 0, 0, \gamma) = \mathcal{H}_\Sigma^\alpha$ and $S_\Sigma(1, 1, 0, 0, 0, \gamma; \beta) = \mathcal{H}_\Sigma(\beta)$ are the subclasses introduced by Srivastava et al. [16].

Lemma 1.4. [4] If $h \in \mathcal{P}$, then the estimates $|c_n| \leq 2, n = 1, 2, 3, \ldots$ are sharp, where $\mathcal{P}$ is the family of all functions h which are analytic in $\mathbb{U}$ for which $h(0) = 1$ and $Re(h(z)) > 0$ $(z \in \mathbb{U})$ where,

$$h(z) = 1 + c_1 z + c_2 z^2 + \ldots, z \in \mathbb{U}.$$

2 Coefficient bounds for the function class $S_\Sigma^\alpha(\gamma, \rho, \lambda, \mu, \xi, \delta)$

Theorem 2.1. If $f \in S_\Sigma^\alpha(\gamma, \rho, \lambda, \mu, \xi, \delta)$ is in $\mathcal{A}$, then

$$|a_2| \leq \frac{2\alpha|\gamma|}{\sqrt{\lambda+1}\sqrt{|2\Omega\alpha\gamma + (1-\alpha)(1+\rho+2\mu-\xi-\xi\delta)^2(\lambda+1)|}} \tag{2.1}$$

and

$$|a_3| \leq min\left\{\frac{4\alpha^2|\gamma|^2}{(1+\rho+2\mu-\xi-\xi\delta)^2(\lambda+1)^2} + \frac{4\alpha|\gamma|}{|1+2\rho+6\mu-\xi-2\delta\xi|(\lambda+1)(\lambda+2)}, \right.$$
$$\left. \frac{4\alpha|\gamma|}{\lambda+1}\left(\frac{\alpha}{2|\Omega|} + \frac{1}{|1+2\rho+6\mu-\xi-2\delta\xi|(\lambda+2)}\right)\right\} \tag{2.2}$$

where

$$\Omega = (1+2\rho+6\mu-\xi-2\xi\delta)\frac{(\lambda+2)}{2} - (1+\rho+2\mu-\xi-\xi\delta)\xi(1+\delta)(\lambda+1). \tag{2.3}$$

Proof . Let $f \in S_\Sigma^\alpha(\gamma, \rho, \lambda, \mu, \xi, \delta)$. Then there exist two analytic functions $h_1(z) = 1 + \sum_{n=1}^\infty p_n z^n$ and $h_2(z) = 1 + \sum_{n=1}^\infty q_n w^n$ with positive real part in the unit disc such that,

$$1 + \frac{1}{\gamma}\left[\frac{(1-\rho)D^\lambda f(z) + (\rho-\mu-\mu\lambda)z(D^\lambda f(z))' + \mu(1+\lambda)z(D^{\lambda+1}f(z))'}{(1-\xi)z + \xi(1-\delta)D^\lambda f(z) + \xi\delta z(D^\lambda f(z))'} - 1\right] = (h_1(z))^\alpha \tag{2.4}$$

and

$$1 + \frac{1}{\gamma}\left[\frac{(1-\rho)D^\lambda g(w) + (\rho-\mu-\mu\lambda)w(D^\lambda g(w))' + \mu(1+\lambda)w(D^{\lambda+1}g(w))'}{(1-\xi)w + \xi(1-\delta)D^\lambda g(w) + \xi\delta w(D^\lambda g(w))'} - 1\right] = (h_2(w))^\alpha. \tag{2.5}$$

Expanding above brackets and comparing coefficients of z, z^2, w and w^2 in both sides of equations (2.4) and (2.5), we get

$$\frac{(1+\rho+2\mu-\xi-\xi\delta)(\lambda+1)a_2}{\gamma} = \alpha p_1, \tag{2.6}$$

$$\frac{(1+2\rho+6\mu-\xi-2\delta\xi)\frac{(\lambda+1)(\lambda+2)}{2}a_3 - (1+\rho+2\mu-\xi-\xi\delta)\xi(1+\delta)(\lambda+1)^2 a_2^2}{\gamma} =$$
$$\alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.7}$$

$$\frac{-(1+\rho+2\mu-\xi-\xi\delta)(\lambda+1)a_2}{\gamma} = \alpha q_1 \tag{2.8}$$

and

$$\frac{(1+2\rho+6\mu-\xi-2\delta\xi)\frac{(\lambda+1)(\lambda+2)}{2}(2a_2^2-a_3)-(1+\rho+2\mu-\xi-\xi\delta)\xi(1+\delta)(\lambda+1)^2a_2^2}{\gamma} = \alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.9}$$

From equations (2.6) and (2.8), we get

$$p_1 = -q_1 \tag{2.10}$$

and

$$a_2 = \frac{\alpha\gamma p_1}{(1+\rho+2\mu-\xi-\xi\delta)(\lambda+1)}. \tag{2.11}$$

Adding (2.7) and (2.9), we obtain

$$\frac{2\Omega(\lambda+1)a_2^2}{\gamma} = \alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2), \tag{2.12}$$

where Ω is given by (2.3). Now, by using (2.10) and (2.11) in (2.12), we get

$$p_1^2 = \frac{(p_2+q_2)(1+\rho+2\mu-\xi-\xi\delta)^2(\lambda+1)}{2\Omega\alpha\gamma+(1-\alpha)(1+\rho+2\mu-\xi-\xi\delta)^2(\lambda+1)},$$

which, on using Lemma (1.4) yields

$$|p_1| \leq \frac{2|1+\rho+2\mu-\xi-\xi\delta|\sqrt{\lambda+1}}{\sqrt{|2\Omega\alpha\gamma+(1-\alpha)(1+\rho+2\mu-\xi-\xi\delta)^2(\lambda+1)|}}. \tag{2.13}$$

By taking modulus of both sides of (2.11) and applying Lemma (1.4) and inequality (2.13), we get desired bound of $|a_2|$ given by (2.1).
Next, to get desire bound of $|a_3|$, we subtract equation (2.9) from (2.7) to get

$$\frac{(1+2\rho+6\mu-\xi-2\delta\xi)(\lambda+1)(\lambda+2)(a_3-a_2^2)}{\gamma} = \alpha(p_2-q_2).$$

This can be written using relation (2.10) as

$$a_3 = a_2^2+\frac{\alpha\gamma(p_2-q_2)}{(1+2\rho+6\mu-\xi-2\delta\xi)(\lambda+1)(\lambda+2)}. \tag{2.14}$$

If we use the value of a_2 given by (2.11) in (2.14), we obtain

$$a_3 = \frac{\alpha^2\gamma^2p_1^2}{(1+\rho+2\mu-\xi-\xi\delta)^2(\lambda+1)^2}+\frac{\alpha\gamma(p_2-q_2)}{(1+2\rho+6\mu-\xi-2\delta\xi)(\lambda+1)(\lambda+2)},$$

in which, using Lemma (1.4) we conclude one of the desired estimates of $|a_3|$. Further, if we use the value of a_2^2 obtained from (2.12) in equation (2.14), we obtain

$$a_3 = \frac{\alpha\gamma(p_2+q_2)+\alpha(\alpha-1)\gamma p_1^2}{2\Omega(\lambda+1)}+\frac{\alpha\gamma(p_2-q_2)}{(1+2\rho+6\mu-\xi-2\delta\xi)(\lambda+1)(\lambda+2)}, \tag{2.15}$$

which, by using Lemma (1.4) proves the second desired estimation of $|a_3|$. □

3 Coefficient bounds for the function class $S_\Sigma(\gamma,\rho,\lambda,\mu,\xi,\delta;\beta)$

Theorem 3.1. If $f \in S_\Sigma(\gamma,\rho,\lambda,\mu,\xi,\delta;\beta)$ is in $\mathcal{A}$, then

$$|a_2| \leq min\left\{\frac{2(1-\beta)|\gamma|}{|1+\rho+2\mu-\xi-\xi\delta|(\lambda+1)}, \sqrt{\frac{2(1-\beta)|\gamma|}{|\Omega|(\lambda+1)}}\right\} \tag{3.1}$$

and

$$|a_3| \leq min\Big\{\frac{4(1-\beta)^2|\gamma|^2}{(1+\rho+2\mu-\xi-\xi\delta)^2(\lambda+1)^2}+\frac{4(1-\beta)|\gamma|}{|1+2\rho+6\mu-\xi-2\xi\delta|(\lambda+1)(\lambda+2)}, \frac{(1-\beta)|\gamma|}{(\lambda+1)}\left(\frac{2}{|\Omega|}+\frac{4}{|1+2\rho+6\mu-\xi-2\xi\delta|(\lambda+2)}\right)\Big\} \tag{3.2}$$

where Ω is given by (2.3).

Proof . Let $f \in S_\Sigma(\gamma,\rho,\lambda,\mu,\xi,\delta;\beta)$. Then there exist two analytic functions $P(z)=\sum_{n=1}^{\infty} p_n z^n$ and $Q(w)=\sum_{n=1}^{\infty} q_n w^n$ with positive real part in the unit disc such that,

$$1+\frac{1}{\gamma}\left[\frac{(1-\rho)D^\lambda f(z)+(\rho-\mu-\mu\lambda)z(D^\lambda f(z))'+\mu(1+\lambda)z(D^{\lambda+1}f(z))'}{(1-\xi)z+\xi(1-\delta)D^\lambda f(z)+\xi\delta z(D^\lambda f(z))'}-1\right] = \beta+(1-\beta)P(z) \tag{3.3}$$

and

$$1+\frac{1}{\gamma}\left[\frac{(1-\rho)D^\lambda g(w)+(\rho-\mu-\mu\lambda)w(D^\lambda g(w))'+\mu(1+\lambda)w(D^{\lambda+1}g(w))'}{(1-\xi)w+\xi(1-\delta)D^\lambda g(w)+\xi\delta w(D^\lambda g(w))'}-1\right] = \beta+(1-\beta)Q(w). \tag{3.4}$$

Expanding above brackets and comparing coefficients of z, z^2, w and w^2 in both sides of equations (3.3) and (3.4), we get

$$\frac{(1+\rho+2\mu-\xi-\xi\delta)(\lambda+1)a_2}{\gamma}=(1-\beta)p_1, \tag{3.5}$$

$$\frac{(1+2\rho+6\mu-\xi-2\delta\xi)\frac{(\lambda+1)(\lambda+2)}{2}a_3-(1+\rho+2\mu-\xi-\xi\delta)\xi(1+\delta)(\lambda+1)^2a_2^2}{\gamma} = (1-\beta)p_2, \tag{3.6}$$

$$\frac{-(1+\rho+2\mu-\xi-\xi\delta)(\lambda+1)a_2}{\gamma}=(1-\beta)q_1 \tag{3.7}$$

and

$$\frac{(1+2\rho+6\mu-\xi-2\delta\xi)\frac{(\lambda+1)(\lambda+2)}{2}(2a_2^2-a_3)-(1+\rho+2\mu-\xi-\xi\delta)\xi(1+\delta)(\lambda+1)^2a_2^2}{\gamma} = (1-\beta)q_2. \tag{3.8}$$

From equations (3.5) and (3.7), we get

$$p_1=-q_1$$

and

$$a_2=\frac{(1-\beta)\gamma p_1}{(1+\rho+2\mu-\xi-\xi\delta)(\lambda+1)}. \tag{3.9}$$

Adding (3.6) and (3.8), we obtain

$$\frac{2\Omega(\lambda+1)a_2^2}{\gamma}=(1-\beta)(p_2+q_2) \tag{3.10}$$

where Ω is given by (2.3). This, by applying Lemma (1.4) gives

$$|a_2|\le\sqrt{\frac{2(1-\beta)|\gamma|}{|\Omega|(\lambda+1)}}. \tag{3.11}$$

Also, equation (3.9) shows that

$$|a_2|\le\frac{2(1-\beta)|\gamma|}{|1+\rho+2\mu-\xi-\xi\delta|(\lambda+1)}. \tag{3.12}$$

Equation (3.11) and (3.12) gives desire bound of $|a_2|$ given by (3.1). Next, to obtain bounds for $|a_3|$, we subtract equation (3.8) from (3.6), to get

$$a_3=a_2^2+\frac{(1-\beta)\gamma(p_2-q_2)}{(1+2\rho+6\mu-\xi-2\delta\xi)(\lambda+1)(\lambda+2)}. \tag{3.13}$$

Using value of a_2 form (3.9) in equation (3.13), we get

$$a_3=\frac{(1-\beta)^2\gamma^2p_1^2}{(1+\rho+2\mu-\xi-\delta\xi)^2(\lambda+1)^2}+\frac{(1-\beta)\gamma(p_2-q_2)}{(1+2\rho+6\mu-\xi-2\delta\xi)(\lambda+1)(\lambda+2)}. \tag{3.14}$$

Using Lemma (1.4), we conclude one of the desired estimate of $|a_3|$. Further, if we use the value of a_2^2 obtained from (3.10) in (3.13), we obtain

$$a_3=\frac{(1-\beta)(p_2+q_2)\gamma}{2\Omega(\lambda+1)}+\frac{(1-\beta)\gamma(p_2-q_2)}{(1+2\rho+6\mu-\xi-2\delta\xi)(\lambda+1)(\lambda+2)}, \tag{3.15}$$

which, in light of Lemma (1.4) gives the desired estimation of $|a_3|$. □

4 Some Corollaries and Consequences

In this section, we have mentioned correlations with some of the known results as consequences of Theorem (2.1) and Theorem (3.1) proved in previous two sections.

By putting $\lambda = 0$ and $\rho = 1$ in the Theorems (2.1) and (3.1), we get modified results considered by Srivastava et al. ([15], Theorem 1 and 2).

Corollary 4.1. Let $f \in \mathcal{H}_\Sigma(\gamma, \mu, \xi, \delta; \alpha)$ given by (1.1) then

$$|a_2| \leq \frac{2\alpha|\gamma|}{\sqrt{|2\Omega_1\alpha\gamma + (1-\alpha)(2+2\mu-\xi-\xi\delta)^2|}}$$

and

$$|a_3| \leq min\left\{\frac{4\alpha^2|\gamma|^2}{(2+2\mu-\xi-\xi\delta)^2} + \frac{2\alpha|\gamma|}{|3+6\mu-\xi-2\xi\delta|}, \frac{2\alpha^2|\gamma|}{|\Omega_1|} + \frac{2\alpha|\gamma|}{|3+6\mu-\xi-2\xi\delta|}\right\}$$

where

$$\Omega_1 = (3+6\mu-\xi-2\xi\delta) - (2+2\mu-\xi-\xi\delta)\xi(1+\delta). \tag{4.1}$$

Corollary 4.2. Let $f \in \mathcal{H}_\Sigma(\gamma, \mu, \xi, \delta; \beta)$ given by (1.1) then

$$|a_2| \leq min\left\{\frac{2(1-\beta)|\gamma|}{|2+2\mu-\xi-\xi\delta|}, \sqrt{\frac{2(1-\beta)|\gamma|}{|\Omega_1|}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{4(1-\beta)^2|\gamma|^2}{(2+2\mu-\xi-\xi\delta)^2} + \frac{2(1-\beta)|\gamma|}{|3+6\mu-\xi-2\xi\delta|}, \frac{2(1-\beta)|\gamma|}{|\Omega_1|} + \frac{2(1-\beta)|\gamma|}{|3+6\mu-\xi-2\xi\delta|}\right\}$$

where $|\Omega_1|$ is given by equation (4.1).

By putting $\gamma = \rho = \xi = 1, \mu = 0$ and $\lambda = 0$ in the Theorems (2.1) and (3.1), we get modified results considered by Murugusundaramoorthy et al. ([9], Theorem 4 and 5).

Corollary 4.3. Let $f \in \mathcal{G}_\Sigma(\alpha, \delta)$ $(0 \leq \delta \leq 1)$ given by (1.1) then

$$|a_2| \leq \frac{2\alpha}{(1-\delta)\sqrt{\alpha+1}}$$

and

$$|a_3| \leq \frac{2\alpha^2}{(1-\delta)^2} + \frac{\alpha}{1-\delta}.$$

Corollary 4.4. Let $f \in \mathcal{M}_\Sigma(\beta, \delta)$ $(0 \leq \delta \leq 1)$ given by (1.1) then

$$|a_2| \leq \frac{\sqrt{2(1-\beta)}}{1-\delta}$$

and

$$|a_3| \leq min\left\{\frac{4(1-\beta)^2}{(1-\delta)^2} + \frac{1-\beta}{1-\delta}, \frac{2(1-\beta)}{(1-\delta)^2} + \frac{1-\beta}{1-\delta}\right\}.$$

By putting $\gamma = \rho = \xi = 1$, $\mu = \delta$ and $\lambda = 0$ in the Theorems (2.1) and (3.1), we get modified results considered by Keerthi and Raja ([18], corollary 2.3 and 3.4).

Corollary 4.5. Let $f \in \mathcal{B}_\Sigma(\alpha, \mu)$ given by (1.1) then

$$|a_2| \leq \frac{2\alpha}{\sqrt{|4\alpha(1+2\mu) + (1-3\alpha)(1+\mu)^2|}}$$

and

$$|a_3| \leq min\left\{\frac{4\alpha^2}{(1+\mu)^2} + \frac{\alpha}{2\mu+1}, \frac{2\alpha^2}{1+2\mu-\mu^2} + \frac{\alpha}{2\mu+1}\right\}.$$

Corollary 4.6. Let $f \in \mathcal{N}_\Sigma(\beta, \mu)$ given by (1.1) then

$$|a_2| \leq min\left\{\frac{2(1-\beta)}{1+\mu}, \sqrt{\frac{2(1-\beta)}{1+2\mu-\mu^2}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{4(1-\beta)^2}{(1+\mu)^2} + \frac{1-\beta}{1+2\mu}, \frac{2(1-\beta)}{1+2\mu-\mu^2} + \frac{1-\beta}{1+2\mu}\right\}.$$

By putting $\lambda = \mu = \xi = 0$ and $\gamma = 1$ in the Theorems (2.1) and (3.1), we get modified results considered by Frasin and Aouf ([6], Theorem 2.2 and 3.2).

Corollary 4.7. Let $f \in \mathcal{B}_\Sigma(\alpha, \rho)$ $(\rho \geq 1)$ given by (1.1) then

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha(1+2\rho-\rho^2)+(1+\rho)^2}}$$

and

$$|a_3| \leq min\left\{\frac{4\alpha^2}{(1+\rho)^2} + \frac{2\alpha}{1+2\rho}, \frac{2\alpha^2}{1+2\rho} + \frac{2\alpha}{1+2\rho}\right\}.$$

Corollary 4.8. Let $f \in \mathcal{B}_\Sigma(\beta, \rho)$ $(\rho \geq 1)$ given by (1.1) then

$$|a_2| \leq min\left\{\frac{2(1-\beta)}{1+\rho}, \sqrt{\frac{2(1-\beta)}{1+2\rho}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{4(1-\beta)^2}{(1+\rho)^2} + \frac{2(1-\beta)}{1+2\rho}, \frac{4(1-\beta)}{1+2\rho}\right\}.$$

By putting $\lambda = \xi = 0$ and $\gamma = \rho = 1$ in the Theorems (2.1) and (3.1), we get modified results considered by Frasin ([5], Theorem 2.2 and 3.2).

Corollary 4.9. Let $f \in \mathcal{H}_\Sigma(\alpha, \mu)$ given by (1.1) then

$$|a_2| \leq \frac{2\alpha}{\sqrt{2(2+\alpha)+4\mu(\alpha-\alpha\mu+2+\mu)}}$$

and

$$|a_3| \leq min\left\{\frac{\alpha^2}{(1+\mu)^2} + \frac{2\alpha}{3(1+2\mu)}, \frac{2\alpha^2+2\alpha}{3(1+2\mu)}\right\}.$$

Corollary 4.10. Let $f \in \mathcal{H}_\Sigma(\mu, \beta)$ given by (1.1) then

$$|a_2| \leq min\left\{\frac{1-\beta}{1+\mu}, \sqrt{\frac{2(1-\beta)}{3(1+2\mu)}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{(1-\beta)^2}{(1+\mu)^2} + \frac{2(1-\beta)}{3(1+2\mu)}, \frac{4(1-\beta)}{3(1+2\mu)}\right\}.$$

By putting $\lambda = \xi = \mu = 0$ and $\gamma = \rho = 1$ in the Theorems (2.1) and (3.1), we get modified results considered by Srivastava et al. ([16], Theorem 1 and 2).

Corollary 4.11. Let $f \in \mathcal{H}_\Sigma^\alpha$ given by (1.1) then

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha+2}}$$

and

$$|a_3| \leq min\left\{\alpha^2 + \frac{2\alpha}{3}, \frac{2\alpha(\alpha+1)}{3}\right\}.$$

Corollary 4.12. Let $f \in \mathcal{H}_\Sigma(\beta)$ given by (1.1) then

$$|a_2| \leq min\left\{1-\beta, \sqrt{\frac{2(1-\beta)}{3}}\right\}$$

and

$$|a_3| \leq min\left\{(1-\beta)^2 + \frac{2(1-\beta)}{3}, \frac{4(1-\beta)}{3}\right\}.$$

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