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Discussion on ordered fixed point results and related applications

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Abstract

The aim of this manuscript is to introduce the notion of $\Re$-extended b-metric spaces and prove some fixed point results in this space. An extensive set of nontrivial examples is given to justify the claims.

Keywords: $\Re$-extended b-metric spaces, fixed point, $\Re$-sequence
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1 Introduction

Since the inception of the celebrated Banach fixed point theorem in 1922, fixed point theory has attracted many researchers, and it has been generalized and extended in different directions, as can be seen in the rich literature of metric fixed point theory. As we know, fixed point theorems play a vital role in proving the existence and uniqueness of the solutions to various mathematical models (integral and partial equations, variational inequalities, etc.). They can be applied to, for example, variational inequalities, optimization, and approximation theory.

On the other hand, the idea of b-metric was initiated by the works of Bourbaki [5] and Bakhtin [3]. Czerwik [6] gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with the view of generalizing the Banach contraction principal. Later, Kamran et al. [8] developed the concept of extended b-metric space.

After looking into the structure of $\Re$-metric spaces, introduced by [9], and the binary relation used with a metric, in [2, 16] the notion of $\Re$-extended b-metric spaces has been introduced. We provide non-trivial examples to justify our claim that our results would be more efficient in dealing with many scientific problems than those in the literature. We are also improving and generalizing the concept of orthogonal contractions in the sense of establishing some fixed point theorems for the proposed contractions and discussing some illustrative examples to demonstrate the validity of our findings. Our results generalize and improve the results of [9] and some other well-known results in the literature.

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2 Preliminaries

Definition 2.1. [3] Let Ω be a non empty set and $s \geq 1$. Suppose that the mapping $d : \Omega \times \Omega \to \mathbb{R}^+$ satisfies

(a) $d(\omega, \varpi) = 0$ if and only if $\omega = \varpi$;

(b) $d(\omega, \varpi) = d(\varpi, \omega)$;

(c) $d(\omega, \varpi) \leq s[d(\omega, z) + d(z, \varpi)]$;

for all $\omega, \varpi, z \in \Omega$.

Then the triplet (Ω, d, s) is called a b-metric space.

Definition 2.2. [3] Let Ω be a non empty set and $\theta : \Omega \times \Omega \to [1, \infty)$. Suppose that there exists a mapping $d_\theta : \Omega \times \Omega \to \mathbb{R}^+$ so that:

(1) $d_\theta(\omega, \varpi) = 0$ if and only if $\omega = \varpi$;

(2) $d_\theta(\omega, \varpi) = d_\theta(\varpi, \omega)$;

(3) $d_\theta(\omega, z) \leq \theta(\omega, z)[d_\theta(\omega, \varpi) + d_\theta(\varpi, z)]$, for all $\omega, \varpi, z \in \Omega$.

Then the pair (Ω, d_θ) is called an extended b-metric space.

Remark 2.3. [8] If $\theta(\omega, \varpi) = s$ for an $s \geq 1$, then it becomes a b-metric space.

Example 2.4. [8] Let $\Omega = \{1, 2, 3\}$. Define $\theta : \Omega \times \Omega \to [1, \infty)$ and $d_\theta : \Omega \times \Omega \to \mathbb{R}^+$ as:

$$\begin{aligned} \theta(\omega, \varpi) &= 1 + \omega + \varpi, \\ d_\theta(1,1) &= d_\theta(2,2) = d_\theta(3,3) = 0, \\ d_\theta(1,2) &= 80,\ d_\theta(1,3) = 1000,\ d_\theta(2,3) = 400. \end{aligned}$$

$(d_\theta 1)$ and $(d_\theta 2)$ trivially hold. For $(d_\theta 3)$ we have

$$\begin{aligned} d_\theta(1,2) &= 80, \quad \theta(1,2)[d_\theta(1,3) + d_\theta(3,2)] = 4(1000 + 600) = 6400, \\ d_\theta(1,3) &= 1000, \quad \theta(1,3)[d_\theta(1,2) + d_\theta(2,3)] = 5(80 + 600) = 3400. \end{aligned}$$

Similar calculations hold for $d_\theta(2,3)$. As for all $\omega, \varpi, z \in \Omega$

$$d_\theta(\omega, z) \leq \theta(\omega, z)[d_\theta(\omega, \varpi) + d_\theta(\varpi, z)],$$

hence, (Ω, d_θ) is an extended b-metric space.

Khalehoghli et al. [9] introduced the concept of $\Re$-sets using a binary relation and defined it as:

Definition 2.5. [16] Let Ω be a nonempty set. A subset $\Re$ of Ω^2 is called a binary relation on Ω. For each pair $\omega, \varpi \in \Omega$, we say that ω is $\Re$-related to ϖ, or, ω relates to ϖ under $\Re$ if and only if $(\omega, \varpi) \in \Re$. $(\omega, \varpi) \notin \Re$ means that ω is not $\Re$-related to ϖ, or, ω does not relate to ϖ under $\Re$.

Definition 2.6. [10] A Binary relation $\Re$ defined on a nonempty set Ω is called (a) reflexive if $(\omega, \omega) \in \Re$ for all $\omega \in \Omega$; (b) irreflexive if $(\omega, \omega) \notin \Re$ for some $\omega \in \Omega$; (c) symmetric if $(\omega, \varpi) \in \Re$ implies $(\varpi, \omega) \in \Re$ for all $\omega, \varpi \in \Omega$; (d) antisymmetric if $(\omega, \varpi) \in \Re$ and $(\varpi, \omega) \in \Re$ imply $\omega = \varpi$ for all $\omega, \varpi \in \Omega$; (e) transitive if $(\omega, \varpi) \in \Re$ and $(\varpi, z) \in \Re$ imply $(\omega, z) \in \Re$ for all $\omega, \varpi, z \in \Omega$; (f) preorder if $\Re$ is reflexive and transitive; (g) partial order if $\Re$ is reflexive, antisymmetric and transitive; and (h) equivalence if $\Re$ is reflexive, symmetric and transitive.

Definition 2.7. [2] Let Ω be a nonempty set and $\Re \subseteq \Omega \times \Omega$ be a binary relation. A sequence $\{\omega_n\}$ is called an $\Re$-sequence if

$$\omega_n \Re \omega_{n+1},$$

for all $n \in \mathbb{N}$

Definition 2.8. [1] An $\Re$-sequence $\{\omega_n\}$ in Ω is said to be an $\Re$-Cauchy sequence, if for every $\varsigma > 0$ there exists an integer N such that $d(\omega_m, \omega_n) < \varsigma$ if $n, m \geq N$. It is clear that $\omega_n \Re \omega_m$ or $\omega_m \Re \omega_n$.

Definition 2.9. [7] A mapping $\mathcal{Q} : \Omega \to \Omega$ is $\Re$-continuous in $\omega \in \Omega$ if for each $\Re$-sequence $\{\omega_n\}_{n\in\mathbb{N}}$ in Ω such that $\omega_n \to \omega$ then $\mathcal{Q}(\omega_n) \to \mathcal{Q}(\omega)$. Also, $\mathcal{Q}$ is said to be $\Re-$continuous on Ω if $\mathcal{Q}$ is $\Re-$continuous in each $\omega \in \Omega$.

Definition 2.10. [2] Let (Ω, d) be a metric space and $\Re$ be a binary relation over Ω. Then Ω is said to be $\Re$-complete if every $\Re$-Cauchy sequence is convergent. Briefly, $(\Omega, d, \Re)$ is an $\Re$-complete metric space.

Remark 2.11. (i) Every convergent $\Re$-sequence in Ω is an $\Re$-Cauchy sequence.

(ii) Every continuous mapping $\mathcal{Q} : \Omega \to \Omega$ is $\Re$-continuous, but the converse is not true.

Definition 2.12. [9] A mapping $\mathcal{Q} : \Omega \to \Omega$ is said to be an $\Re$-contraction with the Lipschitz constant $0 < k < 1$ if

$$d(\mathcal{Q}\omega, \mathcal{Q}\varpi) \leq kd(\omega, \varpi) \tag{1.1}$$

for all $\omega, \varpi \in \Omega$ with $\omega \Re \varpi$.

Definition 2.13. [9] Let $\mathcal{Q} : \Omega \to \Omega$ be a mapping. $\mathcal{Q}$ is said to be $\Re$-preserving if $\omega \Re \varpi$, then $\mathcal{Q}\omega \Re \mathcal{Q}\varpi$ for all ω, $\varpi \in \Omega$.

Motivated by the recent works in [11]–[15], we have the following lemma about convergence of sequences in an $\Re$-extended b-metric space.

Lemma 2.14. Let $(\Omega, d_{\theta_\Re}, \Re)$ be an $\Re$-extended b-metric space with continuous control function $\theta_\Re$ and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y, respectively. Then we have

$$\begin{aligned}\frac{1}{[\theta_\Re(x,y)]^2} d_{\theta_\Re}(x,y) &\leq \liminf_{n\to\infty} d_{\theta_\Re}(x_n,y_n) \leq \limsup_{n\to\infty} d_{\theta_\Re}(x_n,y_n)\\ &\leq [\theta_\Re(x,y)]^2 d_{\theta_\Re}(x,y).\end{aligned}$$

In particular, if $d_{\theta_\Re}(x,y) = 0$, then we have $\lim_{n\to\infty} d_{\theta_\Re}(x_n,y_n) = 0$.

Moreover, for each $z \in X$ we have

$$\frac{1}{\theta_\Re(x,y)} d_{\theta_\Re}(x,y) \leq \liminf_{n\to\infty} d_{\theta_\Re}(x_n,y) \leq \limsup_{n\to\infty} d_{\theta_\Re}(x_n,y) \leq \theta_\Re(x,y) d_{\theta_\Re}(x,y).$$

Proof . Using the triangle inequality in an $\Re$-extended b-metric space it is easy to see that

$$d_{\theta_\Re}(x,y) \leq \theta_\Re(x,y) d_{\theta_\Re}(x,x_n) + \theta_\Re(x,y)\theta_\Re(x_n,y) d_{\theta_\Re}(x_n,y_n) + \theta_\Re(x,y)\theta_\Re(x_n,y) d_{\theta_\Re}(y_n,y)$$

and

$$d_{\theta_\Re}(x_n,y_n) \leq \theta_\Re(x_n,y_n) d_{\theta_\Re}(x_n,x) + \theta_\Re(x_n,y_n)\theta_\Re(x,y_n) d_{\theta_\Re}(x,y) + \theta_\Re(x_n,y_n)\theta_\Re(x,y_n) d_{\theta_\Re}(y,y_n).$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the first desired result.

Also,

$$d_{\theta_\Re}(x,y) \leq \theta_\Re(x,y) d_{\theta_\Re}(x,x_n) + \theta_\Re(x,y) d_{\theta_\Re}(x_n,y)$$

and

$$d_{\theta_\Re}(x_n,y) \leq \theta_\Re(x_n,y) d_{\theta_\Re}(x_n,x) + \theta_\Re(x_n,y) d_{\theta_\Re}(x,y).$$

□

3 Main Result

Now, we introduce the notion of $\Re$-extended b-metric spaces and utilize this concept to investigate some fixed point results. Motivated by the work of Khalehoghli et al. [9] and Baghani et al. [2], we introduce the notion of Banach contraction in the sense of $\Re$-extended b-metric space.

Definition 3.1. Let Ω be a non empty set, $\Re$ be a reflexive binary relation on Ω, denoted as $(\Omega, \Re)$ and $\theta_{\Re} : \Omega \times \Omega \to [1, \infty)$. A map $d_{\theta_{\Re}} : \Omega \times \Omega \to \mathbb{R}^+$ is called $\Re$-extended b-metric on the set Ω, if the following conditions satisfied:

1. $d_{\theta_{\Re}}(\omega, \varpi) = 0$ if and only if $\omega = \varpi$;
2. $d_{\theta_{\Re}}(\omega, \varpi) = d_{\theta_{\Re}}(\varpi, \omega)$;
3. $d_{\theta_{\Re}}(\omega, z) \leq \theta_{\Re}(\omega, z)\left[d_{\theta_{\Re}}(\omega, \varpi) + d_{\theta_{\Re}}(\varpi, z)\right]$,

for all $\omega, \varpi, z \in \Omega$ with either ($\omega \Re z$ or $z \Re \omega$), and, ($\omega \Re \varpi$ or, $\varpi \Re \omega$) and either ($\varpi \Re z$ or $z \Re \varpi$).

Then $(\Omega,\ \Re, d_{\theta_{\Re}})$ is called an $\Re$-extended b-metric space.

Remark 3.2. In the above definition, a set Ω is endowed with a reflexive binary relation $\Re$ and a metric $d_{\theta_{\Re}} : \Omega \times \Omega \to \mathbb{R}^+$ which satisfies (1)-(3) but only for those elements which are comparable under the reflexive binary relation $\Re$. Hence, the $\Re$-extended b-metric may not be an extended b-metric, but the converse is true.

As shown in the following simple example, an $\Re$-extended b-metric space does not have to be an extended b-metric space.

Example 3.3. Let $\Omega = \{0, 1, 2\}$ and $d_{\theta_{\Re}} : \Omega \times \Omega \to [0, \infty)$ be given by

$$\begin{aligned} d_{\theta_{\Re}}(0,2) &= d_{\theta_{\Re}}(2,0) = 2, \\ d_{\theta_{\Re}}(1,1) &= d_{\theta_{\Re}}(2,2) = d_{\theta_{\Re}}(0,0) = 0, \\ d_{\theta_{\Re}}(0,1) &= d_{\theta_{\Re}}(1,0) = d_{\theta_{\Re}}(1,2) = d_{\theta_{\Re}}(2,1) = 1. \end{aligned}$$

Define a binary relation $\Re$ on Ω by $\omega \Re \varpi$ iff $\omega > \varpi$ and $\varpi \neq 0$, and define $\theta_{\Re} : \Omega \times \Omega \to \Re^+$ as $\theta_{\Re}(\omega, \varpi) = 1 + \frac{\omega}{\varpi}$ for all $\omega, \varpi \in \Omega$. So,

$$d_{\theta_{\Re}}(0,2) = 2 \geq 1\,[1+1] = \theta_{\Re}(0,2)\left[d_{\theta_{\Re}}(0,1) + d_{\theta_{\Re}}(1,2)\right].$$

Hence, it is not an extended b-metric, but it is an $\Re$-extended b -metric. Indeed, we must take $\omega > \varpi$ for $\theta_{\Re}(\omega, \varpi)$. Therefore,

$$d_{\theta_{\Re}}(2,1) = 1 \leq 3\,[2+1] = \theta_{\Re}(2,1)\left[d_{\theta_{\Re}}(2,0) + d_{\theta_{\Re}}(0,1)\right].$$

Then $(\Omega, d_{\theta_{\Re}}, \Re)$ is an $\Re$-extended b-metric space.

Definition 3.4. Let $(\Omega, d_{\theta_{\Re}}, \Re)$ be an $\Re$-extended b-metric space with $\theta_{\Re} : \Omega \times \Omega \to \Re^+$ ($(\Omega, \Re)$ is an $\Re$-set and $(\Omega, d_{\theta_{\Re}})$ is an extended b-metric space) and $0 < \lambda < 1$. A mapping $\mathcal{Q} : \Omega \to \Omega$ is called a θ-$\Re$-contraction (briefly, ($\Re_\theta$-contraction), with a Lipchitz constant λ if

$$\theta_{\Re}(\omega, \varpi)\, d_{\theta_{\Re}}(\omega, \varpi) \leq \lambda d_{\theta_{\Re}}(\omega, \varpi)$$

for all $\omega, \varpi \in \Omega$ so that $\omega \Re \varpi$.

Definition 3.5. Let $\{\omega_n\}$ be an $\Re$-sequence in $(\Omega, d_{\theta_{\Re}}, \Re)$, that is, $\omega_n \Re \omega_{n+1}$ or $\omega_{n+1} \Re \omega_n$ for each $n \in \mathbb{N}$. Then
(i) $\{\omega_n\}$ is a convergent sequence to some $\omega \in \Omega$ if $\lim\limits_{n\to\infty} d_{\theta_{\Re}}(\omega_n, \omega) = 0$ and $\omega_n \Re \omega$ for each $n \geq N_0$ $(N_0 \in \mathbb{N})$;
(ii) $\{\omega_n\}$ is Cauchy if $\lim\limits_{n,m\to\infty} d_{\theta_{\Re}}(\omega_n, \omega_m)$ exists and is finite.

Definition 3.6. $(\Omega, d_{\theta_{\Re}}, \Re)$ is said to be $\Re$-complete if for every $\Re$-Cauchy sequence in Ω, there is $\omega \in \Omega$ with $\lim\limits_{n,m\to\infty} d_{\theta_{\Re}}(\omega_n, \omega_m) = \lim\limits_{n\to\infty} d_{\theta_{\Re}}(\omega_n, \omega) = 0$ and $\omega_n \Re \omega$ for each $n \geq N_0$ $(N_0 \in \mathbb{N})$.

Definition 3.7. Let $(\Omega, d_{\theta_{\Re}}, \Re)$ be an $\Re$-extended b -metric space. Then $\mathcal{Q} : \Omega \to \Omega$ is said to be $\Re$-continuous (briefly, $\Re_\theta$-continuous) at $\omega \in \Omega$ if for each $\Re$-sequence $\{\omega_n\}_{n\in N}$ in Ω with $\omega_n \to \omega$ then $\mathcal{Q}\omega_n \to \mathcal{Q}\omega$. Also, $\mathcal{Q}$ is $\Re_\theta$-continuous on Ω if $\mathcal{Q}$ is $\Re_\theta$-continuous at each $\omega \in \Omega$.

Lemma 3.8. Let $(\Omega, d_{\theta_\Re}, \Re)$ be an $\Re$-extended b-metric space. If $d_{\theta_\Re}$ is $\Re$-continuous, then every $\Re$-convergent sequence has a unique limit.

Inspired by Theorem 2 of [8] we have the following result in the setting of $\Re$-extended b-metric spaces.

Theorem 3.9. Let $(\Omega, d_{\theta_\Re}, \Re)$ be a $\theta_\Re$-complete $\Re$-extended b-metric space. Suppose that $\mathcal{Q} : \Omega \to \Omega$ is $\Re$-continuous, $\Re$-preserving and $\Re$-contraction with a Lipchitz constant $\lambda \in (0, 1)$ such that for each $\omega \in \Omega$, $\lim_{n,m\to\infty} \theta(\omega_n, \omega_m) < \frac{1}{\lambda}$ where $\omega_n = \mathcal{Q}^n\omega$ for all $n \geq 1$. Then $\mathcal{Q}$ has a fixed point $\omega_* \in \Omega$ provided that $\omega_0\Re\mathcal{Q}\omega_0$, or, $\mathcal{Q}\omega_0\Re\omega_0$ for some ω_0. The fixed point is unique if $\omega_0\Re\omega_1$ or $\omega_1\Re\omega_0$ for all fixed points ω_0 and ω_1.

Proof . Let $\omega_1 = \mathcal{Q}\omega_0$, $\omega_2 = \mathcal{Q}\omega_1 = \mathcal{Q}^2\omega_0$, ..., $\omega_{n+1} = \mathcal{Q}\omega_n = \mathcal{Q}^{n+1}\omega_0$ for all $n \in \mathbb{N}$. Since $\mathcal{Q}$ is $\Re$-preserving, $\{\omega_n\}$ is an $\Re$-sequence. Also, $\mathcal{Q}$ is an $\Re$-contraction, so,

$$\begin{aligned} d_{\theta_\Re}(\omega_{n+1}, \omega_n) &= d_{\theta_\Re}\left(\mathcal{Q}^{n+1}\omega_0, \mathcal{Q}^n\omega_0\right) \\ &\leq \lambda d_{\theta_\Re}(\omega_n, \omega_{n-1}). \end{aligned}$$

Consequently,

$$d_{\theta_\Re}(\omega_{n+1}, \omega_n) \leq \lambda^n d_{\theta_\Re}(\omega_1, \omega_0) \text{ for all } n \in \mathbb{N}.$$

Next, to discuss the Cauchy criteria, we will consider arbitrary integers $m, n \geq 1$, with $m > n$. So,

$$\begin{aligned} & d_{\theta_\Re}(\omega_n, \omega_m) \\ \leq\ & \theta(\omega_n, \omega_m)\lambda^n d_\theta(\omega_0, \omega_1) + \theta(\omega_n, \omega_m)\theta(\omega_{n+1}, \omega_m)\lambda^{n+1} d_\theta(\omega_0, \omega_1) \\ & + \cdots \\ & + \theta(\omega_n, \omega_m)\theta(\omega_{n+1}, \omega_m)\theta(\omega_{n+2}, \omega_m)\cdots\theta(\omega_{m-2}, \omega_m)\theta(\omega_{m-1}, \omega_m)\lambda^{m-1} d_\theta(\omega_0, \omega_1) \\ \leq\ & d_\theta(\omega_0, \omega_1)\left[\theta(\omega_1, \omega_m)\theta(\omega_2, \omega_m)\cdots\theta(\omega_{n-1}, \omega_m)\theta(\omega_n, \omega_m)\lambda^n + \right. \\ & \theta(\omega_1, \omega_m)\theta(\omega_2, \omega_m)\cdots\theta(\omega_n, \omega_m)\theta(\omega_{n+1}, \omega_m)\lambda^{n+1} + \cdots + \\ & \left. \theta(\omega_1, \omega_m)\theta(\omega_2, \omega_m)\cdots\theta(\omega_n, \omega_m)\theta(\omega_{n+1}, \omega_m)\cdots\theta(\omega_{m-2}, \omega_m)\theta(\omega_{m-1}, \omega_m)\lambda^{m-1}\right]. \end{aligned}$$

From the above inequality and using the ratio test and the fact that $\lim_{n,m\to\infty} \theta(\omega_n, \omega_m) < \frac{1}{\lambda}$, we obtain that

$$\lim_{n,m\to\infty} d_{\theta_\Re}(\omega_n, \omega_m) = 0.$$

Therefore, $\{\omega_n\}$ is an $\Re$-Cauchy sequence. Since Ω is $\theta_\Re$-complete, there exists $\omega^* \in \Omega$ such that $\lim_{n\to\infty} d_{\theta_\Re}(\omega_n, \omega^*) = 0$ and $\omega_n\Re\omega^*$ for each $n \geq N_0$ (for some value of N_0). Thus, from the above discussion, we obtain $\lim_{n\to\infty} d_{\theta_\Re}(\omega_n, \omega^*) = 0$ and $\omega_n\Re\omega^*$ for each $n \geq N_0$. As $\omega_n\Re\omega^*$ for each $n \geq N_0$, we get

$$d_{\theta_\Re}(\mathcal{Q}\omega_n, \mathcal{Q}\omega_*) \leq \lambda d_{\theta_\Re}(\omega_n, \omega^*).$$

This inequality and the above findings imply

$$\lim_{n\to\infty} d_{\theta_\Re}(\omega_{n+1}, \mathcal{Q}\omega^*) \leq d_{\theta_\Re}(\omega_n, \omega^*) = 0.$$

As $\mathcal{Q}$ is $\Re$-continuous, we get $\mathcal{Q}\omega^* = \mathcal{Q}\left(\lim_{n\to\infty} \omega_n\right) = \lim_{n\to\infty} \mathcal{Q}\omega_n = \omega^*$ which implies the existence of a fixed point. To prove the uniqueness property of fixed point, let ω_0 be another fixed point for $\mathcal{Q}$. Then we have $\mathcal{Q}^n\omega^* = \omega^*$ and $\mathcal{Q}^n\omega_0 = \omega_0$ for all $n \in \mathbb{N}$. By the choice of ω^* in the first part of proof, we obtain

$$[\omega_0\Re\omega^* \text{ or } \omega^*\Re\omega_0].$$

Since $\mathcal{Q}$ is $\Re$-preserving, we have

$$[\mathcal{Q}^n\omega^*\Re\mathcal{Q}^n\omega_0 \text{ or } \mathcal{Q}^n\omega_0\Re\mathcal{Q}^n\omega^*]$$

for all $n \in \mathbb{N}$. Therefore, by the triangular inequality, we get

$$\begin{aligned} d_{\theta_{\Re}}(\omega^*, \omega_0) &= d_{\theta_{\Re}}(\mathcal{Q}^n\omega^*, \mathcal{Q}^n\omega_0) \\ &\leq \theta(\mathcal{Q}^n\omega^*, \mathcal{Q}^n\omega_0)\left[d_{\theta_{\Re}}(\mathcal{Q}^n\omega^*, \mathcal{Q}^n\omega) + d_{\theta_{\Re}}(\mathcal{Q}^n\omega, \mathcal{Q}^n\omega_0)\right] \\ &\leq \theta(\mathcal{Q}^n\omega^*, \mathcal{Q}^n\omega_0)\, d_{\theta_{\Re}}(\mathcal{Q}^n\omega^*, \mathcal{Q}^n\omega) + \theta(\mathcal{Q}^n\omega^*, \mathcal{Q}^n\omega_0)\, d_{\theta_{\Re}}(\mathcal{Q}^n\omega, \mathcal{Q}^n\omega_0) \\ &\leq \theta(\mathcal{Q}^n\omega^*, \mathcal{Q}^n\omega_0)\, \lambda^n d_{\theta_{\Re}}(\omega^*, \omega) + \theta(\mathcal{Q}^n\omega^*, \mathcal{Q}^n\omega_0)\, \lambda^n d(\omega, \omega_0). \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality,

$$d_{\theta_{\Re}}(\omega^*, \omega_0) = 0$$

which implies that $\omega^* = \omega_0$.

Thus, it has a unique fixed point.

Let $\omega \in \Omega$ be an arbitrary point. Then

$$d_{\theta_{\Re}}(\omega^*, \mathcal{Q}^n\omega) \leq \lambda^n d_{\theta_{\Re}}(\omega_*, \omega).$$

Taking the limit as $n \to \infty$ in the above inequality, we have $d_{\theta_{\Re}}(\omega^*, \mathcal{Q}^n\omega) = 0$. So,

$$\lim_{n\to\infty} \mathcal{Q}^n\omega = \omega^* \text{ for all } \omega \in \Omega.$$

□

Example 3.10. Let $\Omega = [0, 12]$ and $d_\theta : \Omega \times \Omega \to [0, \infty)$ be given by

$$d_{\theta_{\Re}}(\omega, \varpi) = |\omega - \varpi|^2$$

for all $\omega, \varpi \in \Omega$. Define a binary relation $\Re$ on Ω by $\omega\Re\varpi$ if $\omega\varpi \leq \omega$ or ϖ. Then $(\Omega, d_{\theta_{\Re}}, \Re)$ is an $\Re$-extended b-metric space with $\theta = \omega + \varpi + 2$. Define a mapping $\mathcal{Q} : \Omega \to \Omega$ by

$$\mathcal{Q}\omega = \begin{cases} \frac{\omega}{3}, & \text{if } 0 \leq \omega \leq 3, \\ 0, & \text{if } 3 \leq \omega \leq 12. \end{cases}$$

(i) If $\omega = 0$ and $0 \leq \varpi \leq 3$, then $\mathcal{Q}\omega = 0$ and $\mathcal{Q}\varpi = \frac{\varpi}{3}$.

(ii) If $\omega = 0$ and $3 \leq \varpi \leq 12$, then $\mathcal{Q}\omega = 0$ and $\mathcal{Q}\varpi = 0$.

(iii) If $0 \leq \varpi \leq 3$ and $0 \leq \omega \leq 3$, then $\mathcal{Q}\omega = \frac{\omega}{3}$ and $\mathcal{Q}\varpi = \frac{\varpi}{3}$.

(iv) If $0 \leq \varpi \leq 3$ and $3 \leq \omega \leq 12$, then $\mathcal{Q}\varpi = \frac{\varpi}{3}$ and $\mathcal{Q}\omega = 0$.

From $(i) - (iv)$,

$$|\mathcal{Q}\omega - \mathcal{Q}\varpi| \leq \frac{1}{3}|\omega - \varpi|,$$

i.e.,

$$|\mathcal{Q}\omega - \mathcal{Q}\varpi|^2 \leq \frac{1}{9}|\omega - \varpi|^2,$$

that is,

$$d_{\theta_{\Re}}(\mathcal{Q}\omega, \mathcal{Q}\varpi) \leq \frac{1}{9} d_{\theta_{\Re}}(\omega, \varpi).$$

So, $\mathcal{Q}$ is an $\Re$-contraction with $k = \frac{1}{9}$. Note that for each $\omega \in \Omega$, $\mathcal{Q}^n\omega = \frac{\omega^n}{3}$. Thus, we obtain that

$$\lim_{n\to\infty} \theta(\mathcal{Q}^m\omega, \mathcal{Q}^n\omega) < 9.$$

Therefore, all the conditions of Theorem 4.7 are satisfied. Hence, $\mathcal{Q}$ has a unique fixed point.

Theorem 3.11. Let $(\Omega, d_{\theta_\Re}, \Re)$ be a $\theta_\Re$-complete $\Re$-extended b-metric space with $\theta_\Re : \Omega \times \Omega \to [1, \infty)$. Suppose that $\mathcal{Q} : \Omega \to \Omega$ is an $\Re$-continuous and $\Re$-preserving mapping such that

$$d_{\theta_\Re}(\mathcal{Q}\omega, \mathcal{Q}\varpi) \leq \lambda_1 d_{\theta_\Re}(\omega, \varpi) + \lambda_2[d_{\theta_\Re}(\omega, \mathcal{Q}\omega) + d_{\theta_\Re}(\varpi, \mathcal{Q}\varpi)] \tag{2.1}$$

where $\lambda_i \geq 0$, for $i = 1, 2$, $\lim\limits_{n,m\to\infty} \frac{\lambda_1+\lambda_2\theta_\Re(\omega_{n-1},\omega_{n+1})}{1-\lambda_2\theta_\Re(\omega_{n-1},\omega_{n+1})}\theta_\Re(\omega_n, \omega_m) < 1$ where $\omega_n = \mathcal{Q}^n\omega_0$ for an $\omega_0 \in \Omega$. Then $\mathcal{Q}$ has a fixed point $\omega^* \in \Omega$ provided that $\omega_0 \Re \mathcal{Q}\omega_0$ or $\mathcal{Q}\omega_0 \Re \omega_0$ for some ω_0. The fixed point is unique if $\omega_0 \Re \omega_1$ or $\omega_1 \Re \omega_0$ for all fixed points ω_0, ω_1 for $\mathcal{Q}$.

Proof . For an arbitrary $\omega_0 \in \Omega$, define the iterative sequence $\{\omega_n\}_{n=0}^{\infty}$ by $\omega_n = \mathcal{Q}\omega_n = \mathcal{Q}^n\omega_0$ for all $n \geq 1$. If $\omega_n = \omega_{n-1}$, then ω_n is a fixed point of $\mathcal{Q}$ and the proof holds.

So, we suppose that $\omega_n \neq \omega_{n-1}$, for all $n \geq 1$. From (2.1), we have

$$d_{\theta_\Re}(\mathcal{Q}\omega_n, \mathcal{Q}\omega_{n-1}) \leq \lambda_1 d_{\theta_\Re}(\omega_n, \omega_{n-1}) + \lambda_2[d_{\theta_\Re}(\omega_n, \mathcal{Q}\omega_{n-1}) + d_{\theta_\Re}(\omega_{n-1}, \mathcal{Q}\omega_n)].$$

From the triangle inequality, we get:

$$\begin{aligned} d_{\theta_\Re}(\mathcal{Q}\omega_n, \mathcal{Q}\omega_{n-1}) &\leq \lambda_1 d_{\theta_\Re}(\omega_n, \omega_{n-1}) + \lambda_2\theta_\Re(\omega_{n-1}, \omega_{n+1})[d_{\theta_\Re}(\omega_{n-1}, \omega_n) \\ &\quad + d_{\theta_\Re}(\omega_n, \omega_{n+1})], \end{aligned}$$

i.e.,

$$\begin{aligned} d_{\theta_\Re}(\mathcal{Q}\omega_n, \mathcal{Q}\omega_{n-1}) &\leq \lambda_1 d_{\theta_\Re}(\omega_n, \omega_{n-1}) + \lambda_2\theta_\Re(\omega_{n-1}, \omega_{n+1})d_{\theta_\Re}(\omega_{n-1}, \omega_n) \\ &\quad + \lambda_2\theta_\Re(\omega_{n-1}, \omega_{n+1})d_{\theta_\Re}(\omega_n, \omega_{n+1})]. \end{aligned}$$

This implies that:

$$\begin{aligned} d_{\theta_\Re}(\omega_{n+1}, \omega_n) &\leq (\lambda_1 + \lambda_2\theta_\Re(\omega_{n-1}, \omega_{n+1}))d_{\theta_\Re}(\omega_n, \omega_{n-1}) \\ &\quad + \lambda_2\theta_\Re(\omega_{n-1}, \omega_{n+1})d_{\theta_\Re}(\omega_n, \omega_{n+1}), \end{aligned}$$

that is,

$$(1 - \lambda_2\theta_\Re(\omega_{n-1}, \omega_{n+1}))\, d_{\theta_\Re}(\omega_{n+1}, \omega_n) \leq (\lambda_1 + \lambda_2\theta_\Re(\omega_{n-1}, \omega_{n+1}))d_{\theta_\Re}(\omega_n, \omega_{n-1})$$

which yields that

$$d_{\theta_\Re}(\omega_{n+1}, \omega_n) \leq \zeta d_{\theta_\Re}(\omega_n, \omega_{n-1}),$$

where $\zeta = \frac{\lambda_1+\lambda_2\theta_\Re(\omega_{n-1},\omega_{n+1})}{1-\lambda_2\theta_\Re(\omega_{n-1},\omega_{n+1})}$. So,

$$\frac{d_{\theta_\Re}(\omega_{n+1}, \omega_n)}{d_{\theta_\Re}(\omega_n, \omega_{n-1})} \leq \zeta < 1.$$

Following the same lines in Theorem 4.7, $\{\omega_n\}$ is an $\Re$-Cauchy sequence. Since Ω is $\theta_\Re$-complete, there exists $\omega \in \Omega$ such that $\lim\limits_{n\to\infty} \omega_n = \omega$. Then $\lim\limits_{n\to\infty} d_{\theta_\Re}(\omega_n, \omega) = 0$ and $\omega_n \Re \omega^*$ for each $n \geq N_0$ (for some value of N_0).

Next, we will show that ω is a fixed point of $\mathcal{Q}$. From the triangle inequality and using (2.1), we have

$$\begin{aligned} d_{\theta_\Re}(\omega, \mathcal{Q}\omega) &\leq \theta_\Re(\omega, \mathcal{Q}\omega)[d_{\theta_\Re}(\omega, \omega_{n+1}) + d_{\theta_\Re}(\omega_{n+1}, \mathcal{Q}\omega)] \\ &\leq \theta_\Re(\omega, \mathcal{Q}\omega)[d_{\theta_\Re}(\omega, \omega_{n+1}) + \lambda_1 d_{\theta_\Re}(\omega_n, \omega) \\ &\quad + \lambda_2[d_{\theta_\Re}(\omega_n, \mathcal{Q}\omega) + d_{\theta_\Re}(\omega, \mathcal{Q}\omega_n)]], \end{aligned}$$

hence,

$$\begin{aligned} d_{\theta_\Re}(\omega, \mathcal{Q}\omega) &\leq \limsup \theta_\Re(\omega, \mathcal{Q}\omega)[d_{\theta_\Re}(\omega, \omega_{n+1}) + \lambda_1 d_{\theta_\Re}(\omega_n, \omega) \\ &\quad + \lambda_2[d_{\theta_\Re}(\omega_n, \mathcal{Q}\omega) + d_{\theta_\Re}(\omega, \mathcal{Q}\omega_n)]] \leq \lambda_2[\theta_\Re(\omega, \mathcal{Q}\omega)]^2 d_{\theta_\Re}(\omega, \mathcal{Q}\omega) \text{ as } n \to \infty. \end{aligned}$$

This implies that

$$(1 - \lambda_2[\theta_\Re(\omega, \mathcal{Q}\omega)]^2)d_{\theta_\Re}(\omega, \mathcal{Q}\omega) \leq 0.$$

But

$$(1-\lambda_2[\theta_{\Re}(\omega,\mathcal{Q}\omega)]^2)>0,$$

so, we get

$$d_{\theta_{\Re}}(\omega,\mathcal{Q}\omega)=0,\text{i.e.}, \mathcal{Q}\omega=\omega.$$

Hence, ω is a fixed point. Now, we check that ω is the unique fixed point of $\mathcal{Q}$. Let ω_0 be another fixed point of $\mathcal{Q}$. Then, we have

$$\mathcal{Q}\omega=\omega \text{ and } \mathcal{Q}\omega_0=\omega_0.$$

As we know,

$$[\omega_0\Re\omega \text{ or } \omega\Re\omega_0].$$

Since $\mathcal{Q}$ is $\Re$-preserving, we have

$$[\mathcal{Q}\omega_0\Re\mathcal{Q}\omega \text{ or } \mathcal{Q}\omega\Re\mathcal{Q}\omega_0].$$

Therefore, by the triangular inequality we get

$$\begin{aligned} d_{\theta_{\Re}}(\omega,\omega_0) &= d_{\theta_{\Re}}(\mathcal{Q}\omega,\mathcal{Q}\omega_0)\\ &\leq \lambda_1 d_{\theta_{\Re}}(\omega,\omega_0)+\lambda_2[d_{\theta_{\Re}}(\omega,\mathcal{Q}\omega_0)+d_{\theta_{\Re}}(\omega_0,\mathcal{Q}\omega)], \end{aligned}$$

i.e.,

$$(1-\lambda_1-2\lambda_2)\, d_{\theta_{\Re}}(\omega,\omega_0)\leq 0.$$

This implies that

$$(1-\lambda_1-2\lambda_2)d_{\theta_{\Re}}(\omega,\omega_0)\leq 0.$$

As

$$\lambda_1+2\lambda_2\leq \lambda_1+2\lambda_2\lim_{n,m\to\infty}\theta(u_n,u_m)<1,$$

therefore, $1-\lambda_1-2\lambda_2>0$, and so, $d_{\theta_{\Re}}(\omega,\omega_0)=0$, i.e., $\omega=\omega_0$. Hence, $\mathcal{Q}$ has a unique fixed point in Ω. $\square$

4 Application

In this section, we define a directed graph G on Ω, denoted by $G=(V(\Omega),E(\Omega))$, with the vertex set $V(\Omega)=\Omega$ and the edge set $E(\Omega)$ such that $E(\Omega)\subset\Omega\times\Omega$ and $\{(\omega,\omega):\omega\in\Omega\}\subset E(\Omega)$. Also, $E(\Omega)$ has no parallel edge. Note that $\omega P\varpi$ denotes the path between ω and ϖ, that is, there exists a finite sequence $\{\omega_i\}_{i=0}^{j}$, for some finite j, such that $\omega_0=\omega$, $\omega_j=\varpi$ and $(\omega_i,\omega_{i+1})\in E(\Omega)$ for $i\in\{0,1,\cdots,j-1\}$.

Definition 4.1. Assign the $\Omega\neq\emptyset$ to the previously defined G. A map $d_{\theta_G}:\Omega\times\Omega\to\mathbb{R}^+$ is called a G-extended b-metric on the set Ω, if the following conditions are satisfied for all $\omega,\varpi,z\in\Omega$ with $\omega P\varpi$ and $z\in\omega P\varpi$:

$(d_{\theta_G}1)$ $d_{\theta_G}(\omega,\varpi)=0$ if and only if $\omega=\varpi$;

$(d_{\theta_G}2)$ $d_{\theta_G}(\omega,\ \varpi)=d_{\theta_G}(\varpi\ ,\omega)$;

$(d_{\theta_G}3)$ $d_{\theta_G}(\omega,\ \varpi)\leq\theta_G(\omega,z)\,[d_{\theta_G}(\omega,\ z)+d_{\theta_G}(z,\ \varpi)]$.

Then $(\Omega,\ G,\ d_{\theta_G})$ is called a G-extended b-metric space.

Remark 4.2. If $\omega P\varpi$ and $z\in\omega P\varpi$, then we get ωPz and $zP\varpi$. Also, note if ωPz and $zP\varpi$, then we have $\omega P\varpi$.

Thus, P is a preorder relation on Ω. Therefore, $(\Omega,\ G,\ d_{\theta_G})$ is also an $\Re$-extended b-metric space.

Definition 4.3. Let $\{\omega_n\}$ be a G-sequence in (Ω,G,d_{θ_G}), that is, $\omega_nP\omega_{n+1}$ or $\omega_{n+1}P\omega_n$ for each n. Then we say that

(i) $\{\omega_n\}$ is a convergent sequence to $\omega\in\Omega$ if $\lim_{n\to\infty}d_{\theta_G}(\omega_n,\ \omega)=0$ and $\omega_nP\omega$ for each $n\geq k$;

(ii) $\{\omega_n\}$ is Cauchy if $\lim_{n,m\to\infty}d_{\theta_G}(\omega_n,\ \omega_m)=0$.

Definition 4.4. $(\Omega, G, d_{\theta_G})$ is said to be G-complete if for each Cauchy G-sequence in Ω there is $\omega \in \Omega$ with $\lim_{n,m\to\infty} d_{\theta_G}(\omega_n, \omega_m) = \lim_{n\to\infty} d_{\theta_G}(\omega_n, \omega) = 0$ and $\omega_n P \omega$ for each $n \geq k$.

Note that for a map $\mathcal{Q} : \Omega \to \Omega$, the G-continuity and G-preserving are defined in the same way as explained in the last section.

Theorem 4.5. Let $(\Omega, d_{\theta_G}, G)$ be a θ_G-complete G-extended b-metric space. Suppose that $\mathcal{Q} : \Omega \to \Omega$ is G-continuous, G-preserving (if $(\omega, \varpi) \in E(\Omega)$, then $(\mathcal{Q}\omega, \mathcal{Q}\varpi) \in E(\Omega)$) and G-contraction with a Lipchitz constant $\lambda \in (0, 1)$ such that for each $\omega \in \Omega$, $\lim\limits_{n,m\to\infty} \theta(\omega_n, \omega_m) < \frac{1}{\lambda}$ where $\omega_n = \mathcal{Q}^n\omega$ for all $n \geq 1$. Then $\mathcal{Q}$ has a fixed point $\omega_* \in \Omega$ provided that $\omega_0 P \mathcal{Q}\omega_0$, or, $\mathcal{Q}\omega_0 P \omega_0$ for some ω_0. The fixed point is unique if $\omega_0 P \omega_1$ or $\omega_1 P \omega_0$ for all fixed points ω_0 and ω_1.

By Remark 4.2, we know that P is a preorder relation on Ω and $(\Omega, G, d_{\theta_G})$ is an $\Re$-extended b-metric space. Also, an edge preserving map is path preserving. Hence, all the conditions of Theorem 2.10 hold. Hence, $\mathcal{Q}$ has a fixed point.

We obtain partially-ordered-extended b-metric spaces from $\Re$-extended b-metric spaces in the following by considering $\preceq$ as a partial order on Ω.

Definition 4.6. Let $\Omega \neq \emptyset$ be associated with a partial order $\preceq$, denoted as $(\Omega, \preceq)$. Given a map $d_{\theta_\preceq} : \Omega \times \Omega \to \mathbb{R}^+$. If the following conditions are satisfied for all $\omega, \varpi, z \in \Omega$ with $\omega \preceq \varpi$ and $\omega \preceq z \preceq \varpi$:

$(d_{\theta_\preceq}1)$ $d_{\theta_\preceq}(\omega, \varpi) = 0$ if and only if $\omega = \varpi$;

$(d_{\theta_\preceq}2)$ $d_{\theta_\preceq}(\omega, \varpi) = d_{\theta_\preceq}(\varpi, \omega)$;

$(d_{\theta_\preceq}3)$ $d_{\theta_\preceq}(\omega, \varpi) \leq \theta_\preceq(\omega, z) \left[d_{\theta_\preceq}(\omega, z) + d_{\theta_\preceq}(z, \varpi)\right]$, then $(\Omega, G, d_{\theta_\preceq})$ is called a partially-ordered-extended b-metric space.

According to the above discussion, we have the following result:

Theorem 4.7. Let $\left(\Omega, d_{\theta_\preceq}, \preceq\right)$ be a $\theta_\preceq$-complete $\preceq$-extended b-metric space. Suppose that $\mathcal{Q} : \Omega \to \Omega$ is $\preceq$-continuous, $\preceq$-preserving (if $\omega \preceq \varpi$ then $\mathcal{Q}\omega \preceq \mathcal{Q}\varpi$) and $\preceq$-contraction with a Lipchitz constant $\lambda \in (0, 1)$ such that for each $\omega \in \Omega$, $\lim\limits_{n,m\to\infty} \theta(\omega_n, \omega_m) < \frac{1}{\lambda}$ where $\omega_n = \mathcal{Q}^n\omega$ for all $n \geq 1$. Then $\mathcal{Q}$ has a fixed point $\omega_* \in \Omega$ provided that $\omega_0 \preceq \mathcal{Q}\omega_0$, or, $\mathcal{Q}\omega_0 \preceq \omega_0$ for some ω_0. The fixed point is unique if $\omega_0 \preceq \omega_1$ or $\omega_1 \preceq \omega_0$ for all fixed points ω_0 and ω_1.

Remark 4.8. $\preceq$-completeness is defined in the same way as G-completeness.

5 Conclusion and future work

In this article, we have introduced the concept of $\Re$-extended b-metric space and proved some fixed point theorems in this space. Furthermore, we have obtained results that extend and improve certain comparable results in the existing literature. Moreover, we have provided non-trivial examples to demonstrate the viability of the proposed results. Since our structure is more general, the class of extended b-metric spaces, our results and notions expand and generalize several previous results. In future directions of the studies on the introduced new space, new fixed point results can be investigated for the non-unique fixed points of self-mappings. Geometric properties of the set $Fix\mathcal{Q}$ can be investigated as a future problem in the case of a self-mapping $\mathcal{Q}$ on an $\Re$-extended b-metric space.

Open Problems:

1: We considered an $\Re$-extended b-metric space with a continuous control function. Do these results hold for a discontinuous control function in an $\Re$-extended b-metric space?

2. $\Re$-extended b-metric spaces, also called $\Re$-controlled metric spaces. Do these results hold in an $\Re$-double controlled metric space?

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