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# New sandwich results for univalent functions defined by the Tang-Aouf operator

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**Abstract**

In this paper, we study some differential subordination and subordination results for certain subclass of univalent functions in the open unit disc U using generalized operator $H^{\lambda,\delta}_{\eta,\mu}$. Also, we derive some sandwich theorems.

*Keywords:* Analytic function, Subordinate, Differential Subordination, Dominant, Generalized Operator, Sandwich Theorems.
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## 1. Introduction

Let $Y = Y(U)$ be the class of analytic functions in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. For $n \in N$ and $a \in C$. Let $Y[a, n]$ be the subclass of $Y$ of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., (a \in \mathbb{C}).$$

Let $\zeta$ denote the subclass of $Y$ of functions $f$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U), \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $f$ and $g$ are analytic functions in $Y$, $f$ is said to be subordinate to $g$, or $g$ is said to be superordinate to $f$ in $U$ and write $f \prec g$, if there exists a Shwarz function $w$ in $U$, which with $w(0) = 0$, and $|w(z)| < 1 (z \in U)$, where $f(z) = g(w(z))$. In such a case we write $f \prec g$ or $f(z) \prec g(z) (z \in U)$. If $g$ is univalent in $U$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ ([17, 18]).

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**Definition 1.1.** *[17] Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ and $h(z)$ be univalent in $U$. If $p(z)$ is analytic in $U$ and satisfies the second – order differential subordination:*

$$\phi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.2}$$

*then $p(z)$ is called a solution of the differential subordination (1.2), and the univalent function $q(z)$ is called a dominant of the solution of the differential subordination (1.2), or more simply dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominant $q(z)$ of (1.2) is said to be the best dominant is unique up to a relation of $U$.*

**Definition 1.2.** *[17] Let $p, h \in \zeta$ and $\phi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If $p$ and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent function in $U$ and if $p$ satisfies:*

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

*then $p$ is called a solution of the differential superordination (1.3). An analytic functions $q(z)$, which is called a subordinant of the solutions of the differential subordination (1.3), or more simply a subordinant if $p \prec q$ for all the functions $p$ satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants $q$ of (1.3) is said to be the best subordinant.*

Several researchers [1, 2, 9, 14, 17] obtained sufficient conditions on the functions $h, p$ and $\phi$ for which the following implication holds

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z)$$

then

$$q(z) \prec p(z) \tag{1.4}$$

Making use the results (see [3, 4, 5, 6, 10, 11, 18]) to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where $q_1$ and $q_2$ are given univalent functions in $U$ with $q_1(0) = q_2(0) = 1$.

Also, several researchers (see [1, 3, 5, 6, 7, 8]) derived some differential subordination and superordination results with sandwich results.

Cho et al. [13] introduced the operator $\Im_{0,z}^{\lambda,\mu,\eta}$ due to Goyal and Prajapat [15](see also [21]) as follows:

$$\Im_{0,z}^{\lambda,\mu,\eta} f(z) = \begin{cases} \frac{\Gamma(2-\mu)\Gamma(2-\lambda+\eta)}{\Gamma(2)\Gamma(2-\mu+\eta)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z)(0 \le \lambda < \eta + 2; z \in U) \\ \frac{\Gamma(2-\mu)\Gamma(2-\lambda+\eta)}{\Gamma(2)\Gamma(2-\mu+\eta)} z^{\mu} I_{0,z}^{-\lambda,\mu,\eta} f(z)(-\infty < \lambda < 0 + 2; z \in U), \end{cases} \tag{1.5}$$

where $J_{0,z}^{\lambda,\mu,\eta}$ and $I_{0,z}^{-\lambda,\mu,\eta}$ are the generalized fractional derivative and integral operators, respectively, due to Srivastava et al. [25](see also [19, 22]). For $f \in \zeta$ of form Equation (1.1), we have

$$\begin{aligned} \Im_{0,z}^{\lambda,\mu,\eta} f(z) &= z_3F_2 = (1, 2, 2 + \eta - \mu; 2 - \mu, 2 + \eta - \lambda; z) \\ &= z + \sum_{n=2}^{\infty} \frac{(2)_n(2 - \mu + \eta)_n}{(2 - \mu)_n(2 - \lambda + \eta)_n} a_n z^n, \quad (\mu, \eta \in \mathbb{R}; \mu < 2; -\infty < \lambda < \eta + 2), \end{aligned} \tag{1.6}$$

where $qF_S(q \leq s+1; q,s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is the well– Known generalized hypergeometric function (for details, see [20, 24]),the symbol $*$ stands for convolution of two analytic functions [17] and $(v)_n$ is the Pochhammer symbol [16, 20].

Setting

$$G^{\lambda}_{\eta,\mu}(z) = z + \sum_{n=2}^{\infty} \frac{(2)_n(2-\mu+\eta)_n}{(2-\mu)_n(2-\lambda+\eta)_n} z^n, \quad (\mu,\eta \in \mathbb{R}; \mu < \min\{2, 2+\eta\}; -\infty < \lambda < \eta+2) \tag{1.7}$$

and

$$G^{\lambda}_{\eta,\mu}(z) * [G^{\lambda,\delta}_{\eta,\mu}(z)] = \frac{z}{(1-z)^{\delta+1}}, \quad (\delta < -1; z \in U).$$

Tang et al. [26](see also [23]) defined the operator $H^{\lambda,\delta}_{\eta,\mu} : \zeta \to \zeta$ by $H^{\lambda,\delta}_{\eta,\mu} f(z) = [G^{\lambda,\delta}_{\eta,\mu}(z)] * f(z)$. then for $f \in \zeta$, we have

$$H^{\lambda,\delta}_{\eta,\mu} f(z) = z + \sum_{n=2}^{\infty} \frac{(\delta+1)_n(2-\mu)_n(2-\lambda+\eta)_n}{(1)_n(2)_n(2-\mu+\eta)_n} a_n z^n. \tag{1.8}$$

It is easy to verify that

$$z(H^{\lambda,\delta}_{\eta,\mu} f(z))' = (\delta+1) H^{\lambda,\delta+1}_{\eta,\mu} f(z) - \delta H^{\lambda,\delta}_{\eta,\mu} f(z),$$
$$z(H^{\lambda+1,\delta}_{\eta,\mu} f(z))' = (1+\eta-\lambda) H^{\lambda,\delta}_{\eta,\mu} f(z) - (\eta-\lambda) H^{\lambda,\delta+1}_{\eta,\mu} f(z). \tag{1.9}$$

The specific aim of this idea is to find sufficient condition for certain normalized analytic function $f$ to satisfy:

$$q_1(z) \prec \left( \frac{H^{\lambda,\delta}_{\eta,\mu} f(z)}{z} \right)^{\beta} \prec q_2(z),$$

and

$$q_1(z) \prec \left( \frac{H^{\lambda,\delta+1}_{\eta,\mu} f(z)}{H^{\lambda,\delta}_{\eta,\mu} f(z)} \right)^{\beta} \prec q_2(z),$$

## 2. Preliminaries

In order to prove our subordination and superordination results, we need the following lemmas and definitions.

**Definition 2.1.** *[17] denote by $Q$ the class of all functions $q$ that are analytic and injective on $\overline{U} \backslash E(q)$, where $\overline{U} = U \cup \{z \in \partial U\}$ and $E(q) = \{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\}$ and are such that $q(\zeta)' \neq 0$ for $\zeta \in partialU \backslash E(q)$. Further, let the subclass of $Q$ for which $q(0) = a$ be denoted by $Q(a), Q(0) = Q_0$ and $Q(1) = Q_1 = \{q \in Q : U : q(0) = 1\}$.*

**Lemma 2.2.** *[18] Let $q(z)$ be a convex univalent function in $U$ let $\gamma \in \mathbb{C}, \zeta \in \mathbb{C} \backslash \{0\}$ and suppose that $Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max\{0, -Re\frac{\gamma}{\zeta}\}$,*

*If $g(z)$ is analytic in $U$ and $\gamma g(z) + \zeta z\acute{g}(z) \prec \gamma q(z) + \zeta z\acute{q}(z)$,*

*then $g(z) \prec q(z)$ and $q$ is the best dominant.*

**Lemma 2.3.** *[4] Let $q$ be univalent in $U$ and let $\emptyset$ and $\theta$ be analytic in the domain $D$ containing $q(U)$ with $\emptyset(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = z\acute{q}(z)\emptyset(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$, suppose that*

1. *$Q$ is starlike univalent in $U$,*
2. *$Re\left\{\frac{zh'(z)}{Q'(z)}\right\} > 0, z \in U$.*

*If $g$ is analytic in $U$ with $g(0) = q(0)$, $g(U) \subseteq D$ and $\emptyset(g(z)) + z\acute{g}(z)\emptyset(g(z)) \prec \emptyset(q(z)) + z\acute{q}(z)\emptyset(q(z))$, then $g(z) \prec q(z)$, and $q$ is the best dominant.*

**Lemma 2.4.** *[12] Let $q(z)$ be a convex univalent function in the unit disk $U$ and let $\theta$ and $\emptyset$ be analytic in the domain $D$ containing $q(U)$ suppose that:*

1. *$Re\left\{\frac{\acute{\theta}(q(z))}{\phi(q(z))}\right\} > 0, z \in U$.*
2. *$Q(z) = z\acute{q}(z)\emptyset(q(z))$ is starlike univalent in $U$.*

*If $g \in H[q(0), 1] \cap Q$, with $g(U) \subseteq D$, and $\theta(g(z)) + z\acute{g}(z)\emptyset(g(z))$ is univalent in $U$, and $\theta(q(z)) + z\acute{q}(z)\emptyset(q(z)) \prec \theta(g(z)) + z\acute{g}(z)\emptyset(g(z))$, then $q(z) \prec g(z)$, and $q$ is the best subordinate.*

**Lemma 2.5.** *[18] Let $q(z)$ be a convex univalent function in $U$ and $q(0) = 1$, let $\beta \in \mathbb{C}$, that $Re\{\beta\} > 0$ if $g(z) \in H[q(0), 1] \cap Q$ and $g(z) + \beta z\acute{g}(z)$ is univalent in $U$, then*

$$q(z) + \beta z\acute{q}(z) \prec g(z) + \beta\acute{g}(z),$$

*which implies that $q(z) \prec g(z)$ and $q(z)$ is the best subordinate.*

## 3. Differential Subordination Results

**Theorem 3.1.** *Let $q$ be convex univalent function in $U$ with $q(0) = 1, \alpha \in \mathbb{C}\backslash\{0\}, \beta \in \mathbb{C}$ and suppose that $q$ satisfies:*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, Re\left(\frac{\beta}{\alpha}\right)\right\} \tag{3.1}$$

*If $f \in \zeta$ satisfies the subordination condition:*

$$[1-\alpha(\delta+1)]\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta} + \alpha(\delta+1)\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right) \prec q(z) + \frac{\alpha}{\beta}zq'(z), \tag{3.2}$$

*then*

$$\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta} \prec q(z), \tag{3.3}$$

*and $q$ is the best dominant.*

**Proof .** Define the function $g$ by

$$g(z)=\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}, \tag{3.4}$$

then the function $g(z)$ is analytic in $U$ and $g(0)=1$, therefore, differentiating (3.4) with respect to $z$ and using the identity (1.9) in the resulting equation, we obtain

$$\frac{zg'(z)}{g(z)}=\beta(\delta+1)\left[\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}-1\right].$$

Hence $\frac{zg'(z)}{\beta}=(\delta+1)\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}\left[\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}-1\right]$.

The subordination (3.2) from the hypothesis becomes

$$g(z)+\frac{\alpha}{\beta}zg'(z)\prec q(z)+\frac{\alpha}{\beta}zq'(z).$$

An application of Lemma 2.2, we obtain (3.3) with $\zeta=\frac{\alpha}{\beta}$ and $\gamma=1$. □

Putting $q(z)=\left(\frac{1+z}{1-z}\right)$, in Theorem 3.1, we obtain the following corollary.

**Corollary 3.2.** *Let $0\neq\alpha\in\mathbb{C},\beta>0$ and $Re\left\{1+\frac{2z}{1-z}\right\}>\max\left\{0,-Re\left(\frac{\beta}{\alpha}\right)\right\}$. If $f\in\zeta$ satisfies the subordination condition:*

$$[1-\alpha(\delta+1)]\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}+\alpha(\delta+1)\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\beta}\left(\frac{1-z^2+2\frac{\alpha}{\beta}z}{(1-z)^2}\right),$$

*then $\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}\prec\left(\frac{1+z}{1-z}\right)$,*
*and $q(z)=\left(\frac{1+z}{1-z}\right)$ is the best dominant.*

**Theorem 3.3.** *Let $q$ be convex univalent function in $U$ with $q(0)=1, q'(z)\neq 0\ \ (z\in U)$ and assume that $q$ satisfies:*

$$Re\left\{1+\frac{\gamma}{v}q(z)+q^2(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0,\left\{0,Re\left(\frac{\beta}{\alpha}\right)\right\} \tag{3.5}$$

*where $\alpha,v\in\mathbb{C}\backslash\{0\},\ c,\gamma,\beta\in\mathbb{C}$ and $z\in U$.*
*Assume that $v\frac{zq''(z)}{q'(z)}$ is starlike univalent in $U$. If $f\in\zeta$ satisfies:*

$$\chi(c,\gamma,v,\beta,\lambda,\delta,\eta,\mu;z)\prec c+\gamma q(z)+q^2(z)+v\frac{zq'(z)}{q(z)} \tag{3.6}$$

$$\chi(c,\gamma,v,\beta,\lambda,\delta,\eta,\mu;z)=c+\gamma\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\beta}+\left(\frac{H_{\eta,\mu}^{\lambda,\delta+2}f(z)}{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}\right)+v\beta(\delta+1)\left(\frac{H_{\eta,\mu}^{\lambda,\delta+2}f(z)}{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}-\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right), \tag{3.7}$$

*then $\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\beta}\prec q(z)$,*
*and $q(z)$ is the best dominant of (3.6).*

**Proof .** Consider a function $g$ by

$$g(z) = \left( \frac{H_{\eta,\mu}^{\lambda,\delta+1} f(z)}{H_{\eta,\mu}^{\lambda,\delta} f(z)} \right)^{\beta}. \tag{3.8}$$

Then the function $g(z)$ is analytic in $U$ and $g(0) = 1$ differentiating (3.8) with respect to $z$ and using the identity (1.9), we get,

$$\frac{zg'(z)}{g(z)} = \beta(\delta+1) \left( \frac{H_{\eta,\mu}^{\lambda,\delta+2} f(z)}{H_{\eta,\mu}^{\lambda,\delta+1} f(z)} - \frac{H_{\eta,\mu}^{\lambda,\delta+1} f(z)}{H_{\eta,\mu}^{\lambda,\delta} f(z)} \right),$$

By setting $\theta(w) = c + \gamma w + w^2$ and $\emptyset(w) = \frac{v}{w},\ w \neq 0$.
We see that $\theta(w)$ is analytic in $\mathbb{C}$ and $\emptyset(w)$ is analytic $\mathbb{C}\backslash\{0\}$ and that $\emptyset(w) \neq 0,\ w \in \mathbb{C}\backslash\{0\}$.
Also, we obtain $R(z) = zq'(z)\emptyset(q(z)) = zq'(z)\frac{v}{q(z)} = v\frac{zq'(z)}{q(z)}$,
and $S(z) = \theta(q(z)) + R(z) = c + \gamma q(z) + (q(z))^2 + v\frac{zq'(z)}{q(z)}$.
We find $R(z)$ is starlike univalent in $U$, we have

$$S'(z) = \gamma q'(z) + 2(q(z)q'(z)) + \frac{vzq''(z) + vq'(z)}{q(z)}$$

$$\frac{zS'(z)}{R(z)} = \frac{\gamma}{v}q(z) + \frac{2}{vz}(q(z))^2 + \frac{vzq''(z) + vq'(z)}{vzq(z)} = \frac{\gamma}{v}q(z) + \frac{2}{vz}(q(z))^2 + \frac{zq''(z)}{q'(z)} + \frac{zq'(z)}{q(z)},$$

$$Re\left(\frac{zS'(z)}{R(z)}\right) = Re\left\{1 + \frac{\gamma}{v}q(z) + \frac{2}{vz}(q(z))^2 + \frac{zq''(z)}{q'(z)} + \frac{zq'(z)}{q(z)}\right\} > 0.$$

By a straightforward computing, we get

$$c + \gamma q(z) + q^2(z) + v\frac{zq'(z)}{q(z)} = \chi(c, \gamma, v, \beta, \lambda, \delta, \eta, \mu; z), \tag{3.9}$$

where $\chi(c, \gamma, v, \beta, \lambda, \delta, \eta, \mu; z)$ is given by (3.7).
From (3.6) and (3.9), we have

$$c + \gamma g(z) + g^2(z) + v\frac{zg'(z)}{g(z)} \prec c + \gamma q(z) + q^2(z) + v\frac{zq'(z)}{q(z)}.$$

Therefore, by Lemma 2.3, we get $g(z) \prec q(z)$ by using (3.4), we obtain the result. □

Putting $q(z) = \left(\frac{1+Az}{1+Bz}\right) \quad -1 \leq B < A \leq 1$ in the Theorem (3.3), we get the following corollary:

**Corollary 3.4.** *Let* $-1 \leq B < A \leq 1$ *and* $Re\left\{1 + \frac{\gamma}{v}\left(\frac{1+Az}{1+Bz}\right) + \frac{2}{vz}\left(\frac{1+Az}{1+Bz}\right)^2 + \frac{2Bz}{1+Bz} + \frac{(A-B)z}{(1+Bz)(1+Az)}\right\} >$ 0,
*where* $v \in \mathbb{C}\backslash\{0\}$ *and* $z \in U$, *if* $f \in \zeta$ *satisfies*

$$\chi(c, \gamma, v, \beta, \lambda, \delta, \eta, \mu; z) \prec c + \gamma\left(\frac{1+Az}{1+Bz}\right) + \left(\frac{1+Az}{1+Bz}\right)^2 + \frac{v(A-B)z}{(1+Bz)(1+Az)},$$

*and* $\chi(c, \gamma, v, \beta, \lambda, \delta, \eta, \mu; z)$ *is given by* (3.7), *then*

$$\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\beta} \prec \left(\frac{1+Az}{1+Bz}\right),$$

*and* $q(z)=\left(\frac{1+Az}{1+Bz}\right)$ *is the best dominant.*

## 4. Differential Superordination Results

**Theorem 4.1.** *Let* $q$ *be a convex univalent function in* $U$ *with* $q(0)=1\beta>0$ *and* $Re\{\alpha\}>0$. *Let* $f\in\zeta$ *satisfies:*

$$\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta} \in H[q(0),1]\cap Q$$

*and* $[1-\alpha(\delta+1)]\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}+\alpha(\delta+1)\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right)$ *be univalent in* $U$.
*If*

$$q(z)+\frac{\alpha}{\beta}zq'(z)\prec[1-\alpha(\delta+1)]\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}+\alpha(\delta+1)\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right), \tag{4.1}$$

*then* $q(z)\prec\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}$, *and* $q$ *is the best subordinate of* (4.1).

**Proof .** Define the function $g$ by

$$g(z)=\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}. \tag{4.2}$$

Differentiating (4.2) with respect to $z$, we obtain

$$v\frac{zg'(z)}{g(z)}=\beta\left(\frac{z\left(H_{\eta,\mu}^{\lambda,\delta}f(z)\right)'}{H_{\eta,\mu}^{\lambda,\delta}f(z)}-1\right). \tag{4.3}$$

After some computations and using (1.9), from (4.3), we get

$$[1-\alpha(\delta+1)]\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}+\alpha(\delta+1)\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right)=g(z)+\frac{\alpha}{\beta}zg'(z)$$

and now, by using Lemma 2.5, we get the desired result. □

Putting $q(z)=\left(\frac{1+z}{1-z}\right)$ in Theorem 4.1, we obtain the following corollary:

**Corollary 4.2.** *Let* $\beta>0$ *and* $Re\{\alpha\}>0$, *if* $f\in\zeta$ *satisfies:*

$$\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta} \in H[q(0),1]\cap Q$$

*and* $[1-\alpha(\delta+1)]\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}+\alpha(\delta+1)\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right)$ *be univalent in* $U$.

$$\left(\frac{1-z^2+2\frac{\alpha}{\beta}z}{(1-z)^2}\right)\prec[1-\alpha(\delta+1)]\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}+\alpha(\delta+1)\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right),$$

*then* $\left(\frac{1+z}{1-z}\right)\prec\left(\frac{H_{\eta,\mu}^{\lambda,\delta}f(z)}{z}\right)^{\beta}$,
*and* $q(z)=\left(\frac{1+z}{1-z}\right)$ *is the best subordinant.*

**Theorem 4.3.** *Let* $q$ *be a convex univalent function in* $U$ *with* $q(0)=1,\ q'(z)\neq 0$ *and assume that* $q$ *satisfies:*

$$Re\left\{\frac{\gamma}{v}q(z)q'(z)\right\}>0 \tag{4.4}$$

*where* $v\in\mathbb{C}\backslash\{0\}$ *and* $z\in U$.
*Suppose that* $v\frac{zq'(z)}{q(z)}$ *is starlike univalent function in* $U$. *Let* $f\in\zeta$ *satisfies:*
$\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\beta}\in H[q(0),1]\cap Q$ *and* $\chi(c,\gamma,v,\beta,\lambda,\delta,\eta,\mu;z)$ *is univalent function in* $U$,
*where* $\chi(c,\gamma,v,\beta,\lambda,\delta,\eta,\mu;z)$ *is given by* (3.7). *If*

$$c+\gamma q(z)+q^2(z)+v\frac{zq'(z)}{q(z)}\prec\chi(c,\gamma,v,\beta,\lambda,\delta,\eta,\mu;z), \tag{4.5}$$

*then* $q(z)\prec\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\beta}$,
*and* $q$ *is the best subordinant of* (4.5).

**Proof .** Consider a function $g$ by $g(z)=\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1}f(z)}{H_{\eta,\mu}^{\lambda,\delta}f(z)}\right)^{\beta}$.
By setting: $\theta(w)=c+\gamma w+w^2$ and $\varnothing(w)=\frac{v}{w},\ w\neq 0$.
We see that $\theta(w)$ is analytic in $\mathbb{C}$ and $\varnothing(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\varnothing(w)\neq 0,\ w\in\mathbb{C}\backslash\{0\}$.
Also, we obtain $R(z)=zq'(z)\varnothing(q(z))=v\frac{zq'(z)}{q(z)}$.
It is clear that $R(z)$ is starlike univalent function in $U$,

$$Re\left\{\frac{\theta'(q(z))}{\varnothing(q(z))}\right\}=Re\left\{\frac{\gamma}{v}q(z)q'(z)\right\}>0.$$

By straightforward computation, we get:

$$\chi(c,\gamma,v,\beta,\lambda,\delta,\eta,\mu;z)=c+\gamma q(z)+q^2(z)+v\frac{zq'(z)}{q(z)}, \tag{4.6}$$

where $\chi(c,\gamma,v,\beta,\lambda,\delta,\eta,\mu;z)$ is given by (3.7).
From (4.5) and (4.6), we have

$$c+\gamma q(z)+q^2(z)+v\frac{zq'(z)}{q(z)}\prec c+\gamma g(z)+g^2(z)+v\frac{zg'(z)}{g(z)}.$$

Therefore, by Lemma 2.4, we get $q(z)\prec g(z)$. $\square$

## 5. Sandwich Results

**Theorem 5.1.** *Let $q_1$ be a convex univalent function in $U$ with $q_1(0) = 1$, $Re\{\alpha\} > 0$, $\alpha \in \mathbb{C}\backslash\{0\}$, $\beta > 0$ and let $q_2$ be univalent function in $U$, $q_2(0) = 1$ and satisfies*

$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, Re\left(\frac{\beta}{\alpha}\right)\right\}$. *If $f \in \zeta$ satisfies:*

$$\left(\frac{H_{\eta,\mu}^{\lambda,\delta} f(z)}{z}\right)^{\beta} \in H[1,1] \cap Q,$$

*and* $[1-\alpha(\delta+1)]\left(\frac{H_{\eta,\mu}^{\lambda,\delta} f(z)}{z}\right)^{\beta} + \alpha(\delta+1)\left(\frac{H_{\eta,\mu}^{\lambda,\delta} f(z)}{z}\right)^{\beta}\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1} f(z)}{H_{\eta,\mu}^{\lambda,\delta} f(z)}\right)$ *be univalent in $U$.*

*If* $q_1(z) + \frac{\alpha}{\beta} z q_1'(z) \prec [1-\alpha(\delta+1)]\left(\frac{H_{\eta,\mu}^{\lambda,\delta} f(z)}{z}\right)^{\beta} + \alpha(\delta+1)\left(\frac{H_{\eta,\mu}^{\lambda,\delta} f(z)}{z}\right)^{\beta}\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1} f(z)}{H_{\eta,\mu}^{\lambda,\delta} f(z)}\right) \prec q_2(z) + \frac{\alpha}{\beta} z q_2'(z)$,

*then* $q_1(z) \prec \left(\frac{H_{\eta,\mu}^{\lambda,\delta} f(z)}{z}\right)^{\beta} \prec q_2(z)$ *and $q_1(z)$ and $q_2(z)$ are respectively, the best subordinant and the best dominant.*

**Theorem 5.2.** *Let $q_1$ be a convex univalent function in $U$ with $q_1(0) = 1$ and satisfies $Re\left\{\frac{\gamma}{v} q(z) q'(z)\right\} > 0$. Let $q_2$ be univalent function in $U$ with $q_2(0) = 1$ satisfies*

$$Re\left\{1 + \frac{\gamma}{v} q(z) + q_2(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0$$

*Let $f \in \zeta$ satisfies:* $\left(\frac{H_{\eta,\mu}^{\lambda,\delta+1} f(z)}{H_{\eta,\mu}^{\lambda,\delta} f(z)}\right)^{\beta} H[1,1] \cap Q$,

*and $\chi(c,\gamma,v,\beta,\lambda,\delta,\eta,\mu;z)$ is univalent in $U$, where $\chi(c,\gamma,v,\beta,\lambda,\delta,\eta,\mu;z)$ is given by (3.7). If $\alpha q_1(z) - \beta z q_1'(z) \prec \chi(c,\gamma,v,\beta,\lambda,\delta,\eta,\mu;z) \prec \alpha q_2(z) - \beta z q_2'(z)$,*

*then* $q_1(z) \prec \left(\frac{H_{\eta,\mu}^{\lambda,\delta+1} f(z)}{H_{\eta,\mu}^{\lambda,\delta} f(z)}\right)^{\beta} \prec q_2(z)$, *and $q_1(z)$ and $q_2(z)$ are respectively, the best subordinant and the best dominant.*

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