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The (p, q)-analogue of sigmoid function in the mirror of bi-univalent functions coupled with subordination

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Abstract

The aim of this study is to introduce the new subclasses of bi-univalent functions coupled with subordination in the mirror of (p, q)-analogue of the modified sigmoid function in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. The first two immediate Taylor-Maclaurin coefficients for the function belonging to these newly introduced classes are obtained. The results are new and a number of corollaries are developed by varying the parameters involved.

Keywords: Analytic function, Bi-univalent function, Coefficient bounds, (p,q)-analogue of the sigmoid function
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1 Introduction and Motivation

Hypergeometric function is an important tool used to construct subclasses of analytic functions. Recently, scholars are giving more attention because of its usefulness in Mathematics and Physics, in particular in the fields of ordinary fractional calculus, operator theory, transform analysis and so on. Heine (see [9, 10]) introduced and studied the hypergeometric function of the form

$$1 + \frac{(q^{\alpha} - 1)(q^{\beta} - 1)}{(q - 1)(q^{\gamma} - 1)} z + \frac{(q^{\alpha} - 1)(q^{\alpha+1} - 1)(q^{\beta} - 1)(q^{\beta+1} - 1)}{(q - 1)(q^2 - 1)(q^{\gamma} - 1)(q^{\gamma+1} - 1)} z^2 + \cdots . \tag{1.1}$$

Much later, Jackson, Bailey, Agrawal, Slater, Andrews and many other contributors concentrated in the study of such function. Many authors have studies the q-analogue of some special functions like q-Gamma, q-Beta, q-Bernoulli, q-Zeta functions and so on as a part of q-calculus. The extension of the q-calculus to post-quantum calculus denoted by (p, q)-calculus can not be obtained directly by substitution of q by $\frac{q}{p}$ in $q-$calculus. Taking $p = 1$ in (p,q)-calculus, the q-calculus may be obtained. For recent expository work so called post-quantum calculus or (p, q) calculus, see [4]. Afterwards, researchers like Al-Hawary et al.[1] and Yousef and Salleh[23] have taken their time to look at the post quantum calculus and their findings are too voluminous in literature to discuss.

A special function that receives the attention of researchers now-a-days is the sigmoid function of the form $\frac{1}{1+e^{-z}}$. Sigmoid function is the most popular activation function in the hardware implementation of artificial neural network.

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It is differentiable, monotonic and very useful in squashing outputs. Authors like Fadipe-Joseph et al.[11], Murugusundaramoorthy and Janani[15], Panigrahi [19] and Olatunji[17], mention a few have worked tirelessly on this function in the space of univalent functions theory and their results are available in literature. In recent time, Ezeafulukwe et al.[8] used the $q-$calculus to study and introduce a modified $q-$sigmoid function in the space of univalent $\lambda-$pseudo starlike functions. The early few coefficients are derived which they used to obtain the relevant connection to Fekete-Szegö inequality.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.2}$$

which are analytic and univalent in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ that are univalent in $\mathbb{U}$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*$ of starlike functions of order zero if and only if $\frac{zf'(z)}{f(z)} > 0 \quad (z \in \mathbb{U})$. Further, a function $f(z) \in \mathcal{A}$ is a said to be in the class $\mathcal{K}$ of convex function if and only if $zf'(z)$ is starlike.

Let the functions f and g be analytic in $\mathbb{U}$. We say $f(z)$ is subordinate to $g(z)$ denoted by $f(z) \prec g(z)$ if there exist an analytic function w satisfying the condition of Schwarz lemma such that $f(z) = g(w(z)) \quad (z \in \mathbb{U})$. In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to (see [14])

$$f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

The well-known Koebe one-quarter theorem (see[7]) asserts that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus, the inverse of $f \in \mathcal{A}$ is a univalent analytic function on the disk $\mathbb{U}_\rho := \{z : z \in \mathbb{C} \text{ and } |z| < \rho;\ \rho \geq \frac{1}{4}\}$. Therefore, for each function $f(z) = w$ there is an inverse function $f^{-1}(w)$ of $f(z)$ defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad (w \in \mathbb{U}_\rho)$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.3}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent function in $\mathbb{U}$ given by (1.2). The concept of bi-univalent analytic functions was introduced by Lewin [13] in 1967 and he showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$. On the other hand, Netanyahu [16] showed that $\max_{f \in \Sigma} |a_2| = \frac{4}{3}$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n| \quad (n \in \mathbb{N} \setminus \{1, 2\})$ is presumably still an open problem. Recently, various researchers namely; Altinkaya and Yalcin [3], Bulut et al.[6], Panigrahi and Murugusundarmoorthy [20](also, see [18]), Srivastava et al. [21] mention a few, have introduced different subclasses of bi-univalent functions and obtained non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$.

The Touchard polynomial studied by Touchard [22] is also known as the exponential polynomials or Bell polynomials, comprise a polynomial sequence of binomial type . If X is a random variable with Poisson distribution with expected value m, then its nth moment is $E(X_n) = T_n(m)$ given by

$$T_n(m) = e^{-m} \sum_{k=0}^{\infty} \frac{m^k k^n}{k!} \quad (n \geq 0). \tag{1.4}$$

Recently, Al-Shaqsi [2] introduced a function $F_n(z, m)$ given by

$$F_n(z, m) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}(k-1)^n e^{-m}}{(k-1)!} z^k \quad (m > 0,\ n \geq 0) \tag{1.5}$$

in univalent functions theory. This polynomial has not been deeply studied in this field like Chebychev polynomial of the first and second kinds. It can be shown by ratio test that polynomial is convergent and has radius of convergence at infinity.

For a function f given by (1.2) and g of the form

$$g(z) = z + b_2z^2 + b_3z^3 + b_4z^4 + \cdots = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.6}$$

the convolution of f and g denoted by $f * g$ and is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.7}$$

Define the linear operator $\mathcal{I}_n^m : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$I_n^m f(z) = F_n(z,m) * f(z) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}(k-1)^n e^{-m}}{(k-1)!} a_k z^k \quad (m > 0, n \geq 0). \tag{1.8}$$

We need the following definitions in order to introduce the function class. Let $\mathcal{P}$ denote the class of analytic functions in $\mathbb{U}$ satisfying the condition $p(0) = 1$ and $\Re(p(z)) > 0$ and of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{1.9}$$

Here $p(z)$ is called the Caratheodory function (see[7]).
Definition 2: [11] The sigmoid function is defined as

$$G(z) = \frac{1}{1+e^{-z}} = \frac{1}{2} + \frac{z}{4} - \frac{z^3}{48} + \frac{z^5}{480} - \frac{17z^7}{80640} + \cdots . \tag{1.10}$$

The modified sigmoid function is of the form

$$\gamma(z) = \frac{2}{1+e^{-z}} = 1 + \frac{z}{2} - \frac{z^3}{24} + \frac{z^5}{240} - \frac{17z^7}{40320} + \cdots . \tag{1.11}$$

Definition 3: [12] For any fixed real number $q > 0$, non negative integer t, the q-integers of the number t is defined by

$$[t]_q = \begin{cases} \dfrac{1-q^t}{1-q} & q \neq 1 \\ t & q = 1 \\ 0 & t = 0. \end{cases} \tag{1.12}$$

The so-called (p,q)-bracket or twin-basic number is defined as

$$[t]_{p,q} = \frac{p^t - q^t}{p-q} \quad (0 < q < p \leq 1). \tag{1.13}$$

The twin-basic number is a natural generalization of the q-number i.e. $\lim_{p \longrightarrow 1} [t]_{p,q} = [t]_q$.
Definition 4: [12] The q-fractional is defined in the following

$$[t]_q! = \begin{cases} [t]_q[t-1]_q ... [1]_q & q \neq 1 \\ 1 & t = 0. \end{cases} \tag{1.14}$$

and

$$[t]_{p,q}! = \begin{cases} [t]_{p,q}[t-1]_{p,q} ... [1]_{p,q} & q \neq 1 \\ 1 & t = 0. \end{cases} \tag{1.15}$$

Note that $[1]_{p,q} = 1$ and $[1]_{p,q}! = 1$.
Definition 5: [12] A q-analogue of the ordinary exponential function $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is defined by

$$e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}. \tag{1.16}$$

Definition 7: [8] The q-sigmoid function is defined as

$$G_q(z) = \frac{1}{1+e_q^{-z}}. \tag{1.17}$$

The modified q-sigmoid function is of the form

$$\gamma_q(z) = \frac{2}{1+e_q^{-z}} = 1 + \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{[n]_q!} z^n \right]^k \right). \tag{1.18}$$

Employing the above definitions, one will have
Definition 8: The modified (p, q)-sigmoid function is of the form

$$\gamma_{p,q}(z) = \frac{2}{1+e_{(p,q)}^{-z}} = 1 + \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{[n]_{(p,q)}!} z^n \right]^k \right). \tag{1.19}$$

where $p \neq q$ and $0 < q < p \leq 1$.
Using modified (p,q)-sigmoid function we define the function class as follows:
Definition 9: A function $f \in \mathcal{A}$ given by (1.2) is in the class $\mathcal{S}_{\Sigma,\alpha}^{n,m}(\gamma_{p,q})$ $(0 < q < p \leq 1,\ m > 0,\ n \geq 0, 0 \leq \alpha \leq 1)$ if

$$\frac{z(\mathcal{I}_n^m f(z))'}{(1-\alpha)z + \alpha \mathcal{I}_n^m f(z)} \prec t(z) \tag{1.20}$$

and

$$\frac{w(\mathcal{I}_n^m g(w))'}{(1-\alpha)w + \alpha \mathcal{I}_n^m g(w)} \prec t(w) \quad (z,\ w \in \mathbb{U}), \tag{1.21}$$

where the function $g = f^{-1}$ is defined by (1.3).
Remark 10: Taking $\alpha = 0$, we obtain the class $\mathcal{S}_{\Sigma,0}^{n,m}(\gamma_{p,q}) = \mathcal{S}_{\Sigma}^{n,m}(\gamma_{p,q})$ consists of functions $f \in \Sigma$ satisfying the condition

$$(\mathcal{I}_n^m f(z))' \prec t(z)$$

and

$$(\mathcal{I}_n^m g(w))' \prec t(w) \quad (z,\ w \in \mathbb{U}).$$

Remark 11: Taking $\alpha = 1$, we obtain the class $\mathcal{S}_{\Sigma,1}^{n,m}(\gamma_{p,q}) = \mathcal{T}_{\Sigma}^{n,m}(\gamma_{p,q})$ consists of functions $f \in \Sigma$ satisfying the condition

$$\frac{z(\mathcal{I}_n^m f(z))'}{I_n^m f(z)} \prec t(z)$$

and

$$\frac{w(\mathcal{I}_n^m g(w))'}{I_n^m g(w)} \prec t(w) \quad (z,\ w \in \mathbb{U}).$$

Definition 12: A function $f \in \mathcal{A}$ given by (1.2) belongs to the class $\mathcal{K}_{\Sigma,\alpha}^{n,m}(\gamma_{p,q})$ if the following conditions holds:

$$\frac{z(\mathcal{I}_n^m f(z))' + z^2(I_n^m f(z))''}{(1-\alpha)z + \alpha z(I_n^m f(z))'} \prec t(z) \tag{1.22}$$

and

$$\frac{w(I_n^m g(w)))' + w^2(I_n^m g(w))''}{(1-\alpha)w + \alpha w(I_n^m g(w))'} \prec t(w) \quad (z,\ w \in \mathbb{U}). \tag{1.23}$$

Remark 13: Putting $\alpha = 0$, we get the class $\mathcal{K}_{\Sigma,0}^{n,m}(\gamma_{p,q}) = \mathcal{K}_{\Sigma}^{n,m}(\gamma_{p,q})$ consists of functions $f \in \Sigma$ that satisfy the condition

$$(\mathcal{I}_n^m f(z))' + z(I_n^m f(z))'' \prec t(z)$$

and

$$(I_n^m g(w))' + w(I_n^m g(w))'' \prec t(w) \quad (z,\ w \in \mathbb{U}).$$

Remark 14: Letting $\alpha = 1$, we get the class $\mathcal{K}_{\Sigma,1}^{n,m}(\gamma_{p,q}) = \mathcal{R}_{\Sigma}^{n,m}(\gamma_{p,q})$ consists of functions $f \in \Sigma$ that satisfy the condition

$$1 + \frac{z(\mathcal{I}_n^m f(z))''}{(I_n^m f(z))'} \prec t(z)$$

and

$$1 + \frac{w(\mathcal{I}_n^m g(w))''}{(I_n^m g(w))'} \prec t(w) \quad (z,\ w \in \mathbb{U}).$$

In the present investigation, the authors introduce the new subclasses of bi-univalent functions coupled with subordination in the mirror of (p,q)-analogue of the modified sigmoid function in the unit disc $\mathbb{U}$. The initial coefficient bounds are obtained for the new classes of functions defined. The literature show that so far no individual has worked in this direction.

2 MAIN RESULTS

Theorem 1: Let the function $f \in \mathcal{A}$ be in the class $\mathcal{S}_{\Sigma,\alpha}^{m,n}(\gamma_{p,q})$. Then

$$|a_2| \leq \frac{1}{2m\sqrt{e^{-m}\left|2^n(3-\alpha) - 2e^{-m}(2\alpha - \alpha^2)\right|}}, \tag{2.1}$$

and

$$|a_3| \leq \frac{1}{m^2 e^{-m} 2^n (3-\alpha)} \left(\frac{2^n(3-\alpha)}{4(2-\alpha)^2 e^{-m}} + \frac{1}{2} + \frac{4}{[2]_{p,q}!} \right). \tag{2.2}$$

where $m > 0, n \geq 0,\ 0 \leq \alpha \leq 1$ and $0 < q < p \leq 1$.

Proof: Suppose $f \in \mathcal{S}_{\Sigma,\alpha}^{m,n}(\gamma_{p,q})$. Then by the definition of subordination, there exists two analytic functions $u(z), v(\omega)$ such that $u(0) = v(0) = 0$, $|u(z)| < 1$ and $|v(\omega)| < 1 \quad (z, \omega \in \mathbb{U})$ such that

$$\frac{z(\mathcal{I}_n^m f(z))'}{(1-\alpha)z + \alpha \mathcal{I}_n^m f(z)} = t(u(z)) \tag{2.3}$$

and

$$\frac{w(\mathcal{I}_n^m g(w))'}{(1-\alpha)w + \alpha \mathcal{I}_n^m g(w)} = t(v(w)). \tag{2.4}$$

Define the functions

$$\gamma_{p,q}(z) = \tfrac{1+u(z)}{1-u(z)} = 1 + \tfrac{z}{2} + \left(\tfrac{1}{4} - \tfrac{1}{2[2]_{p,q}!}\right) z^2 + \left(\tfrac{1}{2[3]_{p,q}!} - \tfrac{1}{2[2]_{p,q}!} + \tfrac{1}{8}\right) z^3 + ... \tag{2.5}$$

and

$$\gamma_{p,q}(w) = \tfrac{1+v(w)}{1-v(w)} = 1 + \tfrac{w}{2} + \left(\tfrac{1}{4} - \tfrac{1}{2[2]_{p,q}!}\right) w^2 + \left(\tfrac{1}{2[3]_{p,q}!} - \tfrac{1}{2[2]_{p,q}!} + \tfrac{1}{8}\right) w^3 + ... \tag{2.6}$$

Thus,

$$u(z) = \tfrac{\gamma_{p,q}(z)-1}{\gamma_{p,q}(z)+1} = \tfrac{z}{4} + \left(\tfrac{1}{16} - \tfrac{1}{4[2]_{p,q}!}\right) z^2 + \left(\tfrac{1}{4[3]_{p,q}!} - \tfrac{1}{8[2]_{p,q}!} + \tfrac{1}{64}\right) z^3 + ... \tag{2.7}$$

and

$$v(w) = \tfrac{\gamma_{p,q}(w)-1}{\gamma_{p,q}(w)+1} = \tfrac{w}{4} + \left(\tfrac{1}{16} - \tfrac{1}{4[2]_{p,q}!}\right) w^2 + \left(\tfrac{1}{4[3]_{p,q}!} - \tfrac{1}{8[2]_{p,q}!} + \tfrac{1}{64}\right) w^3 + \cdots, \tag{2.8}$$

which leads to

$$t(u(z)) = 1 + \frac{c_1 z}{4} + \frac{1}{4}\left(\frac{c_2}{4} + \frac{c_1}{4} - \frac{c_1}{[2]_{p,q}!}\right) z^2 + \cdots, \tag{2.9}$$

and

$$t(v(w)) = 1 + \frac{d_1 \omega}{4} + \frac{1}{4}\left(\frac{d_2}{4} + \frac{d_1}{4} - \frac{d_1}{[2]_{p,q}!}\right) w^2 + \cdots. \tag{2.10}$$

From (1.8) one can easily obtain

$$\frac{z(\mathcal{I}_n^m f(z))'}{(1-\alpha)z + \alpha \mathcal{I}_n^m f(z)} = 1 + (2-\alpha)me^{-m}a_2 z + [(\alpha^2 - 2\alpha)m^2 e^{-2m} a_2^2 + (3-\alpha)m^2 2^{n-1} e^{-m} a_3] z^2 + \cdots. \tag{2.11}$$

Similarly, using (1.3) in (1.8) and after simplification gives

$$\begin{aligned}\frac{w(\mathcal{I}_n^m g(w))'}{(1-\alpha)w + \alpha \mathcal{I}_n^m g(w)} = 1 - (2-\alpha)me^{-m}a_2 w + [(2a_2^2 - a_3)(3m^2 2^{n-1} e^{-m} - \alpha m^2 2^{n-1} e^{-m}) \\ +(\alpha^2 - 2\alpha)m^2 e^{-2m} a_2^2] w^2 + \cdots.\end{aligned} \tag{2.12}$$

Using (2.9) and (2.11) in (2.3) and then comparing the coefficients of like power terms on both sides we obtain

$$(2-\alpha)me^{-m}a_2 = \frac{c_1}{4}. \tag{2.13}$$

$$\frac{(3-\alpha)2^n m^2 e^{-m}}{2} a_3 - (2\alpha - \alpha^2)m^2 e^{-2m} a_2^2 = \frac{1}{4}\left(\frac{c_2}{4} + \frac{c_1}{4} - \frac{c_1}{[2]_{p,q}!}\right). \tag{2.14}$$

Similarly, making use of (2.10) and (2.12) in (2.4) and then comparing coefficients of w and w^2 we get

$$-(2-\alpha)me^{-m}a_2 = \frac{d_1}{4} \tag{2.15}$$

$$\frac{(3-\alpha)2^n m^2 e^{-m}}{2}(2a_2^2 - a_3) - (2\alpha - \alpha^2)m^2 e^{-2m} a_2^2 = \frac{1}{4}\left(\frac{d_2}{4} + \frac{d_1}{4} - \frac{d_1}{[2]_{p,q}!}\right) \tag{2.16}$$

From (2.13) and (2.15) one will get

$$c_1 = -d_1 \tag{2.17}$$

$$2(2-\alpha)^2 m^2 e^{-2m} a_2^2 = \frac{c_1^2 + d_1^2}{16}. \tag{2.18}$$

Further, adding (2.14)and (2.16) and making use of (2.17) in the resulting relation we get

$$m^2 e^{-m}\left(2^n(3-\alpha) - 2e^{-m}(2\alpha - \alpha^2)\right) a_2^2 = \frac{c_2 + d_2}{16},$$

which implies

$$a_2^2 = \frac{c_2 + d_2}{16 m^2 e^{-m}\left[2^n(3-\alpha) - 2e^{-m}(2\alpha - \alpha^2)\right]}. \tag{2.19}$$

Applying coefficient inequalities $|c_2| \le 2$ and $|d_2| \le 2$ (see [7]) in (2.19) we get the desire estimate (2.1) .

Next, in order to find the bound on $|a_3|$, subtracting (2.16) from (2.14) we obtain

$$m^2 e^{-m} 2^n (3-\alpha)(a_3 - a_2^2) = \frac{c_2 - d_2}{16} + \frac{c_1 - d_1}{16} - \frac{c_1 - d_1}{[2]_{p,q}!}. \tag{2.20}$$

Substituting the value of a_2^2 from (2.18) in (2.20) yields

$$a_3 = \frac{1}{m^2e^{-m}2^n(3-\alpha)}\left[\frac{2^n(3-\alpha)(c_1^2+d_1^2)}{32(2-\alpha)^2e^{-m}} + \frac{c_2-d_2}{16} + \frac{c_1-d_1}{16} - \frac{c_1-d_1}{[2]_{p,q}!}\right]. \tag{2.21}$$

Using well-known inequalities $|c_i| \leq 2$ and $|d_i| \leq 2,\ i = 1,2$ in (2.21) one will have the assertion (2.2) as stated in the theorem. This completes the proof of Theorem 1.

Taking $\alpha = 0$ in Theorem 1 we get the following result.
Corollary 2: Let $f \in \mathcal{S}_{\Sigma}^{m,n}(\gamma_{p,q})$. Then

$$|a_2| \leq \frac{1}{2m\sqrt{3\ 2^ne^{-m}}}, \quad |a_3| \leq \frac{1}{3m^22^ne^{-m}}\left[\frac{3\ 2^{n-4}}{e^{-m}} + \frac{1}{2} + \frac{4}{[2]_{p,q}!}\right].$$

Taking $\alpha = 1$ in Theorem 1 we get the following result.
Corollary 3: Let $f \in \mathcal{T}_{\Sigma}^{m,n}(\gamma_{p,q})$. Then

$$|a_2| \leq \frac{1}{2m\sqrt{e^{-m}|2^{n+1}-2e^{-m}|}}, \quad |a_3| \leq \frac{1}{2^{n+1}m^2e^{-m}}\left|\frac{2^{n-1}}{e^{-m}} + \frac{1}{2} + \frac{4}{[2]_{p,q}!}\right|.$$

Theorem 4: Let the function $f \in \mathcal{A}$ given by (1.2) be in the class $\mathcal{K}_{\sum,\alpha}^{m,n}(\gamma_{p,q})$. Then

$$|a_2| \leq \frac{1}{2m\sqrt{e^{-m}\,|3\ 2^n(3-\alpha) - 8e^{-m}(2\alpha-\alpha^2)|}}, \tag{2.22}$$

and

$$|a_3| \leq \frac{1}{3m^2e^{-m}2^n(3-\alpha)}\left(\frac{3\ 2^n(3-\alpha)}{16(2-\alpha)^2e^{-m}} + \frac{1}{2} + \frac{1}{[2]_{p,q}!}\right) \tag{2.23}$$

where $m > 0, n \geq 0,\ 0 \leq \alpha \leq 1$ and $0 < q < p \leq 1$.
Proof: Let $f \in \mathcal{K}_{\Sigma,\alpha}^{m,n}(\gamma_{p,q})$. Proceeding as in the line of Theorem 1 we have

$$\frac{z(I_n^mf(z))' + z^2(I_n^mf(z))''}{(1-\alpha)z + \alpha z(I_n^mf(z))'} = t(u(z)), \tag{2.24}$$

$$\frac{w(I_n^mg(w))' + w^2(I_n^mg(w))''}{(1-\alpha)w + \alpha w(I_n^mg(w))'} = t(v(w)), \quad (z,\ w \in \mathbb{U}). \tag{2.25}$$

A simple calculation shows

$$\begin{aligned}\frac{z(I_n^mf(z))' + z^2(I_n^mf(z))''}{(1-\alpha)z + \alpha z(I_n^mf(z))'} &= 1 + 2me^{-m}(2-\alpha)a_2z + [4m^2e^{-2m}(\alpha^2-2\alpha)a_2^2 \\ &\quad +3m^22^{n-1}e^{-m}(3-\alpha)a_3]z^2 + \cdots,\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}\frac{w(I_n^mg(w))' + w^2(I_n^mg(w))''}{(1-\alpha)w + \alpha w(I_n^mg(w))''} &= 1 - 2me^{-m}(2-\alpha)a_2w + [4m^2e^{-2m}(\alpha^2-2\alpha)a_2^2 \\ &\quad +3(2a_2^2-a_3)m^2e^{-m}2^n(3-\alpha)]w^2 + \cdots.\end{aligned} \tag{2.27}$$

Using (2.9) and (2.26) in (2.24) and comparing the coefficients of like power terms on both sides we get

$$2me^{-m}(2-\alpha)a_2 = \frac{c_1}{4}, \tag{2.28}$$

and

$$4m^2e^{-2m}(\alpha^2-2\alpha)a_2^2 + 3m^2e^{-m}2^{n-1}(3-\alpha)a_3 = \frac{1}{4}\left[\frac{c_2}{4} + \frac{c_1}{4} - \frac{c_1}{[2]_{p,q}!}\right]. \tag{2.29}$$

In the similar manner, using (2.9) and (2.27) in (2.25) and comparing the coefficients of w and w^2 on both sides we obtain

$$-2me^{-m}(2-\alpha)a_2 = d_1, \tag{2.30}$$

$$4m^2e^{-2m}(\alpha^2-2\alpha)a_2^2 + 3(2a_2^2-a_3)m^2e^{-m}2^{n-1}(3-\alpha) = \frac{1}{4}\left[\frac{d_2}{4}+\frac{d_1}{4}-\frac{d_1}{[2]_{p,q}!}\right]. \tag{2.31}$$

From (2.28) and (2.30) it follows that

$$c_1 = -d_1. \tag{2.32}$$

Also,

$$8m^2(2-\alpha)^2e^{-2m}a_2^2 = \frac{c_1^2+d_1^2}{16}. \tag{2.33}$$

Further, adding (2.29) and (2.31) and using relation(2.32) in the resulting equation yields

$$2m^2e^{-m}[32^{n-1}(3-\alpha) - 4e^{-m}(2\alpha-\alpha^2)]a_2^2 = \frac{c_2+d_2}{16}.$$

This implies

$$a_2^2 = \frac{c_2+d_2}{32m^2e^{-m}[3\ 2^{n-1}(3-\alpha) - 4e^{-m}(2\alpha-\alpha^2)]} \tag{2.34}$$

Applying the coefficient bounds to c_2 and d_2 in (2.34) gives the desire estimate as stated in (45). In order to determine the coefficient bounds for a_3 we may proceed as follows: Subtracting (2.31) from (2.29) we get

$$6m^2e^{-m}2^{n-1}(3-\alpha)a_3 - 6m^2e^{-m}2^{n-1}(3-\alpha)a_2^2 = \frac{1}{16}(c_2-d_2) + \frac{c_1-d_1}{16} - \frac{c_1-d_1}{4[2]_{p,q}!}. \tag{2.35}$$

Putting the value of a_2^2 from (56) in (2.35) we obtain

$$a_3 = \frac{1}{3m^22^ne^{-m}(3-\alpha)}\left[\frac{3\ 2^{n-1}(3-\alpha)}{64(2-\alpha)^2e^{-m}}(c_1^2+d_1^2) + \frac{c_2-d_2}{16} + \frac{c_1-d_1}{16} - \frac{c_1-d_1}{4[2]_{p,q}!}\right]. \tag{2.36}$$

Applying triangle inequality to both sides of (2.36) and using well-known bounds for $c_i,\ d_i,\ i=1,2$ we get the desire estimate. Thus the proof of Theorem 4 is completed.
Letting $\alpha = 0$ in Theorem 4 we get the following result.
Corollary 5 Let $f \in \mathcal{K}_\Sigma^{m,n}(\gamma_{p,q})$. Then

$$|a_2| \le \frac{1}{6m\sqrt{2^n\ e^{-m}}}$$

and

$$|a_3| \le \frac{1}{9m^22^ne^{-m}}\left[\frac{9\ 2^{n-1}}{32e^{-m}} + \frac{1}{2} + \frac{1}{[2]_{p,q}!}\right].$$

Corollary 6: Let $f \in \mathcal{R}_\Sigma^{m,n}(\gamma_{p,q})$. Then

$$|a_2| \le \frac{1}{2m\sqrt{2e^{-m}|3\ 2^n - 4e^{-m}|}},$$

and

$$|a_3| \le \frac{1}{3m^22^{n+1}e^{-m}}\left[\frac{3\ 2^n}{8e^{-m}} + \frac{1}{2} + \frac{1}{[2]_{p,q}!}\right].$$

Conclusion: A good amount of literature exists for finding the bound of initial coefficients of Taylor-Maclaurin series of the function $f(z)$ of the form (1.2) for different subclasses of analytic univalent and bi-univalent functions associated with various region or functions by means of subordination. In the present paper, we obtain estimates of $|a_2|$ and $|a_3|$ for the class $\mathcal{S}_{\Sigma,\alpha}^{m,n}(\gamma_{p,q})$ and $\mathcal{K}_{\Sigma,\alpha}^{m,n}(\gamma_{p,q})$ associated with normalized Touchard polynomial related to (p,q)-analogue of sigmoid functions.

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