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Controllability of Sobolev type stochastic differential equations driven by fBm with non-instantaneous impulses

S. Madhuri, G.V.S.R. Deekshitulu*

Department of Mathematics, UCEK, JNTUK, Kakinada, A.P., India

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Abstract

In this paper, we examine the controllability results for a class of multi-valued Sobolev type neutral stochastic differential equations that are steered by fractional Brownian motion B_t^H with non-instantaneous impulses for $H \in (\frac{1}{2}, 1)$ by assuming the controllability of the linear system. The results are obtained by utilizing the fixed-point theorem for multi-valued operators and stochastic analysis. At last, an example is given to represent the results of the theorem.

Keywords: Controllability, fractional Brownian motion, fixed point theorem, non-instantaneous impulses, Sobolev type stochastic differential equations
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1 Introduction

Controllability plays a vigorous role in each of deterministic dynamical systems as well as stochastic control theory. Controllability permits one to move the dynamic system from an approximate starting state to the last state by utilizing the collection of admissible controls. Additionally, the deterministic models frequently fluctuate because of natural clamor which is irregular. In this manner to possess better execution within the systems, it is essential to consider stochastic systems. In particular, stochastic differential equations(SDE's) are essentially used in stock markets, noise in communication networks, control engineering, and physical sciences etc., [2, 6, 27]. The qualitative and quantitative properties of nonlinear stochastic differential systems like existence, stability and controllability results have been considered in [13, 20, 21, 23, 31, 34].

fractional Brownian motion(fBm) is a collection of Gaussian processes that appears to have extensive physical applicability. This kind of process was acquainted by Kolmogorov [19] and later concentrated by Mandelbrot VanNess [22], where a stochastic integral of fBm is represented by using standard Brownian motion. Due to the fact that the fBm represented by B_t^H is certainly not a semi martingale if $H \neq \frac{1}{2}$ [5], we can no longer use traditional tools of Itô theory in studying stochastic calculus has driven by fBm.

Presently many real-world processes, as well as phenomena exist to describe extreme changes, sudden discontinuous jumps in the system, known as impulses. The impulses are said to be non-instantaneous if they start suddenly at time t_i and its motion proceed on a finite time interval. In such a phenomenon, it is fascinating and vital to have a look at the effect of SDE's with non-instantaneous impulses. Recently, some works have been performed on the existence of solutions of SDE's with non-instantaneous impulses [2, 18, 35].

*Corresponding author
Email addresses: sunkavilli.madhuri@gmail.com (S. Madhuri), dixitgvsr@hotmail.com (G.V.S.R. Deekshitulu)

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Additionally, Sobolev type equations are mainly used in various applications like flow of fluid through fissures in rocks [4], thermodynamics [8] and propagation of long waves of small amplitude [12]. Sobolev type differential equations play an important role in the structure of operator differential equations where an operator coefficient is multiplied by the highest derivative [25]. Many authors or scientists study extensively on Sobolev type differential equations. The reader may refer [1, 3, 17]. The authors in [24, 32] studied, approximate controllability results for 'neutral integro-differential inclusions of Sobolev-type with infinite delay' and 'non-densely defined Sobolev-type Hilfer fractional neutral delay differential system'. In [33], the authors studied approximate controllability results for second-order Sobolev-type impulsive neutral differential evolution inclusions with infinite delay. Recently there are several contributions on various properties like existence, stability and controllability for SDE's of Sobolev type [7, 11, 14, 15, 23, 26]. Stochastic Sobolev type differential equations play an important role in the theory of control of dynamical systems, when the controlled system is pecularized by a Sobolev type differential equation [10, 11, 23, 29, 30].

The inspiration of the present work is derived from the above literature survey as much of the work is not available on controllability of Sobolev type SDE's driven by fBm with non-instantaneous impulses by utilizing fixed point method. The fixed point approach is most powerful to find out about existence and controllability of differential systems. So, in this paper we study the controllability problem by utilizing Dhage [9] fixed point theorem for Sobolev type stochastic differential equations driven by fractional Brownian motion(fBm) with non-instantaneous impulses in the form of

$$d[\mathcal{L}x(t)] = [Ax(t) + Bu(t) + F(t, x_t)]dt + G(t, x_t)dB_t^H, \quad t \in (s_i, t_{i+1}] = J_i \text{ for } i = 0, 1, ...m, \tag{1.1}$$
$$x(t) = h_i(t, x_t), \quad t \in (t_i, s_i], i = 1, 2, ..., m, \tag{1.2}$$
$$x(t) = \varphi(t) \in \mathcal{B}_h, \text{ for a.e. } t \in (-\infty, 0] = J_0. \tag{1.3}$$

in a real separable Hilbert space $\mathbb{H}$ with inner product $\langle ., . \rangle$ and norm $\|.\|$. We consider the operators $A : \mathcal{D}(A) \to \mathbb{H}$ and $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to \mathbb{H}$ satisfy the following conditions:

i. A and $\mathcal{L}$ are closed linear operators.

ii. $\mathcal{D}(\mathcal{L}) \subset \mathcal{D}(A)$ and $\mathcal{L}$ is bijective.

iii. $\mathcal{L}^{-1} : \mathbb{H} \to \mathcal{D}(\mathbb{H})$ is compact.

From the assumptions (i)-(ii) and closed graph theorem it is clear that $A\mathcal{L}^{-1} : \mathbb{H} \to \mathbb{H}$ is bounded. Consequently $-A\mathcal{L}^{-1}$ generates a semi group $\{\mathbb{S}(t), t \geq 0\}$ in $\mathbb{H}$, with $\|\mathcal{L}^{-1}\|^2 = \tilde{M}_L$ and $\|\mathcal{L}\|^2 = M_L$.

A concise framework of the paper is as follows. In section 2, a few notations and necessary preliminaries are given. Section 3, deals with the controllability result of (1.1)-(1.3) by utilizing fixed point theorem for multi-valued maps given by Dhage [9]. At last, section 4, deals with the example to illustrate the main result.

2 Preliminaries

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ represents complete probability space with natural filtration $\{\mathfrak{F}_t, t \geq 0\}$. We use the following notations through out the paper.

- $\mathbb{H}, \mathbb{K}$ represents separable Hilbert space with norm $\|.\|_{\mathbb{H}}, \|.\|_{\mathbb{K}}$.
- $L_Q^0 := L_Q^0(\mathbb{K}, \mathbb{H})$ represents $Q-$Hilbert-Schmidt operators from $\mathbb{K}$ to $\mathbb{H}$.
- $L^2(\mathfrak{F}, \mathbb{H})$ be the Hilbert space of all strongly $\mathfrak{F}-$ measurable $\mathbb{H}$ valued random variables satisfying $\mathbb{E}\|x\|_{\mathbb{H}}^2 < \infty$.
- $C(J, L^2(\mathfrak{F}, \mathbb{H}))$ be the Banach space of all continuous maps from $J = [0, T]$ into $L^2(\mathfrak{F}, \mathbb{H})$.
- $L^2(J, U)$ be the Hilbert space of admissible control functions for a separable Hilbert space U.

Definition 2.1. [27] Let $\{\beta^H(t) : t \geqslant 0\}$ represents a two sided one dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$ defined on $(\Omega, \mathfrak{F}, \mathbb{P})$ with covariance function

$$R_H(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - \mid s - t \mid^{2H}).$$

Further, β^H is represented by using Wiener process $W = \{W(t) : t \in [0,T]\}$ as follows :

$$\beta_H(t) = \int_0^t K_H(t,s)dW(s),$$

where $K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (y-s)^{H-\frac{3}{2}} y^{H-\frac{1}{2}} dy \ for \ t > s$, with $c_H = \sqrt{\frac{H(2H-1)}{\mathfrak{B}(2-2H, H-\frac{1}{2})}}$, where $\mathfrak{B}(.)$ represents Beta function.

Lemma 2.2. [6] Let $Q \in L(\mathbb{K}, \mathbb{H})$ be a non negative self adjoint trace class operator. Assume that $B_Q^H(t)$ is a $\mathbb{K}$ valued fBm with covariance operator Q. Then the stochastic process $B_Q^H(t)$ given as:

$$B_Q^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) Q^{\frac{1}{2}} e_n, t \geq 0,$$

where $\{e_n\}_{n\in N}$ is a complete ortho normal basis in $\mathbb{K}$.
Let $\phi : [0,T] \to L_Q^0(\mathbb{K}, \mathbb{H})$ be such that

$$\sum_{n=1}^{\infty} \parallel K_H^*(\phi Q^{\frac{1}{2}} e_n) \parallel_{L^{\frac{1}{H}}([0,T];\mathbb{H})} < \infty. \tag{2.1}$$

Definition 2.3. [6] Let $\varphi : [0,T] \to L_Q^0(\mathbb{K}, \mathbb{H})$. Then for $t \geq 0$, the stochastic integral with respect to fBm B_Q^H is defined, as follows:

$$\int_0^t \varphi(s) dB_Q^H(s) := \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{\frac{1}{2}} e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t K_H^*(\varphi Q^{\frac{1}{2}} e_n)(s) dW(s).$$

It can be observed that, if

$$\sum_{n=1}^{\infty} \parallel \varphi Q^{\frac{1}{2}} e_n) \parallel_{L^{\frac{1}{H}}([0,T];H)} < \infty, \tag{2.2}$$

then (2.1)holds.

Lemma 2.4. [2] If $\varphi(s) : [0,T] \to L_Q^0(\mathbb{K}, \mathbb{H})$ be such that $\int_0^T \|\varphi(s)\|^2_{L_Q^0(\mathbb{K},\mathbb{H})} ds < \infty$, then for any $t_1, t_2 \in [0,T]$ we have

$$\mathbb{E}\| \int_{t_1}^{t_2} \varphi(s) dB_s^H \|^2 \leq 2H(t_2 - t_1)^{2H-1} \int_{t_1}^{t_2} \|\varphi(s)\|^2_{L_Q^0(\mathbb{K},\mathbb{H})} ds.$$

Now, we present the abstract phase space $\mathcal{B}_h$ as follows:

$$\mathcal{B}_h = \{\psi : (-\infty, 0] \to \mathbb{H} \text{ for any } a > 0, (\mathbb{E}\|\psi(\theta)\|^2)^{\frac{1}{2}} \text{ is bounded and measurable}$$
$$\text{function on } [-a, 0] \text{ with } \psi(0) = 0 \text{ and } \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} (\mathbb{E}\|\psi(\theta)\|^2)^{\frac{1}{2}} ds < \infty\}.$$

If $\mathcal{B}_h$ is equipped with norm, $\|\psi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} (\mathbb{E}\|\psi(\theta)\|^2)^{\frac{1}{2}} ds, \quad \psi \in \mathcal{B}_h,$
then $(\mathcal{B}_h, \|\psi\|_{\mathcal{B}_h})$ is a Banach space.
Consider

$$\mathcal{B}_T = \{x : (-\infty, T] \to \mathbb{H} \text{ such that } x|J_i \in C(J_i, \mathbb{H}), \text{ there exist}$$
$$x(t_i^+) \text{ and } x(t_i^-) \text{ with } x(t_i) = x(t_i^-), x_0 = \varphi \in \mathcal{B}_h, i = 1, 2...m\},$$

where $x|J_i$ is restriction of x over $J_i = (t_i, t_{i+1}], i = 0, 1, 2, ...m$.
Let $\|.\|_{\mathcal{B}_T}$ represents a semi norm in $\mathcal{B}_T$, defined by

$$\|x\|_{\mathcal{B}_T} = \|x_0\|_{\mathcal{B}_h} + \sup_{0\le t\le T}(\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}}, x \in \mathcal{B}_T.$$

Lemma 2.5. [28] For $t \in [0, T], x_t \in \mathcal{B}_h$, let us assume $x \in \mathcal{B}_T$, then,

$$l(\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}} \le \|x_t\|_{\mathcal{B}_h} \le \|x_0\|_{\mathcal{B}_h} + l \sup_{0\le t\le T}(\mathbb{E}\|x(t)\|^2)^{\frac{1}{2}}, \text{ where } l = \int_{-\infty}^{0} h(t)dt < \infty.$$

We consider following notations for our convenience.

$$P_{cl}(\mathbb{H}) = \{x \in P(\mathbb{H}) : \ x \text{ is closed}\}, P_{bd}(\mathbb{H}) = \{x \in P(\mathbb{H}) : \ x \text{ is bounded}\},$$
$$P_{cv}(\mathbb{H}) = \{x \in P(\mathbb{H}) : \ x \text{ is convex}\}, P_{cp}(\mathbb{H}) = \{x \in P(\mathbb{H}) : x \text{ is compact}\}.$$

Definition 2.6. [34] Let $\chi : \mathbb{H} \to P(\mathbb{H})$ be a multi-valued map then

1. χ is convex(closed) valued if $\chi(x)$ is convex(closed) for all $x \in \mathbb{H}$.
2. if $\chi(D) = \cup_{x\in D}\chi(x)$ is bounded in $\mathbb{H}$ for all $D \in P_{bd}(\mathbb{H})$ then χ is bounded on bounded sets that is $\sup_{x\in D}\{\sup\{||y|| : y \in \chi(x)\}\} < \infty$.
3. χ is upper semi continuous (u.s.c.) on $\mathbb{H}$, if for each $x \in \mathbb{H}$, the set $\chi(x)$ is a non-empty, closed subset of $\mathbb{H}$ and if for each open set U of $\mathbb{H}$ containing $\chi(x)$, there exists open neighborhood V of x such that $\chi(V) \subseteq U$.
4. χ is said to be completely continuous if $\chi(D)$ is relatively compact for every bounded subset $D \in P(\mathbb{H})$.
5. If the multi-valued map χ is completely continuous with non empty compact values, then χ is u.s.c. if and only if χ has a closed graph, i.e., $x_n \to x_*, y_n \to y_*, y_n \in \chi(x_n)$ imply $y_* \in \chi(x_*)$.
6. χ has a fixed point if there is $x \in \mathbb{H}$ such that $x \in \chi(x)$.

Definition 2.7. [29] The multi-valued map $G : J \times \mathcal{B}_h \to P(\mathbb{H})$ is L^2-Carath$\acute{e}$odary if

i. $t \mapsto G(t, \upsilon)$ is measurable for each $\upsilon \in \mathcal{B}_h$;

ii. $\upsilon \mapsto G(t, \upsilon)$ is u.s.c. for $t \in J$;

iii. for each $q > 0$, there exists $g_q \in L^1(J, \mathbb{R}_+)$ such that

$$\|G(t, \upsilon)\|^2 := \sup_{g\in G(t,\upsilon)} \mathbb{E}\|g\|^2 \le h_q(t), \text{ for all } \|\upsilon\|^2_{\mathcal{B}_h} \le q \text{ and } t \in J.$$

To prove our main result the following Lemma given in [16] plays a vital role.

Lemma 2.8. Let J be a compact interval and G be an L^2- Carathéodary multi-valued map with $S_{G,x} = \{g \in L^2(L(\mathbb{K}, \mathbb{H})) : g(t) \in G(t, x_t), t \in J\}$ is non empty. Let $\Gamma : L^2(J, \mathbb{H}) \to C(J, \mathbb{H})$ be a linear continuous mapping then

$$\Gamma o S_G : C(J, \mathbb{H}) \to P_{cp,cv}(\mathbb{H}), x \to (\Gamma o S_G)(x) := \Gamma(S_{G,x}),$$

is a closed graph operator in $C(J, \mathbb{H}) \times C(J, \mathbb{H})$.

Now, we recall fixed point theorem given by Dhage [9].

Theorem 2.9. Let $\Phi_1 : \mathbb{H} \to P_{cl,cv,bd}(\mathbb{H})$ and $\Phi_2 : \mathbb{H} \to P_{cp,cv}(\mathbb{H})$ be two multi-valued operators satisfying

a. Φ_1 is a contraction and

b. Φ_2 is completely continuous.

Then, either

i. the operator equation $x = \Phi_1 x + \Phi_2 x$ has a solution, or

ii. the set $\mathcal{G} = \{x \in \mathbb{H} : x \in \lambda\Phi_1 x + \lambda\Phi_2 x\}$ is unbounded for $\lambda \in (0,1)$.

Definition 2.10. An $\mathfrak{F}_t-$adapted stochastic process $x : (-\infty, T] \to \mathbb{H}$ is called a mild solution of (1.1)-(1.3) if $x_0 = \varphi \in \mathcal{B}_h, u(.) \in L^2_{\mathfrak{F}}(J, U)$ there exists a function $g(t) \in G(t, x(t))$ such that

$$x(t) = \begin{cases} \mathcal{L}^{-1}\mathbb{S}(t)[\mathcal{L}\varphi(0)] + \int\limits_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)Bu(s)ds \\ + \int\limits_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, x_s)ds + \int\limits_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB^H_s, \quad t \in [0, t_1] \\ \mathcal{L}^{-1}\mathbb{S}(t-s_i)[\mathcal{L}h_i(s_i, x_{s_i})] + \int\limits_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)Bu(s)ds \\ + \int\limits_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, x_s)ds + \int\limits_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB^H_s, \quad t \in (s_i, t_{i+1}]. \end{cases} \tag{2.3}$$

Definition 2.11. [18] The system (1.1)-(1.3) is said to be controllable on the interval J, if for every continuous initial function $\varphi \in \mathcal{B}_h$, there exist a control $u \in L^2(J, U)$, such that the mild solution $x(t)$ of (1.1)-(1.3) satisfies $x(T) = x_1$. In order to investigate controllability of non linear system (1.1)-(1.3), let us assume that

$$\begin{aligned} d[\mathcal{L}x(t)] &= [Ax(t) + Bu(t)]dt, \quad t \in J \\ x(0) &= x_0 = \varphi. \end{aligned} \tag{2.4}$$

is controllable on the interval J.

3 Main results

Here we prove the controllability of (1.1)-(1.3) by imposing following assumptions.

H1. $-A\mathcal{L}^{-1}$ is the infinitesimal generator of $\{\mathbb{S}(t)\}_{t\geq 0}$ in $\mathbb{H}$ and there exists a constant M such that $\|\mathbb{S}(t)\|^2 \leq M$, for all $t \geq 0$.

H2. The function $h_i : (t_i, s_i] \times \mathcal{B}_h \to \mathbb{H},\ i = 1, 2...m$ are continuous and there exist $\tilde{M}_{h_i}, M_{h_i} > 0,\ i = 1, 2...m$, such that

$$\mathbb{E}||h_i(t, x) - h_i(t, y)||^2_{\mathbb{H}} \leq M_{h_i}||x - y||^2_{\mathcal{B}_h},\ t \in (t_i, s_i], x, y \in \mathcal{B}_h,$$

$$\mathbb{E}||h_i(t, x)||^2_{\mathbb{H}} \leq \tilde{M}_{h_i}(||x||^2_{\mathcal{B}_h} + 1),\ t \in (t_i, s_i], x \in \mathcal{B}_h.$$

H3. The function F is completely continuous and there exist constants C_1, C_2, M_F such that

$$\begin{aligned} \mathbb{E}||F(t, x)||^2 &\leq C_1\mathbb{E}||x||^2_{\mathcal{B}_h} + C_2,\ \text{for every } t \in J, x \in \mathcal{B}_h, \\ \mathbb{E}||F(t, x) - F(t, y)||^2 &\leq M_F||x - y||^2_{\mathcal{B}_h},\ \text{for } t \in J, x, y \in \mathcal{B}_h. \end{aligned}$$

H4. The multi-valued map $G : J \times \mathcal{B}_h \to P(\mathbb{H})$ is an L^2 carathéodary function and satisfies the following conditions:

i. There exists a constant M_G such that

$$\mathbb{E}||G(t, x_1) - G(t, x_2)||^2 \leq M_G||x_1 - x_2||^2_{\mathcal{B}_h},\ \text{for } t \in J, x_1, x_2 \in \mathbb{H}.$$

ii. There exists an integrable function $p : J \to [0, \infty)$ such that

$$\mathbb{E}||G(t,x)||^2 = \sup_{\upsilon \in G(t,x)} ||\upsilon||^2 \le p(t)\Theta(||\upsilon||^2_{\mathcal{B}_h}), t \in J, \upsilon \in \mathcal{B}_h,$$

where $\Theta : \mathbf{R}_+ \to (0, \infty)$ is a continuous nondecreasing function.

H5. [18] The linear operator $W : L^2((s_i, t_{i+1}], U) \to L^2(\Omega, \mathbb{H})$ defined by

$$W_{s_i}^{t_{i+1}} = \int_{s_i}^{t_{i+1}} \mathcal{L}^{-1}T(t_{i+1} - s)BB^*\mathcal{L}^{-1}T^*(t_{i+1} - s)ds, i = 0, 1, 2...m, \tag{3.1}$$

has an inverse operator $(W_{s_i}^{t_{i+1}})^{-1}$ which takes values in $L^2((s_i, t_{i+1}], U)/KerW$, where $KerW = \{x \in L^2(s_i, t_{i+1}, U), \ Wx = 0\}$ and there exist $M_B > 0, \delta > 0$ such that $\|B\|^2 = M_B$ and $||(W_{s_i}^{t_{i+1}})^{-1}||^2 \le \frac{1}{\delta}$.

Theorem 3.1. Assume that the assumptions (H1)-(H5) hold. If $L_0 = \max\{L_1, L_2, L_3\} < 1$, then the system (1.1)-(1.3) is controllable on $J = [0, T]$.

Proof . To prove our main result we introduce two operators $\bar{\Phi}_1$ and $\bar{\Phi}_2$ as follows:
Consider $\Phi : \mathcal{B}_T \to P(\mathcal{B}_T)$ defined by

$$\Phi(x)(t) = \begin{cases} 0, t \in (-\infty, 0] \\ \mathcal{L}^{-1}\mathbb{S}(t)[\mathcal{L}\varphi(0)] + \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)Bu(s)ds \\ + \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, x_s)ds + \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB_s^H, \quad t \in [0, t_1] \\ \mathcal{L}^{-1}\mathbb{S}(t-s_i)[\mathcal{L}h_i(s_i, x_{s_i})] + \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)Bu(s)ds \\ + \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, x_s)ds + \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB_s^H, \quad t \in (s_i, t_{i+1}]. \end{cases}$$

Now we prove that fixed point exists for the operator Φ . For $\varphi \in \mathcal{B}_h$, define

$$\hat{\varphi}(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0] \\ \mathcal{L}^{-1}\mathbb{S}(t)\mathcal{L}\varphi(0), & t \in [0, T], \end{cases}$$

then $\hat{\varphi}(t) \in \mathcal{B}_T$. Set $x(t) = y(t) + \hat{\varphi}(t); -\infty < t \le T$.
There fore x satisfies (2.3) if and only if y satisfies $y_0 = 0$ and

$$y(t) = \begin{cases} \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)Bu_{y+\hat{\varphi}}(s)ds + \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s + \hat{\varphi}_s)ds \\ + \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB_s^H, & t \in [0, t_1] \\ h_i(t, y_t + \hat{\varphi}_t), & t \in (t_i, s_i] \\ \mathcal{L}^{-1}\mathbb{S}(t-s_i)[\mathcal{L}h_i(s_i, y_{s_i} + \hat{\varphi}_{s_i})] + \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)Bu_{y+\hat{\varphi}}(s)ds \\ + \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s + \hat{\varphi}_s)ds + \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB_s^H, & t \in (s_i, t_{i+1}]. \end{cases}$$

Let $\mathcal{B}_T^0 = \{y \in \mathcal{B}_T : y_0 = 0 \in \mathcal{B}_h\}$; for any $y \in \mathcal{B}_T^0$,

$$||y||_{\mathcal{B}_T} = ||y_0||_{\mathcal{B}_h} + \sup_{0 \le s \le T} \mathbb{E}(||y(s)||^2)^{\frac{1}{2}} = \sup_{0 \le s \le T} (\mathbb{E}||y(s)||^2)^{\frac{1}{2}}.$$

Thus $(\mathcal{B}_T^0, ||.||_T)$ is a Banach space.
Set $\mathcal{B}_q = \{y \in \mathcal{B}_T^0 : \mathbb{E}||y||^2 \le q\}$ for some $q \ge 0$,

then $\mathcal{B}_q \subseteq \mathcal{B}_T^0$ is uniformly bounded.
From Lemma (3.1), for $y \in \mathcal{B}_q$, we have

$$\begin{aligned}||y_t + \hat{\varphi}_t||^2_{\mathcal{B}_h} &\leq 2(||y_t||^2_{\mathcal{B}_h} + ||\hat{\varphi}_t||^2_{\mathcal{B}_h})\\ &\leq 2l^2(q^2 + \tilde{M}_L M M_L \mathbb{E}\|\varphi(0)\|^2) + 2\|\varphi\|^2_{\mathcal{B}_h} = q^*.\end{aligned}$$

Define the multi-valued map $\bar{\Phi} : \mathcal{B}_T^0 \to P(\mathcal{B}_T^0)$ by

$$\bar{\Phi}y(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)Bu_{y+\hat{\varphi}}(s)ds + \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s + \hat{\varphi}_s)ds & \\ + \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB^H_s, & t \in [0, t_1] \\ h_i(t, y_t + \hat{\varphi}_t), & t \in (t_i, s_i] \\ \mathcal{L}^{-1}\mathbb{S}(t-s_i)[\mathcal{L}h_i(s_i, z_{s_i} + \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)Bu_{y+\hat{\varphi}}(s)ds & \\ + \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s + \hat{\varphi}_s)ds + \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB^H_s, & t \in (s_i, t_{i+1}]. \end{cases}$$

Now, we split $\bar{\Phi}$ as $\bar{\Phi} = \bar{\Phi}_1 + \bar{\Phi}_2$, where

$$\bar{\Phi}_1 y(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ \int_0^t L^{-1}\mathbb{S}(t-s)Bu_{y+\hat{\varphi}}(s)ds, & t \in [0, t_1] \\ h_i(t, y_t + \hat{\varphi}_t), & t \in (t_i, s_i] \\ \mathcal{L}^{-1}\mathbb{S}(t-s_i)\mathcal{L}h_i(s_i, y_{s_i} + \hat{\varphi}_{s_i}) + \int_{s_i}^t L^{-1}\mathbb{S}(t-s)Bu_{y+\hat{\varphi}}(s)ds & t \in (s_i, t_{i+1}] \end{cases}$$

$$\bar{\Phi}_2 y(t) = \begin{cases} \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s + \hat{\varphi}_s)ds + \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB^H_s, & t \in [0, t_1] \\ 0, & t \in (t_i, s_i] \\ \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s + \hat{\varphi}_s)ds + \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB^H_s, & t \in (s_i, t_{i+1}]. \end{cases}$$

Now to apply Theorem (2.9) we prove $\bar{\Phi}_1$ is contraction and $\bar{\Phi}_2$ is completely continuous.
To prove $\bar{\Phi}_1$ is contraction, let us define the control process

$$u_x(t) = \begin{cases} B^*\mathcal{L}^{-1}\mathbb{S}^*(t_1 - t)(W_0^{t_1})^{-1}\Big[x_{t_1} - \mathcal{L}^{-1}\mathbb{S}(t)\mathcal{L}\varphi(0) - \int_0^{t_1} \mathcal{L}^{-1}\mathbb{S}(t_1 - s)F(s, x_s)ds & \\ - \int_0^{t_1} \mathcal{L}^{-1}\mathbb{S}(t_1 - s)g(s)dB^H_s\Big], & t \in [0, t_1] \\ 0, & t \in (t_i, s_i] \\ B^*\mathcal{L}^{-1}\mathbb{S}^*(t_{i+1} - t)(W_{s_i}^{t_{i+1}})^{-1}\Big[x_{t_{i+1}} - \mathcal{L}^{-1}\mathbb{S}(t_{i+1} - s_i)[\mathcal{L}\varphi(0)] & \\ - \int_{s_i}^{t_{i+1}} \mathcal{L}^{-1}\mathbb{S}(t_{i+1} - s)F(s, x_s)ds - \int_{s_i}^{t_{i+1}} \mathcal{L}^{-1}\mathbb{S}(t_{i+1} - s)g(s)dB^H_s\Big], & t \in (s_i, t_{i+1}]. \end{cases} \tag{3.2}$$

Case 1: For $t \in [0, t_1]$ and for $y^*, y^{**} \in \mathcal{B}_T^0$, consider

$$\begin{aligned}\mathbb{E}||\bar{\Phi}_1 y^*(t) - \bar{\Phi}_1 y^{**}(t)||^2 &\leq \mathbb{E}||\int_0^t L^{-1}S(t-s)B[u_{y^*+\hat{\varphi}}(s) - u_{y^{**}+\hat{\varphi}}(s)]ds||^2 \\ &\leq \tilde{M}_L M M_B t_1 \int_0^t \mathbb{E}||u_{y^*+\hat{\varphi}}(s) - u_{y^{**}+\hat{\varphi}}(s)ds||^2,\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\mathbb{E}||u_{y^*+\hat{\varphi}}(s)-u_{y^{**}+\hat{\varphi}}(s)||^2 &\leq \frac{2\tilde{M}_B\tilde{M}_LM}{\delta}\Big(4\tilde{M}_LMM_Ft_1^2+8Ht_1^{2H-1}\tilde{M}_LM\Big)||y^*-y^{**}||^2\\ &\leq L_{u_1}||y^*-y^{**}||^2,\end{aligned} \tag{3.4}$$

where $L_{u_1}=\frac{2\tilde{M}_B\tilde{M}_L^2M^2}{\delta}\Big(4M_Ft_1^2+8Ht_1^{2H-1}\Big)$.

Therefore from (3.3),(3.4) we get

$$\begin{aligned}\mathbb{E}||\bar{\Phi}_1y^*(t)-\bar{\Phi}_1y^{**}(t)||^2 &\leq \tilde{M}_LMM_Bt_1^2L_{u_1}||y^*-y^{**}||^2\\ &\leq L_1||y^*-y^{**}||^2,\end{aligned}$$

where $L_1=\tilde{M}_LMM_Bt_1^2L_{u_1}$.

Case 2: For $t\in(t_i,s_i], i=1,2,...m$ and for $y^*,y^{**}\in\mathcal{B}_T^0$, consider

$$\begin{aligned}\mathbb{E}||\bar{\Phi}_1y^*(t)-\bar{\Phi}_1y^{**}(t)||^2 &\leq 2l^2M_{h_i}\mathbb{E}||y^*-y^{**}||^2\\ &\leq L_2||y^*-y^{**}||^2\end{aligned} \tag{3.5}$$

where $L_2=\max\limits_{i=1,2,...m}\big(2l^2M_{h_i}\big)$.

Case 3: For $t\in(s_i,t_{i+1}], i=0,1,2,...m$ and for $y^*,y^{**}\in\mathcal{B}_T^0$, consider

$$\begin{aligned}\mathbb{E}||\bar{\Phi}_1y^*(t)-\bar{\Phi}_1y^{**}(t)||^2 &= \mathbb{E}||L^{-1}S(t-s_i)L[h_i(s_i,y^*_{s_i}+\hat{\varphi}_{s_i})-h_i(s_i,y^{**}_{s_i}+\hat{\varphi}_{s_i})]\\ &\quad+\int_{s_i}^{t}L^{-1}S(t-s)B[u_{y^*+\hat{\varphi}}(s)-u_{y^{**}+\hat{\varphi}}(s)ds]||^2\\ &\leq 4\tilde{M}_LMM_L\mathbb{E}||h_i(s_i,y^*_{s_i}+\hat{\varphi}_{s_i})-h_i(s_i,y^{**}_{s_i}+\hat{\varphi}_{s_i})||^2\\ &\quad+4\tilde{M}_LMM_B(t_{i+1}-s_i)\int_{s_i}^{t}\mathbb{E}||u_{y^*+\hat{\varphi}}(s)-u_{y^{**}+\hat{\varphi}}(s)ds||^2,\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}\mathbb{E}||u_{y^*+\hat{\varphi}}(s)-u_{y^{**}+\hat{\varphi}}(s)||^2 &\leq \frac{2\tilde{M}_B\tilde{M}_LM}{\delta}\Big(4\tilde{M}_LMM_F(t_{i+1}-s_i)^2||y^*-y^{**}||^2\\ &\quad+8H(t_{i+1}-s_i)^{2H-1}\tilde{M}_LM||y^*-y^{**}||^2\Big)\\ &\leq L_{u_2}||y^*-y^{**}||^2,\end{aligned} \tag{3.7}$$

where

$$L_{u_2}=\frac{2\tilde{M}_B\tilde{M}_LM}{\delta}\Big(4\tilde{M}_LMM_F(t_{i+1}-s_i)^2||y^*-y^{**}||^2+8H(t_{i+1}-s_i)^{2H-1}\tilde{M}_LM\Big)$$

Therefore from (3.6),(3.7) we get

$$\begin{aligned}\mathbb{E}||\bar{\Phi}_1y^*(t)-\bar{\Phi}_1y^{**}(t)||^2 &\leq \Big(4\tilde{M}_LMM_LM_{h_i}l^2+4\tilde{M}_LMM_B(t_{i+1}-s_i)^2L_{u_2}\Big)||y^*-y^{**}||^2\\ &\leq L_3||y^*-y^{**}||^2,\end{aligned}$$

where $L_3=\max\limits_{i=0,1,2,...m}\big(4\tilde{M}_LMM_LM_{h_i}l^2+4\tilde{M}_LMM_B(t_{i+1}-s_i)^2L_{u_2}\big)$.

Taking supremum over t, we obtain

$$||\bar{\Phi}_1(y^*)-\bar{\Phi}_1(y^{**})||^2_{\mathcal{B}_T^0}\leq L_0||y^*-y^{**}||^2_{\mathcal{B}_T^0}.$$

Thus $\bar{\Phi}_1$ is contraction on $\mathcal{B}_T^0$.

Now we prove that $\bar{\Phi}_2$ has compact, convex values and it is completely continuous. This will be divided into following steps.

Step 1. For $y\in\mathcal{B}_T^0$, $\bar{\Phi}_2y$ is convex.

If $w_1,w_2\in\bar{\Phi}_2y$, then there exists $g_1,g_2\in S_{G,x}$ such that for $t\in[0,t_1]$, we have

$$w_k(t)=\int_0^t\mathcal{L}^{-1}\mathbb{S}(t-s)F(s,y_s+\hat{\varphi}_s)ds+\int_0^t\mathcal{L}^{-1}\mathbb{S}(t-s)g_k(s)dB_s^H, k=1,2.$$

Let $0 \leq \lambda \leq 1$, we have

$$\lambda w_1(t) + (1-\lambda) w_2(t) = \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s + \hat{\varphi}_s)ds + \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)[\lambda g_1(s) + (1-\lambda)g_2(s)]dB_s^H.$$

Similarly for any $t \in (s_i, t_{i+1}]$,

$$\lambda w_1(t) + (1-\lambda) w_2(t) = \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s + \hat{\varphi}_s)ds + \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)[\lambda g_1(s) + (1-\lambda)g_2(s)]dB_s^H.$$

Since $S_{G,x}$ is convex, we have $\lambda w_1(t) + (1-\lambda) w_2(t) \in \bar{\Phi}_2$.
Step 2. $\bar{\Phi}_2$ maps bounded sets into bounded sets in $\mathcal{B}_T^0$.
Indeed it is enough to show that there exists a positive constant l such that for each $w \in \bar{\Phi}_2 y, y \in \mathcal{B}_q = \{y \in \mathcal{B}_T^0 / ||y||^2_{\mathcal{B}_T} \leq q\},$ *one has* $||w||^2_{\mathcal{B}_T^0} \leq l$. For $y \in \mathcal{B}_q, t \in (s_i, t_{i+1}], i = 0, 1, 2, ...m$, consider

$$\begin{aligned}\mathbb{E}||w(t)||^2 &\leq 4\mathbb{E}||\int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s + \hat{\varphi}_s)ds||^2 + 4\mathbb{E}||\int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB_s^H||^2 \\ &\leq 4\tilde{M}_L M\Big[(t_{i+1} - s_i)^2(C_1 q^* + C_2) + 2H(t_{i+1} - s_i)^{2H-1}||h_{q^*}||_{L^1}\Big] = l,\end{aligned}$$

where

$$l = \max_{0 \leq i \leq m} 4\tilde{M}_L M\Big[(t_{i+1} - s_i)^2(C_1 q^* + C_2) + 2H(t_{i+1} - s_i)^{2H-1}||h_{q^*}||_{L^1}\Big],$$

there fore $||w(t)||^2_{\mathcal{B}_T^0} \leq l$.
Step 3. $\bar{\Phi}_2$ maps bounded sets into equi continuous sets of $\mathcal{B}_T^0$.
Let $\tau_1, \tau_2 \in (s_i, t_{i+1}], i = 0, 1, 2, ..., m, \tau_1 < \tau_2$, we have

$$\begin{aligned}\mathbb{E}||w(\tau_2) - w(\tau_1)||^2 &\leq 6(t_{i+1} - s_i)\tilde{M}_L \int_{s_i}^{\tau_1 - \epsilon} ||\mathbb{S}(\tau_2 - s) - \mathbb{S}(\tau_1 - s)||^2 (C_1 q^* + C_2)ds \\ &+ 6(t_{i+1} - s_i)\tilde{M}_L \int_{\tau_1 - \epsilon}^{\tau_1} ||\mathbb{S}(\tau_2 - s) - \mathbb{S}(\tau_1 - s)||^2 (C_1 q^* + C_2)ds \\ &+ 6(t_{i+1} - s_i)\tilde{M}_L \int_{\tau_1}^{\tau_2} ||\mathbb{S}(\tau_2 - s)||^2 (C_1 q^* + C_2)ds \\ &+ 12H(t_{i+1} - s_i)^{2H-1}\tilde{M}_L \int_{s_i}^{\tau_1 - \epsilon} ||\mathbb{S}(\tau_2 - s) - \mathbb{S}(\tau_1 - s)||^2 ||g(s)||^2 ds \\ &+ 12H(t_{i+1} - s_i)^{2H-1}\tilde{M}_L \int_{\tau_1 - \epsilon}^{\tau_1} ||\mathbb{S}(\tau_2 - s) - \mathbb{S}(\tau_1 - s)||^2 ||g(s)||^2 ds \\ &+ 12H(t_{i+1} - s_i)^{2H-1}\tilde{M}_L \int_{\tau_1}^{\tau_2} ||\mathbb{S}(\tau_2 - s) - \mathbb{S}(\tau_1 - s)||^2 ||g(s)||^2 ds\end{aligned}$$

The right hand side of the above inequality tends to 0 as $\tau_2 - \tau_1 \to 0$ for sufficiently small ϵ.
Since F is completely continuous and the compactness of $\mathbb{S}(t)$ imply the continuity in uniform operator topology. There fore the set $\{\bar{\Phi}_2 y : y \in \mathcal{B}_q\}$ is equi continuous.
Step 4. $\bar{\Phi}_2$ is precompact in $\mathbb{H}$.
Form the above steps 2 to 3, together with Arzelá-Ascoli theorem, it suffices to show that $\bar{\Phi}_2$ maps $\mathcal{B}_q$ into a precompact set in $\mathbb{H}$.

Let $s_i < t < t_{i+1}$ be fixed and let ϵ be a real number satisfying $s_i < \epsilon < t$. For $y \in \mathcal{B}_q$, we define

$$\bar{\Phi}_2^{\epsilon} y(t) = \int_{s_i}^{t-\epsilon} \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s + \hat{\varphi}_s)ds + \int_{s_i}^{t-\epsilon} \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB_s^H.$$

Since $\mathbb{S}(t)$ is a compact operator, the set $V_\epsilon(t) = \{\bar{\Phi}_2^{\epsilon} y(t) : y \in \mathcal{B}_q\}$ is precompact in $\mathbb{H}$. Moreover

$$\mathbb{E}||\bar{\Phi}_2 y(t) - \bar{\Phi}_2^{\epsilon} y(t)||^2 \leq 4(t_{i+1} - s_i) \int_{t-\epsilon}^{t} ||\mathbb{S}(t-s)||^2 \mathbb{E}||F(s, y_s + \hat{\varphi}_s)||^2 ds$$

$$+ 8H(t_{i+1} - s_i)^{2H-1} \int_{t-\epsilon}^{t} ||\mathbb{S}(t-s)||^2 \mathbb{E}||g(s)||^2 ds$$

$$\leq 4(t_{i+1} - s_i) \int_{t-\epsilon}^{t} ||\mathbb{S}(t-s)||^2 (C_1 q + C_2) ds$$

$$+ 8H(t_{i+1} - s_i)^{2H-1} \int_{t-\epsilon}^{t} ||\mathbb{S}(t-s)||^2 h_{q^*}(s) ds.$$

There fore

$$\mathbb{E}||\bar{\Phi}_2 y(t) - \bar{\Phi}_2^{\epsilon} y(t)||^2 \to 0 \ as \ \epsilon \to 0^+.$$

Hence there are precompact sets which are arbitrary close to $\{\bar{\Phi}_2 y(t) : y \in \mathcal{B}_q\}$.

Step 5. $\bar{\Phi}_2$ has closed graph.

Let $y^n \to y^*$ and $w^n \to w^*$ as $n \to \infty$.

We shall prove that $w^* \in \bar{\Phi}_2(y^*)$. since $w^n \in \bar{\Phi}_2(y^n)$, there exists $g^n \in S_{G,y^n}$ such that for $t \in (s_i, t_{i+1}], i = 0, 1, 2, ...m$

$$w^n(t) = \int_{s_i}^{t} \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s^n + \hat{\varphi}_s)ds + \int_{s_i}^{t} \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB_s^H.$$

We need to prove that there exists $g^* \in S_{G,y^*}$ such that

$$w^*(t) = \int_{s_i}^{t} \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s^* + \hat{\varphi}_s)ds + \int_{s_i}^{t} \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB_s^H,$$

since F is continuous, we get

$$||\big(w^n(t) - \int_{s_i}^{t} \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s^n + \hat{\varphi}_s)ds\big) - \big(w^*(t) - \int_{s_i}^{t} \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s^* + \hat{\varphi}_s)ds\big)|| \to 0, \ as \ n \to \infty.$$

Consider the linear continuous operator

$$\Gamma : L^2((s_i, t_{i+1}], \mathbb{H}) \to C((s_i, t_{i+1}], \mathbb{H}), \quad g : \Gamma(g)(t) = \int_{s_i}^{t} \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB_s^H.$$

From Lemma (2.8), it follows that $\Gamma o S_G$ is closed graph operator. Furthermore, we have

$$w^n(t) - \int_{s_i}^{t} \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s^n + \hat{\varphi}_s)ds \in \Gamma(S_{G,y^n}).$$

Since $y^n \to y^*$, from Lemma (2.8)

$$w^*(t) - \int_{s_i}^{t} \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s^* + \hat{\varphi}_s)ds = \int_{s_i}^{t} \mathcal{L}^{-1}\mathbb{S}(t-s)g^*(s)dB_s^H,$$

for $g^* \in S_{G,y^*}$.
Therefore $\bar{\Phi}_2$ is a completely continuous multi-valued map, u.s.c with convex closed, compact values.
Step 6. It remains to show that the set $\mathcal{G} = \{y \in \mathbb{H} : y \in \lambda\bar{\Phi}_1(y) + \lambda\bar{\Phi}_2(y)\}$ is bounded for $\lambda \in (0,1)$.
Let $y \in \mathcal{G}$, then there exists $g \in S_{G,x}$ such that

$$y(t) = \begin{cases} \lambda \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)Bu(s)ds \\ +\lambda \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, x_s)ds + \lambda \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB_s^H, & t \in [0, t_1] \\ \lambda h_i(t, x_t), & t \in (t_i, s_i] \\ \lambda\mathcal{L}^{-1}\mathbb{S}(t-s_i)[\mathcal{L}h_i(s_i, x_{s_i})] + \lambda \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)Bu(s)ds \\ +\lambda \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, x_s)ds + \lambda \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB_s^H, & t \in (s_i, t_{i+1}]. \end{cases} \tag{3.8}$$

For some $0 < \lambda < 1$ and for $t \in [0, t_1]$, we have

$$\begin{aligned} \mathbb{E}||y(t)||^2 &= \mathbb{E}|| \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)Bu(s)ds + \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s + \hat{\varphi}_s)ds + \int_0^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB_s^H||^2 \\ &\le 9\tilde{M}_L M M_B t_1^2 \Big[\frac{16M_B\tilde{M}_L M}{\delta}\big\{\mathbb{E}||x_{t_1}||^2 + \tilde{M}_L M M_L \mathbb{E}||\varphi(0)||^2 + \tilde{M}_L M t_1 \int_0^{t_1} (C_1\mathbb{E}||y_s + \hat{\varphi}_s||^2 \\ &+ C_2)ds + 2\tilde{M}_L M H t_1^{2H-1} \int_0^{t_1} p(s)\Theta(||y_s + \hat{\varphi}_s||^2)ds\big\}\Big] + 9\tilde{M}_L M t_1 \int_0^t (C_1\mathbb{E}||y_s + \hat{\varphi}_s||^2 + C_2)ds \\ &+ 18\tilde{M}_L M H t_1^{2H-1} \int_0^t p(s)\Theta(||y_s + \hat{\varphi}_s||^2)ds \end{aligned} \tag{3.9}$$

For $t \in (t_i, s_i], i = 1, 2, ...m$, we have

$$\begin{aligned} \mathbb{E}||y(t)||^2 &= \mathbb{E}||h_i(t, y_t + \hat{\varphi}_t)||^2 \\ &\le \tilde{M}_{h_i}(1 + ||y_t + \hat{\varphi}_t||^2) \end{aligned} \tag{3.10}$$

For $t \in (s_i, t_{i+1}], i = 1, 2, ...m$, we have

$$\begin{aligned} \mathbb{E}||y(t)||^2 &\le \mathbb{E}||\mathcal{L}^{-1}\mathbb{S}(t-s_i)\mathcal{L}h_i(t, ||y_t + \hat{\varphi}_t||^2) + \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)Bu(s)ds \\ &+ \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)F(s, y_s + \hat{\varphi}_s)ds + \int_{s_i}^t \mathcal{L}^{-1}\mathbb{S}(t-s)g(s)dB_s^H||^2 \\ &\le 16\tilde{M}_L M M_L \tilde{M}_{h_i}(1 + ||y_{s_i} + \hat{\varphi}_{s_i}||^2) + 16\tilde{M}_L M M_B (t_{i+1} - s_i)^2 \Big[\frac{16M_B\tilde{M}_L M}{\delta}\big\{\mathbb{E}||x_{t_{i+1}}||^2 \\ &+ \tilde{M}_L M M_L \mathbb{E}||\varphi(0)||^2 + \tilde{M}_L M (t_{i+1} - s_i) \int_{s_i}^{t_{i+1}} (C_1\mathbb{E}||y_s + \hat{\varphi}_s||^2 + C_2)ds \end{aligned}$$

$$+ 2\tilde{M}_L MH(t_{i+1} - s_i)^{2H-1} \int\limits_{s_i}^{t_{i+1}} p(s)\Theta(||y_s + \hat{\varphi}_s||^2)ds\}\Big]$$

$$+ 16\tilde{M}_L M(t_{i+1} - s_i) \int\limits_{s_i}^{t} (C_1\mathbb{E}||y_s + \hat{\varphi}_s||^2 + C_2)ds \tag{3.11}$$

$$+ 32\tilde{M}_L MH(t_{i+1} - s_i)^{2H-1} \int\limits_{s_i}^{t} p(s)\Theta(||y_s + \hat{\varphi}_s||^2)ds.$$

But

$$\begin{aligned} ||y_t + \hat{\varphi}_t||^2 &\leq 2(||y_t||^2_{\mathcal{B}_h} + ||\hat{\varphi}_t||^2_{\mathcal{B}_h}) \\ &\leq 2l^2\big(\sup_{0\leq t\leq t_1} \mathbb{E}||y(s)||^2 + \tilde{M}_L MM_L\mathbb{E}||\varphi(0)||^2\big) + 2|\varphi||^2_{\mathcal{B}_h} \end{aligned}$$

Let us denote the right hand side of above inequality by $\vartheta(t)$ then

$$||y_t + \hat{\varphi}_t||^2 \leq \vartheta(t).$$

Therefore from the definition of $\vartheta(t)$, for all $t \in [0, T]$ we have

$$\begin{aligned} \vartheta(t) \leq \bar{M} &+ \tilde{M}_{h_i}\vartheta(t) + 16\tilde{M}_L MM_L\tilde{M}_{h_i}\vartheta(t) \\ &+ 16\tilde{M}_L MTC_1\vartheta(t) + 32\tilde{M}_L MHT^{2H-1} \int\limits_0^t p(s)\Theta(\vartheta(s))ds. \end{aligned}$$

Therefore

$$\vartheta(t) \leq K_1 + K_2 \int\limits_0^t p(s)\Theta(\vartheta(s))ds,$$

where

$$\begin{aligned} \bar{M} &= \max_{1\leq i\leq m} \Big\{\tilde{M}_{h_i} + 16\tilde{M}_L MM_L\tilde{M}_{h_i} + 16\tilde{M}_L MM_B T^2\Big[\frac{16M_B\tilde{M}_L M}{\delta}\{\mathbb{E}||x_{t_{i+1}}||^2 \\ &+ \tilde{M}_L MM_L\mathbb{E}||\varphi(0)||^2 + \tilde{M}_L MT^2C_2 + 2\tilde{M}_L MHT^{2H-1} \int\limits_0^T p(s)\Theta(||y_s + \hat{\varphi}_s||^2)ds\}\Big] \\ &+ 16\tilde{M}_L MT^2C_2\Big\}, \\ N_1 &= \tilde{M}_{h_i}\big(1 + 16\tilde{M}_L M(M_L + T^2C_1)\big), K_1 = \frac{\bar{M}}{1 - N_1}, \\ N_2 &= 32\tilde{M}_L MHT^{2H-1}, K_2 = \frac{N_2}{1 - N_1}. \end{aligned}$$

Let us denote the right hand side of above inequality by $\eta(t)$, then

$$\eta(0) = K_1, \vartheta(t) \leq \eta(t)$$

and

$$\eta'(t) = K_2 p(t)\Theta(\vartheta(t))$$

using increasing character of Θ we get

$$\eta'(t) \leq K_2 p(t)\Theta(\vartheta(t)).$$

Therefore

$$\int\limits_{\eta(0)}^{\eta(t)} \frac{ds}{\Theta(s)} \leq K_2 \int\limits_0^T p(s)ds < \int\limits_{K_1}^{\infty} \frac{ds}{\Theta(s)}.$$

This inequality shows that there is a constant K such that $\eta(t) \leq K$

$$||y||^2_{\mathcal{B}_h} \leq \vartheta(t) \leq \eta(t),$$

where K depends on $T, p(.), \Theta(.)$.
Therefore the set $\mathcal{G}$ is bounded. Hence from Theorem (2.9), $\bar{\Phi}$ has a fixed point which is a mild solution of (1.1)-(1.3). Thus the system (1.1)-(1.3) is controllable on $[0, T]$. $\square$

4 Example

Consider stochastic differential equation with non-instantaneous impulses

$$\begin{aligned}
\frac{\partial}{\partial t}\big[X(t,y)\big] &= \big[\frac{\partial^2}{\partial y^2}X(t,y) + \mu(t,y) + F(t,X_t)\big]dt + G(t,X_t)dB^H_t; t \in (s_i, t_{i+1}],\\
X(t,y) &= h_i(t, X_t(.,y)),\ i = 1,2...,m,\\
X(t,0) &= X(t,\pi) = 0, t \in [0,T],\\
X(t,y) &= \varphi(t,y), -\infty \leq t \leq 0,\ 0 \leq y \leq \pi.
\end{aligned} \tag{4.1}$$

Let us take $\mathbb{H} = L^2([0,\pi])$ and define $A : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H}, \mathcal{L} : \mathcal{D}(\mathcal{L}) \subset \mathbb{H} \to \mathbb{H}$ by $AX = X'', \mathcal{L}X = X - X''$, respectively, where $\mathcal{D}(A) = \mathcal{D}(\mathcal{L}) = \{X \in \mathbb{H} : X, X'\ \textit{are absolutely continuous}, X'' \in \mathbb{H}, X(0) = X(\pi) = 0\}$.
Then the operators A and $\mathcal{L}$ are given by

$$AX = \sum_{n=1}^{\infty} n^2 \langle X, X_n\rangle X_n, X \in D(A)$$
$$\mathcal{L}X = \sum_{n=1}^{\infty} (1+n^2)\langle X, X_n\rangle X_n, X \in D(\mathcal{L}),$$

where $X_n(y) = \sqrt{\frac{2}{\pi} sin(ny)}, n = 1,2,3,...$ is the ortho normal set of eigenfunctions of A. Also for $X \in \mathbb{H}$, we have

$$\mathcal{L}^{-1}X = \sum_{n=1}^{\infty} \frac{1}{1+n^2}\langle X, X_n\rangle X_n,$$
$$-A\mathcal{L}^{-1}X = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}\langle X, X_n\rangle X_n,$$
$$\mathbb{S}(t)X = \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}}t\langle X, X_n\rangle X_n.$$

Here $-A\mathcal{L}^{-1}$ generates a strongly continuous semi group $\mathbb{S}(t)$ which is compact with $\|\mathbb{S}(t)\|^2 \leq M$. Hence assumption (H1) is satisfied.

Now, we consider $h(t) = e^{2t}, t < 0$. Then we get $l = \int\limits_{-\infty}^{0} h(s)ds = \frac{1}{2}; s < 0$ and define

$$\|\varphi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(\theta) \sup_{\theta\in[s,0]} (\mathbb{E}\|\varphi(\theta)^2)^{\frac{1}{2}} d\theta.$$

Let us define $B \in L(\mathbb{R}, \mathbb{H})$ as $Bu(t) = b(X)u(t), 0 \leq X \leq \pi, u \in \mathbb{R}, b(X) \in L^2([0,\pi])$. Moreover, the operator

$$W^{t_{i+1}}_{s_i} = \int_{s_i}^{t_{i+1}} \mathcal{L}^{-1}\mathbb{S}(t_{i+1}-s)BB^*\mathcal{L}^{-1}\mathbb{S}^*(t_{i+1}-s)ds, i = 0,1,2...m,$$

$W^{t_{i+1}}_{s_i}$ is a linear bounded operator. Let $KerW^{t_{i+1}}_{s_i} = \{u \in L^2([0,T],\mathbb{R}); W^{t_{i+1}}_{s_i} = 0\}$ be the null space of $W^{t_{i+1}}_{s_i}$, then $\big(W^{t_{i+1}}_{s_i}\big)^{-1}$ is bounded and takes values in $L^2((s_i, t_{i+1}], U)/KerW^{t_{i+1}}_{s_i}$ with $\|\big(W^{t_{i+1}}_{s_i}\big)^{-1}\| \leq \frac{1}{\delta}$, where $s_0 = 0,\ \ t_{i+1} =$

$T, i = 0, 1, 2...m$. Hence assumption (H5) is satisfied.
Let $\psi(\tau)(y) = \psi(\tau, y), (\tau, y) \in (-\infty, 0) \times [0, \pi]$. The functions $F : [0, T] \times \mathcal{B}_h \to \mathbb{H}$, $G : [0, T] \times \mathcal{B}_h \to P(L(\mathbb{K}, \mathbb{H}))$ and $h_i : [0, T] \times \mathcal{B}_h \to \mathbb{H}$ are the operators defined as follows:

$$F(t, \psi)(y) = F(t, \psi(\tau, y)) = \int_{-\infty}^{0} b_1(t, s, y, \psi(s, y))ds,$$

$$G(t, \psi)(y) = G(t, \psi(\tau, y)) = \int_{-\infty}^{0} C_1(t, s, y, \psi(s, y))ds,$$

$$h_i(t, \psi)(y) = \int_{-\infty}^{0} d_i(\tau, y)\psi(\tau)(y)d\tau.$$

Then (4.1) can be written in the form of (1.1)-(1.3). Moreover, we can define F and G to satisfy the assumptions stated in Theorem (3.1). Therefore the system (4.1) is controllable on J.

5 Conclusion

The research presented in this paper focuses on the ”controllability of Sobolev type stochastic differential equations driven by fBm with non-instantaneous impulses”. The results are obtained by utilizing Dhage fixed-point theorem for multi-valued operators, stochastic integrals for fractional Brownian motion. In particular, conditions are formulated and proved by assuming controllability of the linear system. The theoretical results obtained were verified with an example.

Our future work will be focused on investigating approximate controllability results and optimal control results for multi-valued Sobolev type stochastic differential equations driven by fBm. Upon making some appropriate assumptions, by employing the ideas and techniques as in this paper, one can establish the approximate controllability results for Sobolev type stochastic integro differential equations with impulses. By extending this concept one can study second order Sobolev type stochastic differential equations driven by fBm.

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