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Some results in metric modular spaces

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Abstract

A metric modular on a set X is a function $w:(0,\infty)\times X\times X\longrightarrow[0,\infty]$ written as $(\lambda,x,y)\mapsto w_\lambda(x,y)$ satisfying, for all $x,y,z\in X$, the following three properties: $x=y$ if and only if $w_\lambda(x,y)=0$ for all $\lambda>0$; $w_\lambda(x,y)=w_\lambda(y,x)$ for all $\lambda>0$; $w_{\lambda+\mu}(x,y)\le w_\lambda(x,z)+w_\mu(y,z)$ for all $\lambda,\mu>0$. In this paper we define a Hausdorff topology on metric modular spaces and prove some known results of metric spaces including Baire's theorem and Uniform limit theorem for metric modular spaces.

Keywords: Modular, metric modular, Baire's theorem, uniform limit theorem
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1 Introduction

In 1950, Nakano [15] initiated the study of modulars on linear spaces and the related theory of modular linear spaces as generalizations of metric spaces. Next, Luxemburg [8], Mazur, Musielak and Orlicz [10, 11, 12] thoroughly developed it extensively. Since then, the theory of modulars and modular spaces have been widely applied in the study of interpolation theory [7, 9] and various Orlicz spaces [16]. A modular yields less properties than a norm does, but it makes a more sense in many special situations. Recall that the notion of partial modular metric space with some fixed point results are given in [4]. In the formulation given by Kowzslowski [5, 6] a modular on a vector space X is defined as follow.

Definition 1.1. Let X be a linear space over a field $\mathbb{K}$ $(=\mathbb{R}$ or $\mathbb{C})$. A generalized function $\rho:X\longrightarrow[0,\infty]$ is called a *modular* if it satisfies the following three conditions for elements $\lambda,\mu\in\mathbb{K}$, $x,y\in X$

(i) $\rho(x)=0$ if and only if $x=0$;
(ii) $\rho(\lambda x)=\rho(x)$ for all scalar λ with $|\lambda|=1$;
(iii) $\rho(\lambda x+\mu y)\le\rho(x)+\rho(y)$ for all scalar $\lambda,\mu\ge0$ with $\lambda+\mu=1$.

If the condition (iii) is replaced by $\rho(\lambda x+\mu y)\le\lambda^t\rho(x)+\mu^t\rho(y)$ when $\lambda^t+\mu^t=1$ and $\lambda,\mu\ge0$ with an $t\in(0,1]$, then ρ is called an *t-convex modular*. 1-convex modulars are called *convex modulars*. For a modular ρ, there corresponds a linear subspace X_ρ of X, given by $X_\rho:=\{x\in X:\ \rho(\lambda x)\to0\,\text{as}\,\lambda x\to0\}$. In this case X_ρ is called a *modular space*.

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For example if $X = \mathbb{R}$ and $\beta \in (0, 1]$, then the function $\rho : X \longrightarrow [0, \infty]$ defined by $\rho(x) = |x|^\beta$ is a modular.

Example 1.2. *[17]* Let $\psi : [0, \infty) \longrightarrow \mathbb{R}$ be a function defined by $\psi(0) = 0$ and $\psi(t) > 0$ for all $t > 0$, and $\lim_{t\to\infty} \psi(t) = \infty$. If moreover ψ is convex, continuous and nondecreasing, then ψ is called an *Orlicz function*. For a measure space $(X, \sum, \mu)$, suppose that $L^0(\mu)$ is the set of all measurable functions on X. For each $f \in L^0(\mu)$, define $\rho_\psi(f) = \int_X \psi(|f|)d\mu$. Then, ρ_ψ is a modular and the corresponding modular space is called an *Orlicz space* and denoted by

$$L_\psi = \{f \in L^0(\mu) | \ \rho_\psi(\lambda f) \to 0 \,\text{as}\, \lambda \to 0\}.$$

One can check that L_ψ is ρ_ψ-complete.

In 2006, Vyacheslav Chistyakov [2], [3] introduced the concept of a metric modular on a set, inspired partly by the classical linear modulars on function spaces employed by Nakano [13, 14], [15].

Here, we recall the definition of a metric modular on a nonempty set.

Definition 1.3. Let X be a nonempty set. A metric modular on X is a function

$$w : (0, \infty) \times X \times X \longrightarrow [0, \infty],$$

written as $(\lambda, x, y) \mapsto w_\lambda(x, y)$, that satisfies the following three axioms:

(1) $w_\lambda(x, y) = 0$ for all $\lambda > 0$ and $x, y \in X$ if and only if $x = y$.

(2) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$.

(3) $w_{\lambda+\mu}(x, y) \le w_\lambda(x, z) + w_\mu(y, z)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

A metric modular space is an ordered pair (X, w), where X is a set and w is a metric modular on X. Throughout the paper, we suppose that metric modular w has only finite values and $\lambda \mapsto w_\lambda(x, y)$ is continuous.

Example 1.4. *[3]* Let X be a set. Then,

$$w^0_\lambda(x, y) = \begin{cases} \infty & x \ne y, \\ 0 & x = y, \end{cases}$$

define a metric modular on X.

Example 1.5. *[3]* Let (X, d) be a metric space with metric d and at least two points. The following indexed objects w are simple examples of metric modulars on a set X.

(1) $w^1_\lambda(x, y) = d(x, y)$;

(2) $w^2_\lambda(x, y) = \frac{d(x,y)}{\phi(\lambda)}$, where $\phi : (0, \infty) \longrightarrow (0, \infty)$ is a nondecreasing continuous function;

(3) $w^3_\lambda(x, y) = \begin{cases} \infty & d(x, y) \ge \lambda, \\ 0 & d(x, y) < \lambda; \end{cases}$

(4) $w^4_\lambda(x, y) = \begin{cases} 0 & d(x, y) \le \lambda, \\ \infty & d(x, y) > \lambda; \end{cases}$

In the sequel, We will write w^j_λ simply w^j when no confusion can arise. The following theorem states the relation between modulars and metric modules in real linear spaces.

Theorem 1.6. *[3]* Let X be a real linear space.

(a) Given a functional $\rho : X \longrightarrow [0, \infty]$, we set

$$w_\lambda(x, y) = \rho(\frac{x - y}{\lambda}), \qquad \lambda > 0, x, y \in X.$$

Then, ρ is a modular on X in the sense of *Definition 1.1* if and only if w is a metric modular on X.

(b) Suppose that the function $w : (0, \infty) \times X \times X \longrightarrow [0, \infty]$ satisfy the following two conditions:

(I) $w_\lambda(\mu x, 0) = w_{\frac{\lambda}{\mu}}(x, 0)$ for all $\lambda, \mu > 0$ and $x \in X$;

(II) $w_\lambda(x + z, y + z) = w_\lambda(x, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$

Given $x \in X$, we set $\rho(x) = w_1(x, 0)$. Then, w is a metric modular on X if and only if ρ is a modular on X.

Motivated by the above literature, in this paper we define a Hausdorff topology on metric modular spaces and present some well-known results of metric spaces such as Baire's theorem and uniform limit theorem for metric modular spaces.

2 Topology induced by a metric modular

We start this section with a lemma from [1] as follow.

Lemma 2.1. Let w be a metric modular on the set X. Then, for each $x, y \in X$, the function $\lambda \mapsto w_\lambda(x, y)$ is non-increasing.

Definition 2.2. Let (X, w) be a metric modular space and $\lambda > 0$. Define a w-open ball $B_\lambda(x, \epsilon)$ with center $x \in X$ and radius $\epsilon > 0$ as

$$B_\lambda(x, \epsilon) = \{y \in X; w_\lambda(x, y) < \epsilon\}.$$

We say that $A \subseteq X$ is a w-open set in X if and only if for every element $x \in X$ there exist $\lambda > 0$ and $\epsilon > 0$ such that $B_\lambda(x, \epsilon) \subseteq A$.

Theorem 2.3. Eavry w-open ball is a w-open set.

Proof . Consider a w-open ball $B_\lambda(x, \epsilon)$. Then

$$y \in B_\lambda(x, \epsilon) \Rightarrow w_\lambda(x, y) < \epsilon.$$

Assume that there is $\lambda_0 < \lambda$ such that $w_\lambda(x, y) \leq w_{\lambda_0}(x, y) < \epsilon$. Now, consider the ball $B_{\lambda-\lambda_0}(y, \epsilon_0)$ such that $\epsilon_0 < \epsilon - w_{\lambda_0}(x, y)$. We claim that $B_{\lambda-\lambda_0}(y, \epsilon_0) \subseteq B_\lambda(x, \epsilon)$. If $z \in B_{\lambda-\lambda_0}(y, \epsilon_0)$, then $w_{\lambda-\lambda_0}(y, z) < \epsilon_0$. Therefore,

$$w_\lambda(x, z) \leq w_{\lambda_0}(x, y) + w_{\lambda-\lambda_0}(y, z) < w_{\lambda_0}(x, y) + \epsilon_0 < \epsilon.$$

Consequently, $z \in B_\lambda(x, \epsilon)$ and hence $B_{\lambda-\lambda_0}(y, \epsilon_0) \subseteq B_\lambda(x, \epsilon)$. It remains to show that λ_0 exists. Choose $0 < \lambda_1 < \lambda$. By Lemma 2.1, $w_\lambda(x, y) \leq w_{\lambda_1}(x, y)$. If $w_{\lambda_1}(x, y) < \epsilon$, put $\lambda_0 := \lambda_1$. Otherwise, by continuity of $\lambda \to w_\lambda(x, y)$ and intermediate value theorem, there is $\lambda_1 < \lambda_0 < \lambda$ such that $w_\lambda(x, y) \leq w_{\lambda_0}(x, y) < \epsilon < w_{\lambda_1}(x, y)$. □

Example 2.4. Let X be a non-empty set and $A \subseteq X$. Then

(0) A is an open set in (X, w^0) if and only if A is a single set or $A = X$;

(1) A is an open set in (X, w^1) if and only if A is an open set in metric space (X, d);

(2) A is an open set in (X, w^2) if and only if A is an open set in metric space (X, d);

(3) For all $\lambda > 0$, $\epsilon > 0$ and $x \in X$ we have $B_\lambda(x, \epsilon) = \{y \in X : w^3_\lambda(x, y) < \epsilon\} = \{y \in X : d(x, y) < \lambda\}$;

(4) for all $\lambda > 0$, $\epsilon > 0$ and $x \in X$ we have $B_\lambda(x, \epsilon) = \{y \in X : w^4_\lambda(x, y) < \epsilon\} = \{y \in X : d(x, y) \leq \lambda\}$.

The next example is a direct consequence of Theorem 2.3.

Corollary 2.5. Let (X, w) be a metric modular space. Define

$$\tau_w = \{A \subseteq X : x \in A \Leftrightarrow \exists\lambda > 0, \epsilon > 0 \ s.t \ B_\lambda(x, \epsilon) \subseteq A\}.$$

Then, (X, τ_w) is a topological space.

Theorem 2.6. Let (X, w) be a metric modular space. Then, τ_w is Hausdorff.

Proof . Let x, y be two distinct points of X. For any $\lambda > 0$, we have $w_\lambda(x, y) > 0$. Put $w_\lambda(x, y) = r$, for some $r > 0$. Moreover, for $B_{\lambda/2}(x, r/2)$ and $B_{\lambda/2}(y, r/2)$, we get $B_{\lambda/2}(x, r/2) \cap B_{\lambda/2}(y, r/2) = \emptyset$. In other words, if there exists an element z such that $z \in B_{\lambda/2}(x, r/2) \cap B_{\lambda/2}(y, r/2)$, then

$$r = w_\lambda(x, y) \leq w_{\lambda/2}(x, z) + w_{\lambda/2}(z, y) < r,$$

which leads us to a contradiction. Therefore, τ_w is Hausdorff. □

Definition 2.7. Let (X, w) be a metric modular space. A subset A of X is called w-bounded if and only if there exist $\lambda > 0$ and $\epsilon > 0$ such that $w_\lambda(x, y) < \epsilon$ for all $x, y \in A$.

It is easy to see that every subset A of metric modular space (X, w^0) is bounded if and only if A is a single set. In addition, the subset A of metric modular spaces (X, w^i), $i = 1, 2, 3, 4$, is a bounded set if and only if A is a bounded set in metric space (X, d).

Definition 2.8. A metric modular space (X, w) is called w-compact if each of its w-open covers has a finite subcover. Indeed, X is w-compact if for every collection C of w-open subsets of X with $X = \bigcup_{U \in C} U$, there is a finite subset F of C such that $X = \bigcup_{U \in F} U$.

Every w-compact set is w-bounded as it will be shown in the next result.

Theorem 2.9. Let (X, w) be a metric modular space. Then, every w-compact subset A of X is w-bounded. In particular, every w-compact set is w-bounded.

Proof . Suppose that $\lambda > 0, \epsilon > 0$. Consider an open cover $\{B_\lambda(x, \epsilon) : x \in A\}$ of A. Since A is compact, there exist $x_1, x_2, \ldots, x_n \in A$ such that $A \subseteq \bigcup B_\lambda(x_i, \epsilon)$. Let $x, y \in A$. Then, $x \in B_\lambda(x_i, \epsilon)$ and $y \in B_\lambda(x_j, \epsilon)$ for some i, j. Therefore, $w_\lambda(x, x_i) < \epsilon$ and $w_\lambda(y, x_j) < \epsilon$. Set

$$\alpha = \max\{w_\lambda(x_k, x_t) : 1 \leq k \leq n, 1 \leq t \leq n\}.$$

Then, $\alpha > 0$. Now we have

$$w_{3\lambda}(x, y) \leq w_\lambda(x, x_i) + w_\lambda(x_i, x_j) + w_\lambda(x_j, y) \leq 2\epsilon + \alpha.$$

Putting $m > 2\epsilon + \alpha$, we get $w^3_\lambda(x, y) \leq m$ for each $x, y \in A$ and so A is w-bounded. □

Proposition 2.10. Let (X, w) be a metric modular space. Then, $\lim_{n\to\infty} w_\lambda(x_n, x) = 0$, for all $\lambda > 0$ if and only if $x_n \xrightarrow{\tau_w} x$.

Proof . Suppose that $\lim_{n\to\infty} w_\lambda(x_n, x) = 0$, for all $\lambda > 0$. Fix $\lambda > 0$ and $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $w_\lambda(x_n, x) < \epsilon$ for all $n > n_0$. It follows that $x_n \in B_\lambda(x, \epsilon)$. Thus, $x_n \xrightarrow{\tau_w} x$.

Conversely, if $x_n \xrightarrow{\tau_w} x$ then for $\epsilon > 0$ and $\lambda > 0$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B_\lambda(x, \epsilon)$ for all $n > n_0$. This means that $w_\lambda(x_n, x) < \epsilon$, for all $n > n_0$. Therefore, $\lim_{n\to\infty} w_\lambda(x_n, x) = 0$. □

Definition 2.11. Let (X, w) be a metric modular space. We define a w-closed ball with center $x \in X$ and radius $\epsilon > 0$, $\lambda > 0$ as

$$B_\lambda[x, \epsilon] = \{y \in X; B_\lambda(x, y) \leq \epsilon\}.$$

Lemma 2.12. Every w-closed ball is a w-closed set.

Proof . Let $y \in \overline{B_\lambda[x, \epsilon]}$ and $B_1 = B_1(y, 1)$. We know that $B_1 \cap B_\lambda[x, \epsilon] \neq \emptyset$. Choose $y_1 \in B_1$. Set $B_2 = B_{1/2}(y, 1/2) \cap B_1$. Since $B_2 \cap B_\lambda[x, \epsilon] \neq \emptyset$, one can take $y_2 \in B_2 \cap B_\lambda[x, \epsilon]$. This process can be repeated to find $y_n \in B_n \cap B_\lambda[x, \epsilon]$. It is obvious that $y_n \xrightarrow{\tau_w} y$. Now, for each $n \in \mathbb{N}$ we have

$$w_{\lambda+1/n}(y, x) \leq w_{1/n}(y, y_n) + w_\lambda(y_n, x) \leq 1/n + \epsilon.$$

Due to the continuity of the mapping $\lambda \mapsto w_\lambda(y,x)$, we find

$$\lim_{n\to\infty} w_{\lambda+1/n}(y,x) = w_\lambda(y,x).$$

Consequently, $w_\lambda(y,x) = \lim_{n\to\infty} w_{\lambda+1/n}(y,x) \leq \epsilon$. Hence, $y \in B_\lambda[x,\lambda]$ which implies that $B_\lambda[x,\lambda]$ is a w-closed set. □

Definition 2.13. A sequence $\{x_n\}$ in a metric modular space X is said to be a w-Cauchy sequence if and only if for each $\epsilon > 0, \lambda > 0$, there is $n_0 > 0$ such that $w_\lambda(x_{n+m}, x_n) < \epsilon$ for all $n > n_0, m > 0$.
If every w-Cauchy sequence is convergent in τ_w-topology, then X is called w-complete metric modular set.

Theorem 2.14. Let X be a w-complete metric modular set. Then, the intersection of a countable number of dense w-open sets is dense.

Proof . Assume that B_0 is a nonempty w-open set and $D_1, D_2, D_3, \ldots$ dense w-open sets in X. Since $B_o \cap D_1$ is nonempty w-open set, there are $x_1 \in B_o \cap D_1$ and $0 < \lambda_1 < 1, 0 < \epsilon_1 < 1$ such that $B_{\lambda_1}[x_1, \epsilon_1] \subseteq B_o \cap D_1$. Due to being dense D_2, there are $x_2 \in B_{\lambda_1}(x_1, \epsilon_1) \cap D_2$ and $0 < \lambda_2 < 1/2$ and $\epsilon_2 < 1/2$ such that $B_{\lambda_2}[x_2, \epsilon_2] \subseteq B_{\lambda_1}(x_1, \epsilon_1) \cap D_2$. Similarly by induction, we can find $x_n \in B_{\lambda_{n-1}}(x_{n-1}, \epsilon_{n-1}) \cap D_n$ and $0 < \lambda_n < 1/n, 0 < \epsilon_n < 1/n$ such that $B_{\lambda_n}[x_n, \epsilon_n] \subseteq B_{\lambda_{n-1}}(x_{n-1}, \epsilon_{n-1}) \cap D_n$. Given $\lambda > 0$, $\epsilon > 0$, we choose $N_0 > 0$ such that $1/N_0 < \epsilon$ and $1/N_0 < \lambda$. Then for every $n \geq N_0$, we have

$$w_\lambda(x_n, x_{n+m}) \leq w_{1/n}(x_n, x_{n+m}) \leq 1/n < \epsilon.$$

The relation above shows that $\{x_n\}$ is a w-cauchy sequence. Due to the $w-$completenss of X, we obtain $x_n \xrightarrow{\tau_w} x$ for some $x \in X$. On the other hand, $x_{n+m} \in B_{\lambda_n}(x_n, \epsilon_n)$ for all $m > 0$. It follows from Lemma 2.12 that $x \in B_{\lambda_n}[x_n, \epsilon_n] \subseteq B_{n-1}(x_{n-1}, \epsilon_{n-1}) \cap D_n$, for all n. Therefore, $x \in B_0 \cap (\cap D_n) \neq \emptyset$. □

Definition 2.15. Let (X,w) be a metric modular space. A collection of sets $\{A_n\}_{n\in I}$ is said to have modular diameter zero if and only if for each pair $\lambda > 0, \epsilon > 0$, there exists $N \in I$ such that $w_\lambda(x,y) < \epsilon$ for all $x, y \in A_N$.

The next result is a version of Baire's theorem for metric modular spaces.

Theorem 2.16. Let (X,w) be a metric modular space. Then, X is w-complete metric modular set if and only if every nested sequence of nonempty w-closed sets $\{A_n\}_{n=1}^\infty$ with modular diameter zero have nonempty intersection.

Proof . Assume that X is w-complete metric modular set and $\{A_n\}_{n=1}^\infty$ is a nested sequence of nonempty w-closed sets with modular diameter zero. Choose $x_n \in A_n$ for $n \in \mathbb{N}$. Since $\{A_n\}$ has modular diameter zero for each $\epsilon > 0$ and $\lambda > 0$ there exists $N > 0$ such that $w_\lambda(x,y) < \epsilon$ for all $x, y \in A_N$. For every $n, m \geq N$, we choose $x_n \in A_n \subseteq A_N$ and $x_m \in A_m \subseteq A_N$. Thus, $\{x_n\}$ is a $w-$cauchy sequence. By assumption, x_n converges to x for some $x \in X$. For each $n \in \mathbb{N}$ and $k > n$ we have $x_k \in A_n$ and hence $x \in \overline{A_n} = A_n$ for every n and $x \in \cap_{n=1}^\infty A_n$.

Conversely, suppose that every nested sequence of nonempty w-closed sets $\{A_n\}_{n=1}^\infty$ with modular diameter zero have non-empty intersection. Let $\{x_n\}$ be a w-Cauchy sequence in X. Put $B_n = \{x_n, x_{n+1}, \ldots\}$ and $A_n = \overline{B_n}$. We wish to show that $\{A_n\}$ has modular diameter zero. Let $\epsilon > 0$ and $\lambda > 0$. Since $\{x_n\}$ is a w-Cauchy sequence, there is $N > 0$ such that $w_{\lambda/3}(x,y) < \epsilon/3$ for all $x, y \in B_N$. Take $x, y \in A_N$. Then, there exist sequences $\{x_n^1\}$ and $\{y_n^1\}$ in B_N such that x_n^1 converges to x and y_n^1 converges to y, and so for sufficiently large n, we have $x_n^1 \in B_{\lambda/3}(x, \epsilon/3)$ and $y_n^1 \in B_{\lambda/3}(y, \epsilon/3)$. Hence

$$w_\lambda(x,y) \leq w_{\lambda/3}(x, x_n^1) + w_{\lambda/3}(x_n^1, y_n^1) + w_{\lambda/3}(y_n^1, y) < \epsilon.$$

Consequently, $\{A_n\}$ has modular diameter zero and hence $\cap_{n=1}^\infty A_n \neq \emptyset$. Take $x \in \cap_{n=1}^\infty A_n$. Then for $\epsilon > 0, \lambda > 0$, there exists N_1 such that $w_{\lambda-\lambda/3}(x_{N_1}, x) < \epsilon/3$. Thus, for all $n > N_1$,

$$w_\lambda(x_n, x) \leq w_{\lambda/3}(x_n, x_{N_1}) + w_{\lambda-\lambda/3}(x_{N_1}, x) < \epsilon.$$

Hence, x_n converges to x. Therefore, X is w-complete metric modular set. □

Definition 2.17. Let X be a non-empty set and (Y,w) be a metric modular space. We say a sequence $\{f_n\}$ of functions from X to Y converges w-uniformly to a function f from X to Y if for given $\epsilon > 0, \lambda > 0$, there exists $n_0 \in \mathbb{N}$ such that $w_\lambda(f_n(x), f(x)) < \epsilon$ for all $n \geq n_0$ and for all $x \in X$.

Theorem 2.18. Let $f_n : X \longrightarrow Y$ be a sequence of continuous functions from a topological space X to a metric modular set (Y, w). If $\{f_n\}$ converges w-uniformly to f, then f is continuous.

Proof . Suppose that V is an w-open set. Let $x_0 \in f^{-1}(V)$. Since V is open, we can find $\epsilon > 0$ and $\lambda > 0$ such that $B_\lambda(f(x_0), \epsilon) \subseteq V$. Since $\{f_n\}$ converges w-uniformly to f, there exists $n_0 \in \mathbb{N}$ such that $w_{\lambda/3}(f_n(x), f(x)) < \epsilon/3$ for all $n \geq n_0$ and for all $x \in X$. On the other hand, f_{n_0} is continuous and so we can find a neighborhood U of x_0 such that $f_{n_0}(U) \subseteq B_{\lambda/3}(f_{n_0}(x_0), \epsilon/3)$. Hence, for all $x \in U$ we have

$$w_\lambda(f(x), f(x_0)) \leq w_{\lambda/3}(f(x), f_{n_0}(x)) + w_{\lambda/3}(f_{n_0}(x), f_{n_0}(x_0)) + w_{\lambda/3}(f_{n_0}(x_0), f(x_0)) < \epsilon.$$

It follows from the relation above that $f(U) \subseteq B_\lambda(f(x_0), \epsilon) \subseteq V$. □

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References

[1] H. Abobakr and R.A. Ryan, *Modular Metric Spaces,* Irish Math. Soc. Bull. **80** (2017), 35–44.

[2] V. Chistyakov, *Metric modulars and their application,* Dokl. Akad. Nauk. **406** (2006), no. 2, 165–168.

[3] V. Chistyakov,*Modular metric spaces. I. Basic concepts*, Nonlinear Anal. **72** (2010), no. 1, 1–14.

[4] H. Hosseinzadeh and V. Parvaneh, *Meir-Keeler type contractive mappings in modular and partial modular metric spaces*, Asian-European J. Math. **13** (2020), no. 05, 2050087.

[5] M.A. Khamsi, *A convexity property in modular function spaces*, Math. Japon. **44** (1996), no. 2, 269–279.

[6] W.M. Kozlowski, *Modular function spaces*, Monographs and Text- books in Pure and Applied Mathematics, vol. 122, Marcel Dekker, Inc., New York, 1988.

[7] M. Krbec, *Modular interpolation spaces*, I. Z. Anal. Anwend. **1** (1982), 25–40.

[8] W.A.J. Luxemburg, *Banach function spaces*, Ph.D. Thesis, Delft University of Technology, Delft, The Netherlands, 1955.

[9] L. Maligranda, *Orlicz spaces and interpolation.* Semin. Math., 5, Universidade Estadual de Campinas, Departamento de Matematica, Campinas, 1989.

[10] S. Mazur and W. Orlicz, *On some classes of linear spaces*, Studia Math. **17** (1958), 97–119.

[11] J. Musielak and W. Orlicz, *On modular spaces*, Studia Math. **18** (1959), 49–65.

[12] J. Musielak and W. Orlicz, *Some remarks on modular spaces*, Bull. Acad. Polon. Sci. Sr. Math. Astron. Phys. **7** (1959), 661–668.

[13] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Tokyo Math. Book Ser., 1, Maruzen Co., Tokyo, 1950.

[14] H. Nakano, *Topology and Linear Topological Spaces*, Tokyo Math. Book Ser., 3, Maruzen Co., Tokyo, 1951.

[15] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Maruzen Co., Ltd., Tokyo, 1950.

[16] W. Orlicz, *Collected Papers*, vols. I, II. PWN, Warszawa, 1988.

[17] C. Park, J.M. Rasias, A. Bodaghi and S.O. Kim, *Approximate homomorphisms from ternary semigroups to modular spaces*, RACSAM **113** (2019), 2175–2188.