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Fuglede-Putnam type theorems for extension of $*$-class A operators

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Abstract

In this article, we consider k-quasi-$*$-class A operator $T \in \mathcal{B}(\mathcal{H})$ such that $TX = XS$ for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and prove the Fuglede-Putnam type theorem when adjoint of $S \in \mathcal{B}(\mathcal{K})$ is k-quasi-$*$-class A or dominant operators. Among other things, we prove that two quasisimilar k-quasi-$*$-class A operators have equal essential spectra.

Keywords: Fuglede-Putnam theorem, $*$-class A operators, k-quasi-$*$-class A operators, quasisimilar operators
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1 Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be separable complex Hilbert spaces, and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. When $\mathcal{H} = \mathcal{K}$, we write $\mathcal{B}(\mathcal{H})$ for $\mathcal{B}(\mathcal{H}, \mathcal{H})$. Throughout this paper, the range and the null space of an operator T will be denoted by $\mathcal{R}(T)$ and $\ker(T)$, respectively. Let $\overline{\mathcal{M}}$ and $\mathcal{M}^{\perp}$ be the norm closure and the orthogonal complement of the subspace $\mathcal{M}$ of $\mathcal{H}$. The classical Fuglede-Putnam theorem [10, Problem 152] asserts that if $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ are normal operators such that $TX = XS$ for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $T^*X = XS^*$. The references [25, 24, 26, 29, 15] are among the various extensions of this celebrated theorem for non-normal operators. According to [32] an operator $T \in \mathcal{B}(\mathcal{H})$ is dominant if

$$\mathcal{R}(T - \lambda I) \subseteq \mathcal{R}(T - \lambda I)^* \text{ for all } \lambda \in \sigma(T),$$

where $\sigma(T)$ denote the spectrum of T. From [6], it is seen that this condition is equivalent to the existence of a positive constant M_λ such that

$$(T - \lambda I)(T - \lambda I)^* \leq M_\lambda^2 (T - \lambda I)^*(T - \lambda I)$$

for each $\lambda \in \mathbb{C}$. An operator T is called M-hyponormal if there is a constant M such that $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$. If $M = 1$, T is hyponormal. Hence we have the following inclusion:

$$\{\text{Hyponormal}\} \subseteq \{M\text{-hyponormal}\} \subseteq \{\text{Dominant}\}.$$

Recall [2, 7] that $T \in \mathcal{B}(\mathcal{H})$ is called hyponormal if $T^*T \geq TT^*$, paranormal (resp., $*$-paranormal) if $\|T^2x\| \geq \|Tx\|^2$ (resp., $\|T^2x\| \geq \|T^*x\|^2$) for all unit vectors $x \in \mathcal{H}$. Following [7] and [13] we say that $T \in \mathcal{B}(\mathcal{H})$ belongs to class A if

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$|T^2| \geq |T|^2$, where $|T|^2 = T^*T$. Recently, B. P. Duggal al et. [5] considered the following new class of operators: We say that $T \in \mathcal{B}(\mathcal{H})$ belongs to $*$-class A if $|T^2| \geq |T^*|^2$. From [2] and [7], it is well known that

$$\{\text{Hyponormal}\} \subset \{\text{Class } A\} \subset \{\text{Paranormal}\}$$

and

$$\{\text{Hyponormal}\} \subset \{*\text{-class } A\} \subset \{*\text{-paranormal}\}.$$

More recently, the authors of [14] have extended $*$-class A operators to quasi-$*$-class A operators. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be quasi-$*$-class A if $T^*|T^2|T \geq T^*|T^*|^2T$, and quasi-$*$-paranormal if $\|T^*Tx\|^2 \leq \left\|T^3x\right\| \|Tx\|$ for all $x \in \mathcal{H}$. Hence we have the following inclusion:

$$\{\text{Hyponormal}\} \subset \{*\text{-class } A\} \subset \{*\text{-paranormal}\} \subset \{\text{quasi-}*\text{-paranormal}\}.$$

As a further generalization, Mecheri [20] introduced the class of k-quasi-$*$-class A operators. An operator T is said to be a k-quasi-$*$-class A operator if

$$T^{*k}(|T^2| - |T^*|^2)T^k \geq 0,$$

where k is a positive integer number.

For $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$, we say that FP-theorem holds for the pair (T, S) if $TX = XS$ implies $T^*X = XS^*$, $\mathcal{R}(X)$ reduces T, and $\ker(X)^\perp$ reduces S, the restrictions $T|_{\overline{\mathcal{R}(X)}}$ and $S|_{\ker(X)^\perp}$ are unitary equivalent normal operators for all $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. We say that an operator S is quasi-affine transform of an operator T if there exists an injective operator X with dense range such that $TX = XS$. Two operators $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ are quasisimilar if there exist quasiaffinities $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $Y \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $XT = SX$ and $YS = TY$. In general quasisimilarity need not preserve the spectrum and essential spectrum. However, in special classes of operators quasisimilarity preserves spectra. For instance, it is well known that two quasisimilar hyponormal operators have equal spectrum and equal essential spectrum.
Recently in [21, 25, 26, 29, 30, 32], the author investigated some extensions of Fuglede-Putnam theorems involving class A, w-hyponormal, dominant, and spectral operators.

Recall [18] that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to have the single-valued extension property (SVEP) if for every open subset D of $\mathbb{C}$ and any analytic function $f : D \to \mathcal{H}$ such that $(T - z)f(z) \equiv 0$ on D, it results $f(z) \equiv 0$ on D. We say that a Hilbert space operator satisfies Bishop property (β) if, for every open subset D of $\mathbb{C}$ and every sequence $f_n : D \to \mathcal{H}$ of analytic functions with $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of D, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of D. It is well known that,

$$\text{Bishop property } (\beta) \Longrightarrow \text{single-valued extension property (SVEP)},$$

see [4, 17] for further details.

In the present article, we seek some extensions of Fuglede-Putnam type theorems involving k-quasi-$*$-class A and dominant operators. Let U be an open set in $\mathbb{C}$. Stampfli [32] investigated the equation $(T - \lambda I)f(\lambda) \equiv x$ for some non-zero $x \in \mathcal{H}$ and $f : U \to \mathcal{H}$ in an effort to discover necessary and/or sufficient condition for analyticity of f when T is a dominant operator. In this note, we show that if $T \in \mathcal{B}(\mathcal{H})$ be k-quasi-$*$-class A, if $0 \notin \delta \subseteq \mathbb{C}$ be closed, and if there exists a bounded function $f : \mathbb{C} \setminus \delta \to \mathcal{H}$ such that $(T - \lambda I)f(\lambda) \equiv x$ for some nonzero $x \in \mathcal{H}$, then f is analytic at every non zero point and hence f has analytic extension everywere on $\mathbb{C} \setminus \delta$. In section 3, we show that if $T, S \in \mathcal{B}(\mathcal{H})$ are quasisimilar k-quasi-$*$-class A operators, then they have equal spectrum and essential spectrum.

2 Fuglede-Putnam Type Theorems

Throughout this article we would like to present some known results as propositions which will be used in the sequel.

Proposition 2.1. [34] Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$. Then the following assertions are equivalent.

1. If $TX = XS$, where $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $T^*X = XS^*$,
2. If $TX = XS$, where $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $\overline{\mathcal{R}(X)}$ reduces T, $\ker(X)^\perp$ reduces S, the restrictions $T|_{\overline{\mathcal{R}(X)}}$ and $S|_{\ker(S)^\perp}$ are normal.

Proposition 2.2. If $T \in \mathcal{B}(\mathcal{H})$ is a $*$-class A operator, then T is a $*$-paranormal operator

It is well known that a normal part of hyponormal is reducing. This result remains true for $*$-class A operators.

Proposition 2.3. [19, 20, 28, 31] Let $T \in \mathcal{B}(\mathcal{H})$ be $*$-class A operator and let $\mathcal{M}$ be an invariant subspace of T. Then the following assertions hold.

(i) The restriction $T|_{\mathcal{M}}$ is $*$-class A operator.
(ii) If the restriction $T|_{\mathcal{M}}$ is normal, then $\mathcal{M}$ reduces T.

As a consequence of Proposition 2.2 and Theorem 5 of [3], we have

Proposition 2.4. Let T and S be $*$-class A operators and $TX = XS^*$. Then

(i) $\overline{\mathcal{R}(X)}$ reduces T and $\ker(X)$ reduces S.
(ii) $T|_{\overline{\mathcal{R}(X)}}$ and $S^*|_{\ker(S)^\perp}$ are unitarily equivalent normal operators.

Recall from [27] that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a k-quasi-$*$-paranormal operator if

$$\left\|T^*T^k x\right\|^2 \le \left\|T^{k+2}x\right\| \left\|T^k x\right\|$$

for all unit vector $x \in \mathcal{H}$, where k is a positive integer number.

Proposition 2.5. [27, Theorem 2.4] Let $T \in \mathcal{B}(\mathcal{H})$. If T is k-quasi-$*$-class A operator, then T is k-quasi-$*$-paranormal operator

Theorem 2.6. Let $T \in \mathcal{B}(\mathcal{H})$. If T is a k-quasi-$*$-class A with dense range, then T is $*$-class A operator.

Proof . Since T has dense range, $\overline{\mathcal{R}(T^k)} = \mathcal{H}$. Then there exists a sequence $\{x_n\} \subset \mathcal{H}$ such that $\lim\limits_{n\longrightarrow\infty} T^k x_n = y$. Since T is a k-quasi-$*$-class A, we have

$$\begin{aligned} &\left\langle T^k|T^2|T^k x_n, x_n\right\rangle \ge \left\langle T^k|T^*|^2 T^k x_n, x_n\right\rangle \\ &\left\langle |T^2|T^k x_n, T^k x_n\right\rangle \ge \left\langle |T^*|^2 T^k x_n, T^k x_n\right\rangle \text{ for all } n \in \mathbb{N} \end{aligned}$$

By the continuity of the inner product, we have

$$\left\langle (|T^2| - |T^*|^2)y, y\right\rangle \ge 0.$$

Therefore T is a $*$-class A operator. □

Corollary 2.7. Let $T \in \mathcal{B}(\mathcal{H})$. If T is a k-quasi-$*$-class A and not $*$-class A, then T is not invertible.

Corollary 2.8. Suppose that T is non-zero k-quasi-$*$-class A and it has no nontrivial T-invariant closed subspace. Then T is $*$-class A operator.

Proof . Since T has no non-trivial invariant closed subspace, it has no non-trivial hyperinvariant subspace. But $\ker(T^k)$ and $\overline{\mathcal{R}(T^k)}$ are hyperinvariant subspaces, and $T \neq 0$, hence, $\ker(T^k) \neq \mathcal{H}$ and $\overline{\mathcal{R}(T^k)} \neq \{0\}$. Therefore $\ker(T^k) = \{0\}$ and $\overline{\mathcal{R}(T^k)} = \mathcal{H}$. In particular, T has dense range. It follows from Corollary 2.6 that T is $*$-class A operator. □ It is well-known that if T is $*$-class A and a closed subspace $\mathcal{M}$ of $\mathcal{H}$ is T-invariant, then $T|_{\mathcal{M}}$ is $*$-class A. We obtain a similar result for a k-quasi-$*$-class A operator.

Proposition 2.9. The restriction $T|_{\mathcal{M}}$ of a k-quasi-$*$-class A operator T to a T-invariant closed subspace $\mathcal{M}$ of $\mathcal{H}$ is k-quasi-$*$-class A operator.

Proof . Let P be the projection of $\mathcal{H}$ onto $\mathcal{M}$. Thus we can represent T as the following matrix with respect to the decomposition $\mathcal{M} \oplus \mathcal{M}^\perp$,

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

Put $A = T|_{\mathcal{M}}$ and we have

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP.$$

Since T is k-quasi-$*$-class A, we have

$$PT^{*k}(|T^2| - |T^*|^2)T^kP \geq 0.$$

We remark that

$$\begin{aligned} PT^{*k}|T^*|^2T^kP &= PT^{*k}P|T^*|^2PT^kP = PT^{*k}PTT^*PT^kP \\ &= \begin{pmatrix} A^{*k}|A^*|^2A^k + |B^*A^k|^2 & 0 \\ 0 & 0 \end{pmatrix} \\ &\geq \begin{pmatrix} A^{*k}|A^*|^2A^k & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and by Hansen's inequality, we have

$$\begin{aligned} PT^{*k}|T^2|T^kP &= PT^{*k}P(T^{*2}T^2)^{\frac{1}{2}}PT^kP \\ &\leq PT^{*k}(PT^{*2}T^2P)^{\frac{1}{2}}T^kP \\ &= \begin{pmatrix} A^{*k} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |A^2|^2 & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} A^k & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A^{*k} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |A^2| & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^k & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A^{*k}|A^2|A^k & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Then

$$\begin{aligned} \begin{pmatrix} A^{*k}|A^2|A^k & 0 \\ 0 & 0 \end{pmatrix} &\geq PT^{*k}|T^2|T^kP \\ &\geq PT^{*k}|T^*|^2T^kP \geq \begin{pmatrix} A^{*k}|A^*|^2A^k & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and so A is k-quasi-$*$-class A operator on $\mathcal{M}$. □
We give a structure for k-quasi-$*$-class A operators.

Theorem 2.10. [28] Let $T \in \mathcal{B}(\mathcal{H})$ be a k-quasi-$*$-class A operator. If the range of T^k is not dense and

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \ker(T^{*k}),$$

then T_1 is $*$-class A, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

For a $*$-class A operator T we have $\ker(T-\lambda) \subseteq \ker(T-\lambda)^*$ for every $\lambda \in \mathbb{C}$. We have a similar result for k-quasi-$*$-class A under restricted condition on λ as follows.

Theorem 2.11. Suppose that T is a k-quasi-$*$-class A. Then $\ker(T - \alpha) \subseteq \ker(T - \alpha)^*$ for each $\alpha \neq 0$.

Proof . We may assume that $x \neq 0$. Let $\mathcal{M}$ be a span of $\{x\}$. Then $\mathcal{M}$ is an invariant subspace of T and let

$$T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}.$$

Let P be the projection of $\mathcal{H}$ onto $\mathcal{M}$, where $T|_{\mathcal{M}} = \lambda \neq 0$. To end the proof, it is suffices to show that $T_2 = 0$. Since T is k-quasi-$*$-class A operator and $x = T^k\left(\frac{x}{\lambda^k}\right) \in \overline{\mathcal{R}(T^k)}$, we have $P(|T^2| - |T^*|^2)P \geq 0$. By Hansen's inequality, we have

$$\begin{aligned} \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix} &= \left(PT^{*2}T^2P\right)^{\frac{1}{2}} \\ &\geq P|T^2|P \geq P|T^*|^2P = \begin{pmatrix} |\lambda|^2 + |T_2^*|^2 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and so $T_2 = 0$. □ From this theorem we obtain the following corollary.

Corollary 2.12. Suppose that T is a k-quasi-$*$-class A and $\alpha, \beta \in \sigma_p(T) \setminus \{0\}$ with $\alpha \neq \beta$. Then $\ker(T-\alpha) \perp \ker(T-\beta)$.

Proof . Let $x \in \ker(T-\alpha)$ and $y \in \ker(T-\beta)$. Then $Tx = \alpha x$ and $Ty = \beta y$. Therefore

$$\alpha \langle x, y\rangle = \langle \alpha x, y\rangle = \langle Tx, y\rangle = \langle x, T^* y\rangle = \langle x, \overline{\beta} y\rangle = \beta \langle x, y\rangle .$$

Hence $\alpha \langle x, y\rangle = \beta \langle x, y\rangle$ and so $(\alpha-\beta)\langle x, y\rangle = 0$. But $\alpha \neq \beta$, hence $\langle x, y\rangle = 0$. Consequently $\ker(T-\alpha) \perp \ker(T-\beta)$. □

Theorem 2.13. If T is k-quasi-$*$-class A, has the representation $T = \lambda \oplus T_1$ on $\ker(T-\lambda) \oplus \ker(T-\lambda)^\perp$, where $\lambda \neq 0$ is an eigenvalue of T, then T_1 is k-quasi-$*$-class A with $\ker(T_1 - \lambda) = \{0\}$.

Proof . Since $T = \lambda \oplus T_1$, then $T = \begin{pmatrix} \lambda & 0 \\ 0 & T_1 \end{pmatrix}$ and we have

$$\begin{aligned} T^{*k}|T^2|T^k - T^{*k}|T^*|^2T^k &= \begin{pmatrix} |\lambda|^{2(k+1)} & 0 \\ 0 & T_1^{*k}|T_1^2|T_1^k \end{pmatrix} - \begin{pmatrix} |\lambda|^{2(k+1)} & 0 \\ 0 & T_1^{*k}|T_1^*|^2T_1^k \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & T_1^{*k}|T_1^2|T_1^k - T_1^{*k}|T_1^*|^2T_1^k \end{pmatrix} \end{aligned}$$

Since T is k-quasi-$*$-class A, then T_1 is k-quasi-$*$-class A. Let $x \in \ker(T_1 - \lambda)$. Then

$$(T-\lambda)\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & T_1 - \lambda \end{pmatrix}\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $x_2 \in \ker(T_1 - \lambda)$. Since $\ker(T_1-\lambda) \subseteq \ker(T-\lambda)^\perp$ and hence $x_2 = 0$. □

Theorem 2.14. [28] If T is a k-quasi-$*$-class A, then T has Bishop's property (β). Hence T has the single-valued extension property (SVEP).

Lemma 2.15. [31] If the restriction $T|_{\mathcal{M}}$ of the k-quasi-$*$-class A operator $T \in \mathcal{B}(\mathcal{H})$ to an invariant subspace $\mathcal{M}$ is injective and normal, then $\mathcal{M}$ reduces T.

Remark 2.16. The condition $T|_{\mathcal{M}}$ is injective in Lemma 2.15 is indispensable because $\ker(T)$ for k-quasi-$*$-class A operator T is not always reducing.

In [25], the author considered the situation S and T^* are w-hyponormal operators and proved FP-theorem for (S, T) if either S or T is injective. Now we study FP-theorem for the case that T and S^* are k-quasi-$*$-class A operators with the condition that either T or S^* is injective.

Theorem 2.17. Let $T \in \mathcal{B}(\mathcal{H})$ and $S^* \in \mathcal{B}(\mathcal{K})$ be k-quasi-$*$-class A operators such that $TX = XS$ for $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. If T or S^* is injective, then FP-theorem holds for (T, S).

Proof . Suppose T and S^* are k-quasi-$*$-class A operators and $TX = XS$ for any operator $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Since $\overline{\mathcal{R}(X)}$ is invariant under T and $\ker(X)^\perp$ is invariant under S^*, we decompose T, S and X into

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{\mathcal{R}(X)} \oplus \overline{\mathcal{R}(X)}^\perp,$$
$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \text{ on } \mathcal{K} = \ker(X)^\perp \oplus \ker(X),$$

and

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ on } \ker(X)^\perp \oplus \ker(X) \to \overline{\mathcal{R}(X)} \oplus \overline{\mathcal{R}(X)}^\perp,$$

where T_1 and S_1^* are $*$-class A operators by Theorem 2.10, and

$$X_1 : \ker(X)^\perp \to \overline{\mathcal{R}(X)}$$

is injective with dense range. From $TX = XS$, we have

$$T_1X_1 = X_1S_1. \tag{2.1}$$

First consider the case where T is injective. Clearly, T_1 is injective. It is not difficult to show from (2.1) that S_1 is injective or equivalently, $\mathcal{R}(S_1^*)$ is dense. Incidently, S_1^* turns out to be a $*$-class A operator. In particular, $\ker(S_1^*) \subset \ker(S_1)$ and hence $\ker(S_1^*) = \{0\}$. From (2.1), it is easy to see that T_1^* is injective, thereby T_1 is $*$-class A. Next consider the case that S^* is injective. Then S_1^* is injective and so T_1^* is injective by (2.1). Obviously, T_1 is an injective $*$-class A operator, and by (2.1), S_1 is injective. Therefore, S_1^* is $*$-class A. Ultimately, if either T or S^* is injective, then T_1 and S_1^* are both $*$-class A operators. Then by Proposition 2.1, Proposition 2.4 and Equation 2.1, we obtain

$$T_1^*X_1 = X_1S_1^*$$

and T_1, S_1 are normal operators. Since T_1 and S_1 are injective, $T_2 = S_2 = 0$ by Lemma 2.15. Hence

$$T^*X = T_1^*X_1 = X_1S_1^* = XS^*.$$

The rest of the proof follows from Proposition 2.1. □

Corollary 2.18. Let $T \in \mathcal{B}(\mathcal{H})$ and $S^* \in \mathcal{B}(\mathcal{K})$ be k-quasi-$*$-class A operators with reducing kernels. Then FP-theorem holds for (T, S).

Proof . By hypothesis, we can write $T = T_1 \oplus T_2$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $S = S_1 \oplus S_2$ with respect to $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, where T_1 and S_1 are normal parts and T_2 and S_2 are pure parts. Let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \text{ on } \mathcal{K}_1 \oplus \mathcal{K}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2.$$

From $TX = XS$, we have

$$\begin{pmatrix} T_1X_1 & T_1X_2 \\ T_2X_3 & T_2X_4 \end{pmatrix} = \begin{pmatrix} X_1S_1 & X_2S_2 \\ X_3S_1 & X_4S_2 \end{pmatrix}.$$

The underlying kernel conditions ensures of T_2 and S_2^* are injective. The operator T_2 is injective k-quasi-$*$-class A and S_1 normal. From the above matrix relation, we have $T_2X_3 = X_3S_1$. Then by applying Theorem 2.17, we have $T_2^*X_3 = X_3S_1^*$, $\mathcal{R}(X_3)$ reduces T_2 and $T_2|_{\overline{\mathcal{R}(X_3)}}$ is normal and so $X_3 = 0$. In a similar manner we obtain $X_2 = 0$ from $T_1X_2 = X_2S_2$ and $X_4 = 0$ from $T_2X_4 = X_4S_2$. Since T_1 and S_1 are normal and since $T_1X_1 = X_1S_1$, required result follows from classical Fuglede-Putnam theorem and Proposition 2.1. □

Theorem 2.19. If $T^* \in \mathcal{B}(\mathcal{H})$ is $*$-class A, $S \in \mathcal{B}(\mathcal{K})$ is dominant, and if $XT = SX$ for $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then $XT^* = S^*X$.

Proof . From $XT = SX$ we know that $\ker(X)^\perp$ and $\overline{\mathcal{R}(X)}$ are invariant subspaces of T^* and S, respectively. Hence $T^*|_{\ker(X)^\perp}$ is $*$-class A and $S|_{\overline{\mathcal{R}(X)}}$ is also dominant by [36, Lemma 2]. By the decompositions $\mathcal{H} = \ker(X)^\perp \oplus \ker(X)$, $\mathcal{K} = \overline{\mathcal{R}(X)} \oplus \overline{\mathcal{R}(X)}^\perp$, we have

$$T = \begin{pmatrix} T_1 & 0 \\ * & T_2 \end{pmatrix}, \ S = \begin{pmatrix} S_1 & * \\ 0 & S_2 \end{pmatrix}, \ X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here $T_1^* = T|_{\ker(X)^\perp}$ is $*$-class A,$S_1 = S|_{\overline{\mathcal{R}(X)}}$ is dominant and X_1 is injective with dense range. We obtain $X_lT_1 = S_1X_1$ from $XT = SX$. Hence, T_1 and S_1 are normal by Lemma 2.15 and $X_1T_1^* = S_1^*X_1$, by the Famous Putnam-Fuglede theorem. Then, by [36, Lemma 1] and [19, Theorem 2.2] , $\ker(X)^\perp$ and $\overline{\mathcal{R}(X)}$ reduces T^* and S to normal operators, respectively. Therefore, we have

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \ S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}.$$

Hence we obtain $XT^* = S^*X$. □

Now we consider the situation that where T is a k-quasi-$*$-class A operator and S^* is a dominant operator.

Theorem 2.20. Let $T \in \mathcal{B}(\mathcal{H})$ be k-quasi-$*$-class A and let $S^* \in \mathcal{B}(\mathcal{K})$ be dominant such that $TX = XS$ for $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. If T or S^* is injective, then FP-theorem holds for (T, S).

Proof . Suppose that $T \in \mathcal{B}(\mathcal{H})$ be k-quasi-$*$-class A and $S^* \in \mathcal{B}(\mathcal{K})$ is dominant such that $TX = XS$ for $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Since $\mathcal{R}(X)$ is invariant under T and $\ker(X)^\perp$ is invariant under S^*, we can write T, S and X as follows:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{\mathcal{R}(X)} \oplus \overline{\mathcal{R}(X)}^\perp,$$

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \text{ on } \mathcal{K} = \ker(X)^\perp \oplus \ker(X),$$

and

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ on } \ker(X)^\perp \oplus \ker(X) \to \overline{\mathcal{R}(X)} \oplus \overline{\mathcal{R}(X)}^\perp.$$

From $TX = XS$, we have

$$T_1X_1 = X_1S_1, \tag{2.2}$$

where T_1 is $*$-class A by Theorem 2.10, S_1^* is dominant by Lemma 2 of [36] and

$$X_1 : \ker(X)^\perp \to \overline{\mathcal{R}(X)}$$

is injective with dense range. First assume that T is injective. Then , T_1 is injective. From Equation 2.2, S_1 is injective. Since S_1^* is dominant, it turns out to be injective. By Equation 2.2, we have T_1^* is injective. Ultimately, T_1 is $*$-class A. Applying Proposition 2.19 to Equation 2.2, we obtain

$$T_1^*X_1 = X_1S_1^*$$

and T_1, S_1 are normal operators. Since T_1 injective, $T_2 = 0$ by Lemma 2.15. Also $S_2 = 0$ by Proposition 2.3. Next assume S^* is injective. Then S_1^* is injective. Then by Equation 2.2, T_1^* is injective. Ultimately, T_1 turns out to be $*$-class A. Conclude as before that

$$T_1^*X_1 = X_1S_1^*$$

and T_1, S_1 are injective normal operators and so $T_2 = S_2 = 0$. Hence,

$$T^*X = T_1^*X_1 = X_1S_1^* = XS^*.$$

The rest of the proof follows from Proposition 2.1. □

Corollary 2.21. Let $T \in \mathcal{B}(\mathcal{H})$ be dominant and let $S^* \in \mathcal{B}(\mathcal{K})$ be k-quasi-$*$-class A operator such that $TX = XS$ for $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. If T or S^* is injective, then FP-theorem holds for (T, S).

Proof . From $TX = XS$, we have $S^*X^* = X^*T^*$. Applying Theorem 2.20, it follows that $SX^* = X^*T$. The rest of the proof follows from Proposition 2.1. □

Corollary 2.22. Let $T \in \mathcal{B}(\mathcal{H})$ be k-quasi-$*$-class A operator with reducing kernel and let $S^* \in \mathcal{B}(\mathcal{K})$ be dominant operator such that $TX = XS$ for $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Then FP-theorem holds for (T, S).

Proof . Let $T \in \mathcal{B}(\mathcal{H})$ be k-quasi-$*$-class A operator with reducing kernel and let $S^* \in \mathcal{B}(\mathcal{K})$ be dominant operator. We decompose T, S and X as follows:

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ on } \mathcal{H} = \ker(T)^\perp \oplus \ker(T),$$

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ on } \mathcal{K} = \ker(S)^\perp \oplus \ker(S).$$

Let

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \text{ on } \ker(S)^\perp \oplus \ker(S) \to \ker(T)^\perp \oplus \ker(T).$$

From $TX = XS$, we have

$$\begin{pmatrix} T_1X_1 & T_1X_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_1S_1 & 0 \\ X_3S_1 & 0 \end{pmatrix}.$$

The equations $T_1X_2 = 0$ and $X_3S_1 = 0$ yields $X_2 = X_3 = 0$ because T_1 and S_1^* are injective. Applying Theorem 2.20 to $T_1X_1 = X_1S_1$, it follows $T_1^*X_1 = X_1S_1^*$. This achieves the proof. □
Stampfli and Wadhwa [32] proved if T be dominant and S a normal operator and if $TX = XS$ where $X \in \mathcal{B}(\mathcal{H})$ has dense range, then T is a normal operator. This remarkable result for k-quasihyponormal operators has been studied by Gupta and P.B. Ramanujan [9]. Now we show this result remains true for k-quasi-$*$-class A operators.

Theorem 2.23. Let T be a k-quasi-$*$-class A and let S a normal operator. If S is quasi-affine transform of T, then T is a normal operator unitarily equivalent to S.

Proof . Let T be a k-quasi-$*$-class A. By Theorem 2.10, decompose T on $\mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \ker(T^{*k})$ as follows:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

where $T_1 = T|_{\overline{\mathcal{R}(T^k)}}$ is $*$-class A and $T_3^k = 0$. Let $S_1 = S|_{\overline{\mathcal{R}(S^k)}}$. Decompose

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Obviously, S_1 is normal. Let $X_1 = X|_{\overline{\mathcal{R}(S^k)}}$. Then

$$X_1 : \overline{\mathcal{R}(S^k)} \to \overline{\mathcal{R}(T^k)}$$

is injective and has dense range. From $TX = XS$, we have $T_1X_1 = X_1S_1$. Since T_1 is $*$-class A and since S_1 is normal, it follows from [19, Theorem 2.2] that T_1 is normal operator unitary equivalent to S_1. Consequently, $\overline{\mathcal{R}(T^k)}$ reduces T and so $T_2 = 0$ by Lemma 2.15. Since $X^*(\ker(T^{*k})) \subset \ker(S^{*k}) = \ker(S^*)$,

$$X^*T_3^*x = X^*T^*x = S^*X^*x,$$

for each $x \in \ker(T^{*k})$. Since X has dense range, X^* is one to one. Therefore, $T_3^*x = 0$ for each $x \in \ker(T^{*k})$. Hence, $T_3 = 0$ and so $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$ is normal. This achieves the proof. □

The following result proved for hyponormal operators by Radjabalipour [23]. This result for k-quasihyponormal with a condition $0 \notin \delta$ and its consequences has been studied by Gupta [8].

Proposition 2.24. [33] Let $T \in \mathcal{B}(\mathcal{H})$ be dominant. Let $\delta \subset \mathbb{C}$ be closed. If there exists a bounded function $f(z) : \mathbb{C} \setminus \delta \to \mathcal{H}$ such that $(T - zI)f(z) \equiv x$ for some non-zero $x \in \mathcal{H}$, then $f(z)$ is analytic on $\mathbb{C} \setminus \delta$.

In the following theorem, we show this result hold true in the case of k-quasi-$*$-class A operators.

Theorem 2.25. Let $T \in \mathcal{B}(\mathcal{H})$ be k-quasi-$*$-class A and let $0 \notin \delta \subset \mathbb{C}$ be closed. If there exists a bounded function $f(\lambda) : \mathbb{C} \setminus \delta \to \mathcal{H}$ such that $(T - \lambda I)f(\lambda) \equiv x$ for some non-zero $x \in \mathcal{H}$, then f is analytic at every non zero point and hence f has analytic extension everywhere on $\mathbb{C} \setminus \delta$.

Proof . Suppose that T is a k-quasi-$*$-class A. By Theorem 2.10, decompose T on $\mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \ker(T^{*k})$ as follows:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

where T_1 is $*$-class A and $T_3^k = 0$.
Let $f(\lambda) = f_1(\lambda) \oplus f_2(\lambda)$ and $x = x_1 \oplus x_2$ are the decomposition of f and x, respectively. Then

$$\begin{aligned} (T_1 - \lambda I)f_1(\lambda) + T_2f_1(\lambda) &\equiv x_1 \\ (T_3 - \lambda I)f_2(\lambda) &\equiv x_2 \end{aligned}$$

Since $T_3^k = 0$ and since $0 \notin \delta$, $f_2(\lambda) = (T_3 - \lambda I)x_2$ can be extended to a bounded entire function. Since k-quasi-$*$-class A operators satisfies single valued extension property, we conclude $x_2 = 0$ (see, [18, Proposition 1.2.16 9(f)]). Hence $f_2(\lambda) = 0$. Therefore, for all $\lambda \notin \delta$,

$$(T_1 - \lambda I)f(\lambda) \equiv x_1.$$

T_1 is $*$-class A ensures f is analytic at every non zero point and hence f has analytic extension everywhere on $\mathbb{C} \setminus \delta$ by Proposition 2.24. This achieves the proof. □

Definition 2.26. [1] Let $T \in \mathcal{B}(\mathcal{H})$. Then

(i) the spectral manifold (analytic), denoted by $X_T(\delta)$, of an operator T is defined as follows:

$$X_T(\delta) = \{x \in \mathcal{H} : (T - \lambda I)f(\lambda) \equiv x \text{ for some analytic function } f(\lambda) : \mathbb{C} \setminus \delta \to \mathcal{H}\}.$$

(ii) a closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be hyperinvariant of $T \in \mathcal{B}(\mathcal{H})$ if $\mathcal{M}$ is invariant under every operator which commutes with T.

From Theorem 2.25, $X_T(\delta) \neq \{0\}$ for k-quasi-$*$-class A operators and we know by Theorem 2.14 that k-quasi-$*$-class A operators satisfies single valued extension property. The above results yields the following result by the method of [23, Proposition 2].

Corollary 2.27. Let $T \in \mathcal{B}(\mathcal{H})$ be k-quasi-$*$-class A and let $0 \notin \delta \subset \mathbb{C}$ be closed. If there exists a bounded function $f : \mathbb{C} \setminus \delta \to \mathcal{H}$ such that $(T - \lambda I)f \equiv x$ for some non-zero $x \in \mathcal{H}$, then T has non zero hyperinvariant subspace $\mathcal{M}$ with $\sigma(T|_{\mathcal{M}}) \subseteq \delta$. In particular, $\mathcal{M}$ is a nontrivial invariant subspace of T if δ is proper subset of $\sigma(T)$.

3 Quasisimilarity

Recall that an operator $X \in \mathcal{B}(\mathcal{H})$ is called a quasiaffinity if X is injective and has dense range. For $T, S \in \mathcal{B}(\mathcal{H})$, if there exist quasiaffinities X and $Y \in \mathcal{B}(\mathcal{H})$ such that $TX = XS$ and $YT = SY$, then we say that T and S are quasisimilar. It is well-known that in finite dimensional spaces quasiaffinity coincides with similarity; but in infinite dimensional spaces quasiaffinity is a much weaker relation than similarity. Similarity preserves the spectrum and essential spectrum, but this is not true for quasiaffinity. Many researchers have studied what conditions can insure two quasisimilar operators have equal spectrum and essential spectrum. For instance, R. Yingbin and Y. Zikun [35] proved that quasisimilar p-hyponormal operators have equal spectrum and essential spectrum; I. H. Jeon et al. [11] proved that quasisimilar injective p-quasihyponormal operators have equal spectrum and essential spectrum; A. H. Kim [16] proved that quasisimilar (p, k)-quasihyponormal operators have equal spectrum and essential spectrum respectively. Recently, I. H. Jeon et al. [12] proved that quasisimilar quasi-class A operators have equal spectrum and essential spectrum. In the following, we point out that quasisimilar k-quasi-$*$-class A operators also have equal spectrum and essential spectrum.

Proposition 3.1. [19, Proposition 1.1] If T is a $*$-class A operator, then T has Bishop's property (β).

Proposition 3.2. [22] If both T and S have Bishop's property (β) and if they are quasisimilar, then $\sigma(T) = \sigma(S)$ and $\sigma_e(T) = \sigma_e(S)$ hold.

As a consequence of Proposition 3.1 and Proposition 3.2, we have

Corollary 3.3. If T and S are quasisimilar $*$-class A operators, then they have equal spectrum and essential spectrum.

Also, as a consequence of Theorem 2.14 and Proposition 3.2, we have

Corollary 3.4. If T and S are quasisimilar k-quasi-$*$-class A operators, then they have equal spectrum and essential spectrum.

Two operators $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ are densely similar if there exist $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $Y \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $XT = SX$ and $YS = TY$, and are with dense ranges.

Theorem 3.5. If k-quasi-$*$-class A operators $T, S \in \mathcal{B}(\mathcal{H})$ are densely similar, then they have equal essential spectrum.

Proof . Since T and S are k-quasi-$*$-class A operators, both T and S satisfies Bishop property (β). Then by applying [18, Theorem 3.7.13], it follows that they have equal essential spectrum. □

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