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Solving quadratic programming problem via dynamic programming approach

Naghada Saber*, Nejmaddin Sulaiman

Department of Mathematics, College of Education, Salahaddin University-Erbil, Iraq

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Abstract

In this paper, we define the dynamic programming approach to solve quadratic programming problem when the objective function can be written as the product of two linear factors with single linear constraint. An algorithm is proposed for solving such problems, we also solved the problems by simplex method to obtained the exact solution as dynamic programming technique. To demonstrate our proposed method, numerical examples are also illustrated.

Keywords: Quadratic Programming Problem, Dynamic Programming Approach, Optimal Solution.
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1 Introduction

Non-linear programming is an optimization problem in which either the objective function or some of the constraints are nonlinear functions. Quadratic programming (QP) is a mathematical optimization problem with quadratic objective function and linear inequality (or equality) constraints. QP is viewed as a discipline in Operational Research, it is used in the field of Management Science, Health Science and Engineering. Several techniques have been introduced for solving nonlinear QP problems. Some of them are extensions of the simplex method and others are based on different principles. Wolf's method [12], Swarups simplex method [11] and Gupta and Sharma's method [1] are the most popular methods for solving QP problems.

Many researchers working in this field such as [10] are studied a technique for solving and transforming multi-objective quadratic programming problems. In [3], authors proposed an objective separable method based on a simplex method for solving a QP problem where the objective function can be factorized as two linear functions. The main idea is to transform the QP problem into two linear programming problems and then solve each LP by the simplex method. [7] suggested a new technique for solving QP problems having linearly factorized objective function. The idea is to transform the problem into Multi objective LPP and solve it by Chandra Sen's method. [2] developed a computer technique for determining the optimal solution of QP problem having linearly factorized objective function. [9] proposed a new modified simplex method to solve the Quadratic fractional programming problem. Optimal transform techniques to solve multi-objective linear programming problems have been presented by [8]. Moreover, [5] presented a dynamical system approach for solving quadratic programming problems subject to equality constraints. Dynamic programming approach for solving constrained linear-quadratic regulator problems has been proposed by [4].

*Corresponding author
Email addresses: naghada.sabir@su.edu.krd (Naghada Saber), nejmaddin.sulaiman@su.edu.krd (Nejmaddin Sulaiman)

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In this paper, we show how dynamic programming can be used for determining the optimal solution of the QP problem in which the objective function can be written as the product of two linear functions with a single linear constraint. We transform the problem into two linear programming problems.

2 Mathematical Formulation

2.1 Quadratic Programming Problem

The general form of quadratic programming problem states as follows:

$$\max(\text{or } \min) Z = a + C^{'}x + x^{'}Hx$$

subject to:

$$Ax \begin{bmatrix} \geq \\ \leq \\ = \end{bmatrix} b, \quad x \geq 0.$$

here, $A = (a_{ij})_{m\times n}$ is a matrix of coefficients, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. $b = (b_1, b_2, \ldots, b_m)$, $x = (x_1, x_2, \ldots, x_n)$, $C = (c_1, c_2, \ldots, c_n)$ and $H = (h_{ij})_{n\times n}$ is a positive definite or positive semi-definite symmetric square matrix, moreover the objective function is quadratic with linear constraints.

2.2 Dynamic Programming Approach

Dynamic programming is a mathematical technique dealing with the optimization of multistage decision processes. DP technique converts one problem in n variables into n smaller sub-problems, each in one variable. In the terminology of DP, each sub-problem is referred to as a stage. There are two ways of DP backward recursive approach and forward recursive approach. The advantage of DP is to be easier and has more influence than other optimization techniques.

2.3 Bellman's Principle of Optimality

An optimal policy (set of decisions) has the property that whatever the initial state and decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.
Mathematically, this can be written as:

$$f_N(x) = \max_{d_n \in \{x\}} \left[r(d_n) + f_{N-1}\{T(x, d_n)\} \right],$$

where,
$f_N(x)$ = the optimal return from an N-stage process when initial state is x.
$r(d_n)$ = immediate return due to decision d_n,
$T(x, d_n)$ = the transfer function which gives the resulting state,
$\{x\}$ = set of admissible decisions.

We shall consider the implication of this principle as a multi-stage decision problem. It should always be borne in mind that a problem that does not satisfy the principle of optimality cannot be solved by dynamic programming.

2.4 Definition (decomposable)

An optimization problem is said to be decomposable if it can be solved by recursive optimization through N-stage, at each stage optimization, being done over one decision variable. In other words, the validity of the recursive equation

$$F)j(s_j) = \max_{d_j}\{f_j o F_{j-1}(s_{j-1})\}, \quad 2 \geq j \geq N,$$

with $F_1(s_1) = \max_{d_j} f_1$ implies decomposability.

2.5 Definition (monotonic non-decreasing and monotonic non-increasing)

The function $f(x, y)$ is said to be monotonic non-decreasing function of x for all feasible values of y if

$$x_1 > x_2 \Rightarrow f(x_1, y) \geq f(x_2, y)$$

for every feasible value of y it is said to be monotonic non-increasing if

$$x_1 > x_2 \Rightarrow f(x_1, y) \leq f(x_2, y),$$

for every feasible value of y.

Theorem 2.1. In a serial double-stage minimization or maximization problem, if
i. The objective ψ_2 function is a separable function of stage returns $f_1(s_1, d_1)$ and $f_2(s_2, d_2)$, and
ii. ψ_2 is a monotonic non-decreasing function of f_1 for every feasible value of f_2, then the theorem is decomposable.

Theorem 2.2. If the real valued return function $\psi_N(f_N, f_{N-1}, \ldots, f_1)$ satisfies
i. The condition of separability, i.e.
$\psi_N(f_N, f_{N-1}, \ldots, f_1) = f_N o \psi_{N-2}$ where $\psi_N(f_N, f_{N-1}, \ldots, f_1)$ is real-valued; and
ii. ψ_N is a monotonic non-decreasing function of ψ_{N-1} for every f_N, then ψ_N is decomposable, i.e.

$$\max_{d_N, \ldots, d_1} \psi_N(f_N, \ldots, f_1) = \max_{d_N}[f_N o \max_{d_N, \ldots, d_1} \psi_{N-1}]$$

The two theorems prove that the monotonicity is the sufficient condition for decomposability.

3 Objective Solution Techniques

Let as consider the problem of quadratic objective function with single linear constraint:

$$Opt, Z = (\sum_{j=1}^{n} a_j x_j + a)(\sum_{j=1}^{n} b_j x_j + \beta) \tag{3.1}$$

subject to:

$$\sum_{j=1}^{n} c_j x_j \{=, \leq, \geq\} b, \quad \text{and} \quad x \geq 0.$$

First, we construct two linear programming problems as follows:

$$Opt, Z = \sum_{j=1}^{n} b_j x_j + \beta \tag{3.2}$$

subject to:

$$\sum_{j=1}^{n} c_j x_j \{=, \leq, \geq\} b, \quad \text{and} \quad x \geq 0$$

It is possible to apply the dynamic programming approach to each of them. We sequentially proceed to find the optimal policy by considering the last decision first and proceeding backward to the decision.
Algorithm of the proposed method:
Step 1: Convert the original quadratic programming problem into two linear programming problems.
Step 2: Solve each LP problem separately and apply backward recursive approach.
Step 3: obtain the optimal solution for the given problem by storing the solutions of each LP problem.

4 Numerical Examples

Consider the following examples of quadratic programming with single linear constraint.

Example 4.1. Consider the following

$$\begin{aligned}\max Z &= 3x_2^2 + 2x_1x_2 - 10x_1 - 13x_2 - 10\\ &= (2x_1 + 3x_2 + 2)(x_2 - 5)\end{aligned}$$

subject to: $x_1 + x_2 \leq 1$ and $x_1, x_2 \geq 0$.

We construct two linear programming problem as follows

$$\max Z_1 = 2x_1 + 3x_2 + 2 \tag{4.1}$$

subject to: $x_1 + x_2 \leq 1$ and $x_1, x_2 \geq 0$.

$$\max Z_2 = x_2 - 5 \tag{4.2}$$

Subject to: $x_1 + x_2 \leq 1$ and $x_1, x_2 \geq 0$.
From the constraint $x_2 = 1 - x_1, 0 \leq x_1 \leq 1$, and $0 \leq x_2 \leq 1$. From (4.1), we have

$$\begin{aligned}f_2(1) &= \max_{x_2}\{R_2(x_2)\}\\ &= \max_{0\leq x_2\leq 1}\{3x_2\}\\ &= 3(1 - x_1)\end{aligned}$$

and

$$\begin{aligned}f_1(1) &= \max_{x_1}\{R_1(x_1) + f_2(b - x_1)\}\\ &= \max_{0\leq x_1\leq 1}\{2x_1 + 3(1 - x_1)\}\\ &= \max_{0\leq x_1\leq 1}\{3 - x_1\}.\end{aligned}$$

So, if $x_1 = 0$ then max $= 3$, put $x_1 = 0$ in $x_2 = 1 - x_1$ we get $x_2 = 1$. The optimal solution is $(0, 1)$ and max, $Z_1 = 3 + 2 = 5$. For (4.2)

$$\begin{aligned}f_2(1) &= \max_{x_2}\{R_2(x_2)\}\\ &= \max_{0\leq x_2\leq 1}\{x_2\}\\ &= 1 - x_1\end{aligned}$$

and

$$\begin{aligned}f_1(1) &= \max_{x_1}\{R_1(x_1) + f_2(b - x_1)\}\\ &= \max_{0\leq x_1\leq 1}\{0 + 1 - x_1\}\\ &= \max_{0\leq x_1\leq 1}\{1 - x_1\}\end{aligned}$$

So, if $x_1 = 0$, then max $= 1$, put $x_1 = 0$ in $x_2 = 1 - x_1$ we get $x_2 = 1$. The optimal solution is $(0, 1)$ and max, $Z_2 = 1 - 5 = -4$. The optimal solution for the original problem is $(0, 1)$ and max, $Z = -20$.

Example 4.2. Consider the following

$$\begin{aligned}\min Z &= -40x_1^2 - 60x_2^2 - 140x_1x_2 - 60x_1 - 80x_2 - 20\\ &= (5x_1 + 15x_2 + 5)(-8x_1 - 4x_2 - 4)\end{aligned}$$

subject to: $2x_1 + 3x_2 \leq 6$ and $x_1, x_2 \leq 2$.

From the constraint $x_2 = 2 - \frac{2}{3}x_1, 0 \leq x_1 \leq 3$ and $0 \leq x_2 \leq 2$, for (4.1), we have

$$\begin{aligned} f_2(6) &= \min_{x_2}\{R_2(x_2)\} \\ &= \min_{0\leq x_2\leq 2}(15x_2) \\ &= 15\min_{0\leq x_2\leq 2}\{2 - \frac{2}{3}x_1\} \\ &= 15(2 - \frac{2}{3}x_1) \end{aligned}$$

and

$$\begin{aligned} f_1(6) &= \min_{x_1}\{R_1(x_1) + f_2(b - x_1)\} \\ &= \min_{0\leq x_1\leq 3}\{5x_1 + 15(2 - \frac{2}{3}x_1)\} \\ &= \min_{0\leq x_1\leq 3}\{30 - 5x_1\}. \end{aligned}$$

So, if $x_1 = 3$, then min $= 15$ put $x_1 = 3$ in $x_2 = 2 - \frac{2}{3}x_1$ we get $x_2 = 0$. The optimal solution is $(3, 0)$ and $\min, Z_1 = 15 + 5 = 20$. For (4.1), we have

$$\begin{aligned} f_2(6) &= \min_{x_2}\{R_2(x_2)\} \\ &= \min_{0\leq x_2\leq 2}\{-4x_2\} \\ &= -4\min_{0\leq x_2\leq 2}\{2 - \frac{2}{3}x_1\} \\ &= -4(2 - \frac{2}{3}x_1) \end{aligned}$$

and

$$\begin{aligned} f_1(6) &= \min_{x_1}\{R_1(x_1) + f_2(b - x_1)\} \\ &= \min_{0\leq x_1\leq 3}\{-8x_1 - 4(2 - \frac{2}{3}x_1)\} \\ &= \min_{0\leq x_1\leq 3}\{-8 - \frac{16}{3}x_1\}. \end{aligned}$$

So, if $x_1 = 3$, then min $= -24$ put $x_1 = 3$ in $x_2 = 2 - \frac{2}{3}x_1$ we get $x_2 = 0$. The optimal solution is $(3, 0)$ and $\min Z_2 = -24 - 4 = -28$. The optimal solution for the original problem is $(3, 0)$ and $\min Z = -560$.

The table below show us the comparison result between simplex method and our technique, we obtained the same result.

Table 1: Comparison of numerical results

Examples	Simplex method	Dynamic programming technique
Example(4.1)	$x_1 = 3, x_2 = 0, \max Z = -20$	$x_1 = 3, x_2 = 0, \max Z = -20$
Example(4.2)	$x_1 = 3, x_2 = 0, \max Z = -560$	$x_1 = 3, x_2 = 0, \max Z = -560$

5 Conclusion

A dynamic programming approach is proposed for solving quadratic programming problems where the objective function can be written as the product of two linear functions. After solving the numerical examples by the traditional simplex method, we found that the optimal solution obtained by the DP approach is an exact solution. Further, this work can be extended to multi constrained quadratic programming problems.

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