Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 789–802
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2020.20566.2172

Coupled fixed points of generalized rational type $\mathcal{Z}$-contraction maps in b-metric spaces

Velisela Amarendra Babu[a], Dasari Ratna Babu[b], Nasina Siva Prasad[c,d,*]

[a] *Department of Mathematics, Acharya Nagarjuna University, Guntur - 522 510, India*
[b] *Department of Mathematics, PSCMRCET, Vijayawada - 520 001, India*
[c] *Department of Mathematics, Rayalaseema University, Kurnool - 518 007, India*
[d] *Department of Mathematics, PBR VITS, Kavali- 524 201, India*

(Communicated by Abdolrahman Razani)

Abstract

In this paper, we introduce generalized rational type $\mathcal{Z}$-contraction maps for a single map $f : X \times X \to X$ where X is a b-metric space and prove the existence and uniqueness of coupled fixed points. We extend it to a pair of maps by defining generalized rational type $\mathcal{Z}$-contraction pair of maps and prove the existence of common coupled fixed points in complete b-metric spaces. We provide examples in support of our results.

Keywords: coupled fixed points, b-metric space, generalized rational type $\mathcal{Z}$-contraction maps
2020 MSC: 47H10, 54H25

1 Introduction

Banach contraction principle plays an important role in solving nonlinear functional analysis. In the direction of generalization of contraction condition, Dass and Gupta [13] initiated a contraction condition involving rational expression and established the existence of fixed points in complete metric spaces.

In the direction of generalization of metric spaces, Bourbaki [10] and Bakhtin [5] initiated the idea of b-metric spaces. The concept of b-metric space or metric type space was introduced by Czerwik [11] as a generalization of metric space. Afterwards, many authors studied the existence of fixed points for a single-valued and multi-valued mappings in b-metric spaces under certain contraction conditions. For more details, we refer [2, 3, 8, 9, 12, 14, 16, 18, 23, 24].

In 2006, Bhaskar and Lakshmikantham [6] introduced the notion of coupled fixed point and established the existence of coupled fixed points for mixed monotone mappings in ordered metric spaces. Later, Lakshmikantham and Ćirić [19] introduced the notion of coupled coincidence points of mappings in two variables. Afterwards, many authors studied coupled fixed point theorems, we refer [20, 22, 25, 26].

*Corresponding author
Email addresses: amarendravelisela@ymail.com (Velisela Amarendra Babu), ratnababud@gmail.com (Dasari Ratna Babu), n.sivaprasadmsc@gmail.com (Nasina Siva Prasad)

Received: April 2020 *Accepted:* June 2020

2 Preliminaries

Definition 2.1. [11] Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied: for any $x, y, z \in X$

(i) $0 \le d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) there exists $s \ge 1$ such that $d(x, z) \le s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space with coefficient s.

Every metric space is a b-metric space with $s = 1$. In general, every b-metric space is not a metric space.

Definition 2.2. [9] Let (X, d) be a b-metric space.

(i) A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.
(ii) A sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
(iii) A b-metric space (X, d) is said to be a complete b-metric space if every b-Cauchy sequence in X is b-convergent in X.
(iv) A set $B \subset X$ is said to be b-closed if for any sequence $\{x_n\}$ in B such that $\{x_n\}$ is b-convergent to $z \in X$ then $z \in B$.

In general, a b-metric is not necessarily continuous.

In this paper, we denote $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{N}$ is the set of all natural numbers.

Example 2.3. [15] Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d : X \times X \to \mathbb{R}^+$ as follows:

$$d(m, n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m, n \text{ is odd and the other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then (X, d) is a b-metric space with coefficient $s = \frac{5}{2}$.

Definition 2.4. [9] Let (X, d_X) and (Y, d_Y) be two b-metric spaces. A function $f : X \to Y$ is a b-continuous at a point $x \in X$, if it is b-sequentially continuous at x. i.e., whenever $\{x_n\}$ is b-convergent to x we have fx_n is b-convergent to fx.

Definition 2.5. [6] Let X be a nonempty set and $f : X \times X \to X$ be a mapping. Then we say that an element $(x, y) \in X \times X$ is a coupled fixed point, if $f(x, y) = x$ and $f(y, x) = y$.

Definition 2.6. [19] Let X be a nonempty set. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. An element $(x, y) \in X \times X$ ia called

(i) a coupled coincidence point of the mappings F and g if $F(x, y) = gx$ and $F(y, x) = gy$;
(ii) a common coupled fixed point of mappings F and g if $F(x, y) = gx = x$ and $F(y, x) = gy = y$.

The following lemma is useful in proving our main results.

Lemma 2.7. [1] Let (X, d) be a b-metric space with coefficient $s \ge 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively. Then we have

$$\frac{1}{s^2} d(x, y) \le \liminf_{n\to\infty} d(x_n, y_n) \le \limsup_{n\to\infty} d(x_n, y_n) \le s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover for each $z \in X$ we have

$$\frac{1}{s} d(x, z) \le \liminf_{n\to\infty} d(x_n, z) \le \limsup_{n\to\infty} d(x_n, z) \le s d(x, z).$$

In 2015, Khojasteh, Shukla and Radenović [17] introduced simulation function and defined $\mathcal{Z}$-contraction with respect to a simulation function.

Definition 2.8. [17] A simulation function is a mapping $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ satisfying the following conditions:
(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$ for all $s, t > 0$;
(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Remark 2.9. [4] Let ζ be a simulation function. If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$ then $\limsup_{n\to\infty} \zeta(kt_n, s_n) < 0$ for any $k > 1$.

The following are examples of simulation functions.

Example 2.10. [4] Let $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ be defined by
(i) $\zeta(t,s) = \lambda s - t$ for all $t, s \in \mathbb{R}^+$, where $\lambda \in [0,1)$;
(ii) $\zeta(t,s) = \frac{s}{1+s} - t$ for all $s, t \in \mathbb{R}^+$;
(iii) $\zeta(t,s) = s - kt$ for all $t, s \in \mathbb{R}^+$, where $k > 1$;
(iv) $\zeta(t,s) = \frac{1}{1+s} - (1+t)$ for all $s, t \in \mathbb{R}^+$;
(v) $\zeta(t,s) = \frac{1}{k+s} - t$ for all $s, t \in \mathbb{R}^+$ where $k > 1$.

Definition 2.11. [17] Let (X,d) be a metric space and $f : X \to X$ be a selfmap of X. We say that f is a $\mathcal{Z}$-contraction with respect to ζ if there exists a simulation function ζ such that

$$\zeta(d(fx, fy), d(x,y)) \geq 0 \quad \text{for all} \quad x, y \in X.$$

Theorem 2.12. [17] Let (X,d) be a complete metric space and $f : X \to X$ be a $\mathcal{Z}$-contraction with respect to a certain simulation function ζ. Then for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges in X and $\lim_{n\to\infty} f^n x_0 = u$(say) in X and u is the unique fixed point of f in X.

Recently, Olgun, Bicer and Alyildiz [21] proved the following result in complete metric spaces.

Theorem 2.13. [21] Let (X,d) be a complete metric space and $f : X \to X$ be a selfmap on X. If there exists a simulation function ζ such that

$$\zeta(d(fx, fy), M(x,y)) \geq 0$$

for all $x, y \in X$, where $M(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\}$,then for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges in X and $\lim_{n\to\infty} f^n x_0 = u$(say) in X and u is the unique fixed point of f in X.

In 2018, Babu, Dula and Kumar [4] extended Theorem 1.13 [21] to pair of selfmaps in the setting of b-metric spaces as follows.

Theorem 2.14. [4] Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and $f, g : X \to X$ be a selfmaps on X. If there exists a simulation function ζ such that

$$\zeta(s^4 d(fx, gy), M(x,y)) \geq 0$$

for all $x, y \in X$, where $M(x,y) = \max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2s}\}$, then f and g have a unique common fixed point in X, provided either f or g is b-continuous.

Recently, Bindu and Malhotra [7] proved the existence of common coupled fixed points as follows:

Theorem 2.15. Let (X,d) be a complete b-metric space with parameter $s \geq 1$ and let the mappings $S, T : X \times X \to X$ satisfy

$$\begin{aligned} d(S(x,y), T(u,v)) \leq\ & \alpha_1 \frac{d(x,u)+d(y,v)}{2} + \alpha_2 \frac{d(x,S(x,y))d(u,T(u,v))}{1+d(x,u)+d(y,v)+d(u,S(x,y))} + \alpha_3 \frac{d(u,S(x,y))d(x,T(u,v))}{1+d(x,u)+d(y,v)+d(u,S(x,y))} \\ & + \alpha_4 \frac{d(S(x,y),T(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,S(x,y))} + \alpha_5 \frac{d(S(x,y),T(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,S(x,y))} \\ & + \alpha_6 \frac{d(u,T(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,S(x,y))} + \alpha_7 \frac{d(u,S(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,S(x,y))} \\ & + \alpha_8 \frac{d(u,S(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,S(x,y))} + \alpha_9 \max d(u, S(x,y)), d(S(x,y), T(u,v)) \end{aligned}$$

for all $x, y, u, v \in X$ and $\alpha_i \geq 0, i = 1, 2, \ldots, 9$ with $s\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + s\alpha_9 < 1$ and $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7 + \alpha_8 + \alpha_9 < 1$. Then S and T have a unique common coupled fixed point in X.

Motivated by the works of Bindu and Malhotra [7], in Section 3, we introduce generalized rational type $\mathcal{Z}$-contraction maps for a single map $f : X \times X \to X$ where X is a b-metric space and we extend it to a pair of maps. In Section 4, we prove the existence and uniqueness of coupled fixed points and common coupled fixed points in complete b-metric spaces. Examples are provided in support of our results in Section 5.

3 Generalized rational type $\mathcal{Z}$-contraction maps

The following we introduce generalized rational type $\mathcal{Z}$-contraction maps for a single and a pair of maps in b-metric spaces as follows:

Definition 3.1. Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f : X \times X \to X$ be a map. We say that f is a generalized rational type $\mathcal{Z}$-contraction map, if there exists a simulation function ζ such that

$$\zeta(s^3 d(f(x,y), f(u,v)), M(x,y,u,v)) \geq 0 \text{ for all } x,y,u,v \in X, \tag{3.1}$$

where

$$M(x,y,u,v) = \max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),f(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \max\{d(u,f(x,y)), d(f(x,y),f(u,v))\}\}.$$

Remark 3.2. It is clear that from definition of simulation function that $\zeta(a,b) < 0$, for all $a \geq b > 0$. Therefore if f satisfies (3.1), then

$$s^3 d(f(x,y), f(u,v)) < M(x,y,u,v), \text{ for all } x,y,u,v \in X.$$

Example 3.3. Let $X = [0,1]$ and let $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

Then clearly (X,d) is a b-metric space with coefficient $s = 2$. We define $f : X \times X \to X$ by $f(x,y) = \frac{\log(1+x^2+y^2)}{16}$ for all $x \in [0,1]$ and

$$\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty) \quad \text{by} \quad \zeta(t,s) = \frac{1}{2}s - t, t \geq 0, s \geq 0.$$

We have
$$\begin{aligned} s^3 d(f(x,y), f(u,v)) &= 8[\tfrac{\log(1+x^2+y^2)}{16} + \tfrac{\log(1+u^2+v^2)}{16}]^2 \\ &\leq \tfrac{1}{8}[(x+u)^2 + (y+v)^2] \\ &= \tfrac{1}{2}(\tfrac{d(x,u)+d(y,v)}{2s}) \\ &\leq \tfrac{1}{2}(\max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(u,f(x,y))d(x,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),f(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(f(x,y),f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \max\{d(u,f(x,y)), d(f(x,y),f(u,v))\}\}). \end{aligned}$$

Therefore f is a generalized rational type $\mathcal{Z}$-contraction map.

Definition 3.4. Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f,g : X \times X \to X$ be two maps. We say that the pair (f,g) is a generalized rational type $\mathcal{Z}$-contraction maps, if there exists a simulation function ζ such that

$$\zeta(s^3 d(f(x,y), g(u,v)), M(x,y,u,v)) \geq 0, \text{ for all } x,y,u,v \in X, \tag{3.2}$$

where

$$M(x,y,u,v) = \max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),g(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \max\{d(u,f(x,y)), d(f(x,y),g(u,v))\}\}.$$

Remark 3.5. It is clear that from definition of simulation function that $\zeta(a,b)<0$, for all $a\geq b>0$. Therefore if f satisfies (3.2), then

$$s^3 d(f(x,y),g(u,v)) < M(x,y,u,v), \text{ for all } x,y,u,v\in X.$$

Example 3.6. Let $X=[0,1]$ and let $d: X\times X\to\mathbb{R}^+$ defined by

$$d(x,y)=\begin{cases}0 & \text{if } x=y\\ (x+y)^2 & \text{if } x\neq y.\end{cases}$$

Then clearly (X,d) is a b-metric space with coefficient $s=2$. We define $f,g: X\times X\to X$ by

$$f(x,y)=\begin{cases}\frac{\log(1+x+y)}{16} & \text{if } x,y\in[0,\frac{1}{2})\\ \frac{1}{32} & \text{if } x,y\in[\frac{1}{2},1]\end{cases} \text{ and } g(x,y)=\begin{cases}\frac{xe^y}{8} & \text{if } x,y\in[0,\frac{1}{2})\\ \log(x+y) & \text{if } x,y\in[\frac{1}{2},1].\end{cases}$$

$\zeta:\mathbb{R}^+\times\mathbb{R}^+\to(-\infty,\infty)$ by $\zeta(t,s)=\frac{99}{100}s-t, t\geq 0, s\geq 0.$

Case (i). $x,y,u,v\in[0,\frac{1}{2})$.

$$\begin{aligned} s^3 d(f(x,y),g(u,v)) &= 8[\tfrac{\log(1+x+y)}{16}+\tfrac{ue^v}{8}]^2\\ &\leq \tfrac{99}{400}[(x+u)^2+(y+v)^2]\\ &= \tfrac{99}{100}(\tfrac{d(x,u)+d(y,v)}{2s})\\ &\leq \tfrac{99}{100}(\max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\quad \tfrac{d(f(x,y),g(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\quad \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\quad \max\{d(u,f(x,y)),d(f(x,y),g(u,v))\}\}).\end{aligned}$$

Case (ii). $x,y,u,v\in[\frac{1}{2},1]$.

$$\begin{aligned} s^3 d(f(x,y),g(u,v)) &= 8[\tfrac{1}{32}+\log(u+v)]^2\\ &\leq \tfrac{99}{400}[(x+u)^2+(y+v)^2]\\ &= \tfrac{99}{100}(\tfrac{d(x,u)+d(y,v)}{2s})\\ &\leq \tfrac{99}{100}(\max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\quad \tfrac{d(f(x,y),g(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\quad \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\quad \max\{d(u,f(x,y)),d(f(x,y),g(u,v))\}\})\end{aligned}$$

Case (iii). $x,y\in[\frac{1}{2},1], u,v\in[0,\frac{1}{2})$.

$$\begin{aligned} s^3 d(f(x,y),g(u,v)) &= 8[\tfrac{1}{16}+\tfrac{ue^v}{4}]^2\\ &\leq \tfrac{99}{400}[(x+u)^2+(y+v)^2]\\ &= \tfrac{99}{100}(\tfrac{d(x,u)+d(y,v)}{2s})\\ &\leq \tfrac{99}{100}(\max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\quad \tfrac{d(f(x,y),g(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\quad \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\quad \max\{d(u,f(x,y)),d(f(x,y),g(u,v))\}\})\end{aligned}$$

Case (iv). $x,y\in[0,\frac{1}{2}), u,v\in[\frac{1}{2},1]$.

$$\begin{aligned} s^3 d(f(x,y),g(u,v)) &= 8[\tfrac{\log(1+x+y)}{8}+\log(x+y)]^2\\ &\leq \tfrac{99}{400}[(x+u)^2+(y+v)^2]\\ &= \tfrac{99}{100}(\tfrac{d(x,u)+d(y,v)}{2s})\\ &\leq \tfrac{99}{100}(\max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\quad \tfrac{d(f(x,y),g(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\quad \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\quad \max\{d(u,f(x,y)),d(f(x,y),g(u,v))\}\}).\end{aligned}$$

Therefore the pair (f,g) is a generalized rational type $\mathcal{Z}$-contraction maps.

4 Main results

Theorem 4.1. Let (X,d) be a complete b-metric space with coefficient $s\geq 1$ and $f: X\times X\to X$ be a rational type $\mathcal{Z}$-contraction map. Then f has a unique coupled fixed point in X.

Proof . Let x_0 and y_0 be arbitrary points in X. We define $x_{i+1} = f(x_i, y_i)$ and $y_{i+1} = f(y_i, x_i)$ for $i = 0, 1, 2, \ldots$. We consider

$$\zeta(s^3 d(x_{n+1}, x_{n+2}), M(x_n, y_n, x_{n+1}, y_{n+1})) = \zeta(s^3 d(f(x_n, y_n), f(x_{n+1}, y_{n+1})), M(x_n, y_n, x_{n+1}, y_{n+1})) \geq 0, \tag{4.1}$$

where

$$\begin{aligned} M(x_n, y_n, x_{n+1}, y_{n+1}) = \max\{ & \tfrac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2s}, \tfrac{d(x_n,f(x_n,y_n))d(x_{n+1},f(x_{n+1},y_{n+1}))}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},f(x_n,y_n))}, \\ & \tfrac{d(x_{n+1},f(x_n,y_n))d(x_n,f(x_{n+1},y_{n+1}))}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},f(x_n,y_n))}, \tfrac{d(f(x_n,y_n),f(x_{n+1},y_{n+1}))d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},f(x_n,y_n))}, \\ & \tfrac{d(f(x_n,y_n),f(x_{n+1},y_{n+1}))d(y_n,y_{n+1})}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},f(x_n,y_n))}, \tfrac{d(x_{n+1},f(x_{n+1},y_{n+1}))d(y_n,y_{n+1})}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},f(x_n,y_n))}, \\ & \tfrac{d(x_{n+1},f(x_n,y_n))d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},f(x_n,y_n))}, \tfrac{d(x_{n+1},f(x_n,y_n))d(y_n,y_{n+1})}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},f(x_n,y_n))}, \\ & \max\{d(x_{n+1}, f(x_n, y_n)), d(f(x_n, y_n), f(x_{n+1}, y_{n+1}))\}\} \\ = \max\{ & \tfrac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2s}, \tfrac{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},x_{n+1})}, \\ & \tfrac{d(x_{n+1},x_{n+1})d(x_n,x_{n+2})}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},x_{n+1})}, \tfrac{d(x_{n+1},x_{n+2})d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},x_{n+1})}, \\ & \tfrac{d(x_{n+1},x_{n+2})d(y_n,y_{n+1})}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},x_{n+1})}, \tfrac{d(x_{n+1},x_{n+2})d(y_n,y_{n+1})}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},x_{n+1})}, \\ & \tfrac{d(x_{n+1},x_{n+1})d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},x_{n+1})}, \tfrac{d(x_{n+1},x_{n+1})d(y_n,y_{n+1})}{1+d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(x_{n+1},x_{n+1})}, \\ & \max\{d(x_{n+1}, x_{n+1}), d(x_{n+1}, x_{n+2})\}\} \\ \leq \max\{ & \tfrac{d(x_n,x_{n+1})+d(y_n,y_{n+1})}{2s}, d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

If $M(x_n, y_n, x_{n+1}, y_{n+1}) = d(x_{n+1}, x_{n+2})$ then from (4.1), we have

$$\begin{aligned} 0 \leq \zeta(s^3 d(x_{n+1}, x_{n+2}), M(x_n, y_n, x_{n+1}, y_{n+1})) &= \zeta(s^3 d(x_{n+1}, x_{n+2}), d(x_{n+1}, x_{n+2})) \\ &< d(x_{n+1}, x_{n+2}) - s^3 d(x_{n+1}, x_{n+2}), \end{aligned}$$

which is a contradiction. Therefore

$$d(x_{n+1}, x_{n+2}) \leq \frac{d(x_n, x_{n+1}) + d(y_n, y_{n+1})}{2s} \tag{4.2}$$

for all $n = 0, 1, 2, \ldots$. Similarly, we can prove that

$$d(y_{n+1}, y_{n+2}) \leq \frac{d(y_n, y_{n+1}) + d(x_n, x_{n+1})}{2s} \tag{4.3}$$

for all $n = 0, 1, 2, \ldots$. Adding the inequalities (4.2) and (4.3), we have

$$d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) \leq h[d(x_n, x_{n+1}) + d(y_n, y_{n+1})],$$

where $h = \frac{1}{2s} < 1$. Also, it is easy to see that

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq h[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)].$$

Therefore

$$d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) \leq h^2[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)].$$

Continuing in the same way, we get that

$$d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) \leq h^n[d(x_0, x_1) + d(y_0, y_1)].$$

For $m > n, m, n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_m) + d(y_n, y_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] + s[d(y_n, y_{n+1}) + d(y_{n+1}, y_m)] \\ &\leq s[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &\quad + s^2[d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_m)] \\ &= s[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + s^2[d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})] \\ &\quad + s^2[d(x_{n+2}, x_m) + d(y_{n+2}, y_m)] \ldots + s^{m-n}[d(x_{m-1}, x_m) + d(y_{m-1}, y_m)] \\ &\leq [sh^n + s^2 h^{n+1} + \ldots + s^{m-n} h^{m-1}][d(x_0, x_1) + d(y_0, y_1)] \\ &\leq sh^n[1 + sh + (sh)^2 \ldots + (sh)^{m-1} + \ldots][d(x_0, x_1) + d(y_0, y_1)] \\ &= sh^n(\tfrac{1}{1-sh})[d(x_0, x_1) + d(y_0, y_1)] \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore $\{x_n\}$ and $\{y_n\}$ are b-Cauchy sequences in X. Since X is b-complete, there exist $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. We now prove that $x = f(x, y)$ and $y = f(y, x)$. On the contrary suppose that $x \neq f(x, y)$ and $y \neq f(y, x)$. We now consider

$$\zeta(s^3 d(f(x, y), x_{n+1}), M(x, y, x_n, y_n)) = \zeta(s^3 d(f(x, y), f(x_n, y_n)), M(x, y, x_n, y_n)) \geq 0, \tag{4.4}$$

where $M(x,y,x_n,y_n) = \max\{\frac{d(x,x_n)+d(y,y_n)}{2s}, \frac{d(x,f(x,y))d(x_n,f(x_n,y_n))}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))},$
$\frac{d(x_n,f(x,y))d(x,f(x_n,y_n))}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))}, \frac{d(f(x,y),f(x_n,y_n))d(x,x_n)}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))},$
$\frac{d(f(x,y),f(x_n,y_n))d(y,y_n)}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))}, \frac{d(x_n,f(x_n,y_n))d(y,y_n)}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))},$
$\frac{d(x_n,f(x,y))d(x,x_n)}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))}, \frac{d(x_n,f(x,y))d(y,y_n)}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))},$
$\max\{d(x_n,f(x,y)), d(f(x,y),f(x_n,y_n))\}\}$
$= \max\{\frac{d(x,x_n)+d(y,y_n)}{2s}, \frac{d(x,f(x,y))d(x_n,x_{n+1})}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))},$
$\frac{d(x_n,f(x,y))d(x,x_{n+1})}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))}, \frac{d(f(x,y),x_{n+1})d(x,x_n)}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))},$
$\frac{d(f(x,y),x_{n+1})d(y,y_n)}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))}, \frac{d(x_n,x_{n+1})d(y,y_n)}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))},$
$\frac{d(x_n,f(x,y))d(x,x_n)}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))}, \frac{d(x_n,f(x,y))d(y,y_n)}{1+d(x,x_n)+d(y,y_n)+d(x_n,f(x,y))},$
$\max\{d(x_n,f(x,y)), d(f(x,y),x_{n+1})\}\}.$

On taking limit superior as $n\to\infty$ in $M(x,y,x_n,y_n)$, we have

$$\limsup_{n\to\infty} M(x,y,x_n,y_n) \le sd(x,f(x,y)).$$

On letting limit superior as $n\to\infty$ in (4.4) and using the Lemma 2.7, we have
$0 \le \limsup_{n\to\infty} \zeta(s^3 d(f(x,y),x_{n+1}), M(x,y,x_n,y_n))$
$= \limsup_{n\to\infty} M(x,y,x_n,y_n) - \liminf_{n\to\infty} s^3 d(f(x,y),x_{n+1})$
$\le d(x,f(x,y)) - s^3 \frac{d(x,f(x,y))}{s},$
a contradiction. Therefore $x = f(x,y)$. Similarly we can prove that $y = f(y,x)$. Therefore (x,y) is a coupled fixed point of f. Let $(x',y') \in X\times X$ be another coupled fixed point of f with $(x',y') \ne (x,y)$. We consider

$$\zeta(s^3 d(x,x'), M(x,y,x',y')) = \zeta(s^3 d(f(x,y),f(x',y')), M(x,y,x',y')) \ge 0,$$

where
$M(x,y,x',y') = \max\{\frac{d(x,x')+d(y,y')}{2s}, \frac{d(x,f(x,y))d(x',f(x',y'))}{1+d(x,x')+d(y,y')+d(x',f(x,y))}, \frac{d(x',f(x,y))d(x,f(x',y'))}{1+d(x,x')+d(y,y')+d(x',f(x,y))},$
$\frac{d(f(x,y),f(x',y'))d(x,x')}{1+d(x,x')+d(y,y')+d(x',f(x,y))}, \frac{d(f(x,y),f(x',y'))d(y,y')}{1+d(x,x')+d(y,y')+d(x',f(x,y))},$
$\frac{d(x',f(x',y'))d(y,y')}{1+d(x,x')+d(y,y')+d(x',f(x,y))}, \frac{d(x',f(x,y))d(x,x')}{1+d(x,x')+d(y,y')+d(x',f(x,y))},$
$\frac{d(x',f(x,y))d(y,y')}{1+d(x,x')+d(y,y')+d(x',f(x,y))}, \max\{d(x',f(x,y)), d(f(x,y),f(x',y'))\}\}$
$\le \max\{\frac{d(x,x')+d(y,y')}{2s}, d(x,x')\}.$
If $M(x,y,x',y') = d(x,x')$ then we have

$$\zeta(s^3 d(x,x'), M(x,y,x',y')) = d(x,x') - s^3 d(x,x') \ge 0,$$

which is a contradiction. Therefore

$$d(x,x') \le \frac{d(x,x')+d(y,y')}{2s}. \tag{4.5}$$

Similarly, we can prove that

$$d(y,y') \le \frac{d(x,x')+d(y,y')}{2s}.. \tag{4.6}$$

Adding the inequalities (4.5) and (4.6), we get that

$$d(x,x')+d(y,y') \le \frac{d(x,x')+d(y,y')}{2s} < d(x,x')+d(y,y')$$

it is a contradiction. Therefore $(x,y) = (x',y')$ is the unique coupled fixed point of f in X. □

Proposition 4.2. Let (X,d) be a b-metric space with coefficient $s\ge 1$ and $f,g: X\times X\to X$ be two maps. Assume that the pair (f,g) is generalized rational type $\mathcal{Z}$-contraction maps. Then (u,v) is a coupled fixed point of f if and only if (u,v) is a coupled fixed point of g. Moreover, (u,v) is unique in this case.

Proof . Let (u,v) be a coupled fixed point of f. Then $u = f(u,v)$ and $v = f(v,u)$. Suppose that $u \ne g(u,v)$. We now consider

$$\zeta(s^3 d(u,g(u,v)), M(u,v,u,v)) = \zeta(s^3 d(f(u,v),g(u,v)), M(u,v,u,v)) \ge 0, \tag{4.7}$$

where
$M(u,v,u,v) = \max\{\frac{d(u,u)+d(v,v)}{2s}, \frac{d(u,f(u,v))d(u,g(u,v))}{1+d(u,u)+d(v,v)+d(u,f(u,u))}, \frac{d(u,f(u,v))d(u,g(u,v))}{1+d(u,u)+d(v,v)+d(u,f(u,u))},$
$\frac{d(f(u,v),g(u,v))d(u,u)}{1+d(u,u)+d(v,v)+d(u,f(u,u))}, \frac{d(f(u,v),g(u,v))d(v,v)}{1+d(u,u)+d(v,v)+d(u,f(u,u))}, \frac{d(u,g(u,v))d(v,v)}{1+d(u,u)+d(v,v)+d(u,f(u,u))},$
$\frac{d(u,f(u,v))d(u,u)}{1+d(u,u)+d(v,v)+d(u,f(u,u))}, \frac{d(u,f(u,v))d(v,v)}{1+d(u,u)+d(v,v)+d(u,f(u,u))},$
$\max\{d(u,f(u,v)), d(f(u,v),g(u,v))\}\} = d(u,g(u,v)).$

From the inequality (4.7), we have

$$0 \leq \zeta(s^3 d(u,g(u,v)), M(u,v,u,v)) = d(u,g(u,v)) - s^3 d(u,g(u,v)),$$

which is a contradiction. Therefore $u = g(u,v)$. Similarly, we can prove that $v = g(v,u)$. Hence, (u,v) is a coupled fixed point of g.

In the similar lines as above, it is easy to see that (u,v) is a coupled fixed point of f whenever (u,v) is a coupled fixed point of g. Let $(u,v),(u',v') \in X \times X$ be two coupled fixed points of f and g with $(u.v) \neq (u',v')$. We consider

$$\zeta(s^3 d(u,u'), M(u,v,u',v')) = \zeta(s^3 d(f(u,v), g(u',v')), M(u,v,u',v')) \geq 0,$$

where
$M(u,v,u',v') = \max\{\frac{d(u,u')+d(v,v')}{2s}, \frac{d(u,f(u,v))d(u',g(u',v'))}{1+d(u,u')+d(v,v')+d(u',f(u,v))}, \frac{d(u',f(u,v))d(x,g(u',v'))}{1+d(u,u')+d(v,v')+d(u',f(u,v))},$
$\frac{d(f(u,v),g(u',v'))d(u,u')}{1+d(u,u')+d(v,v')+d(u',f(u,v))}, \frac{d(f(u,v),g(u',v'))d(v,v')}{1+d(u,u')+d(v,v')+d(u',f(u,v))},$
$\frac{d(u',g(u',v'))d(v,v')}{1+d(u,u')+d(v,v')+d(u',f(u,v))}, \frac{d(u',f(u,v))d(u,u')}{1+d(u,u')+d(v,v')+d(u',f(u,v))},$
$\frac{d(u',f(u,v))d(v,v')}{1+d(u,u')+d(v,v')+d(u',f(u,v))}, \max\{d(u',f(u,v)), d(f(u,v),g(u',v'))\}\}$
$\leq \max\{\frac{d(u,u')+d(v,v')}{2s}, d(u,u')\}.$

If $M(u,v,u',v') = d(u,u')$ then we have

$$\zeta(s^3 d(u,u'), M(u,v,u',v')) = d(u,u') - s^3 d(u,u') \geq 0,$$

which is a contradiction. Therefore

$$d(u,u') \leq \frac{d(u,u') + d(v,v')}{2s}. \tag{4.8}$$

Similarly, we can prove that

$$d(v,v') \leq \frac{d(u,u') + d(v,v')}{2s}.. \tag{4.9}$$

Adding the inequalities (4.8) and (4.9), we get that

$$d(u,u') + d(v,v') \leq \frac{d(u,u') + d(v,v')}{2s} < d(u,u') + d(v,v'),$$

it is a contradiction.
Therefore $(u,v) = (u',v')$ is the unique coupled fixed point of f and g in X. □

Theorem 4.3. Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and the pair (f,g) be a generalized rational type $\mathcal{Z}$-contraction maps. Then f and g have a unique coupled fixed point in X.

Proof . Let x_0 and y_0 be arbitrary points in X. We define $x_{2i+1} = f(x_{2i}, y_{2i}), y_{2i+1} = f(y_{2i}, x_{2i})$ and $x_{2i+2} = g(x_{2i+1}, y_{2i+1})$,$y_{2i+2} = g(y_{2i+1}, x_{2i+1})$ for $i = 0,1,2,\ldots$. We consider

$$\begin{aligned}\zeta(s^3 d(x_{2n+1}, x_{2n+2}), M(x_{2n}, y_{2n}, x_{2n+1}, y_{2n+1})) = \zeta(s^3 d(f(x_{2n}, y_{2n}), g(x_{2n+1}, y_{2n+1})),\\ M(x_{2n}, y_{2n}, x_{2n+1}, y_{2n+1})) \geq 0,\end{aligned} \tag{4.10}$$

where
$M(x_{2n}, y_{2n}, x_{2n+1}, y_{2n+1}) = \max\{\frac{d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})}{2s},$
$\frac{d(x_{2n},f(x_{2n},y_{2n}))d(x_{2n+1},g(x_{2n+1},y_{2n+1}))}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},f(x_{2n},y_{2n}))},$
$\frac{d(x_{2n+1},f(x_{2n},y_{2n}))d(x_{2n},g(x_{2n+1},y_{2n+1}))}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},f(x_{2n},y_{2n}))},$
$\frac{d(f(x_{2n},y_{2n}),g(x_{2n+1},y_{2n+1}))d(x_{2n},x_{2n+1})}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},f(x_{2n},y_{2n}))},$
$\frac{d(f(x_{2n},y_{2n}),g(x_{2n+1},y_{2n+1}))d(y_{2n},y_{2n+1})}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},f(x_{2n},y_{2n}))},$

$$
\begin{aligned}
&\frac{d(x_{2n+1},g(x_{2n+1},y_{2n+1}))d(y_{2n},y_{2n+1})}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},f(x_{2n},y_{2n}))},\\
&\frac{d(x_{2n+1},f(x_{2n},y_{2n}))d(x_{2n},x_{2n+1})}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},f(x_{2n},y_{2n}))},\\
&\frac{d(x_{2n+1},f(x_{2n},y_{2n}))d(y_{2n},y_{2n+1})}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},f(x_{2n},y_{2n}))},\\
&\max\{d(x_{2n+1},f(x_{2n},y_{2n})),d(f(x_{2n},y_{2n}),g(x_{2n+1},y_{2n+1}))\}\}\\
=\max\{&\frac{d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})}{2s},\\
&\frac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})+d(x_{2n+1},x_{2n+1})}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},x_{2n+1})},\\
&\frac{d(x_{2n+1},x_{2n+1})d(x_{2n},x_{2n+2})}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},x_{2n+1})},\\
&\frac{d(x_{2n+1},x_{2n+2})d(x_{2n},x_{2n+1})}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},x_{2n+1})},\\
&\frac{d(x_{2n+1},x_{2n+2})d(y_{2n},y_{2n+1})}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},x_{2n+1})},\\
&\frac{d(x_{2n+1},x_{2n+2})d(y_{2n},y_{2n+1})}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},x_{2n+1})},\\
&\frac{d(x_{2n+1},x_{2n+1})d(x_{2n},x_{2n+1})}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},x_{2n+1})},\\
&\frac{d(x_{2n+1},x_{2n+1})d(y_{2n},y_{2n+1})}{1+d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})+d(x_{2n+1},x_{2n+1})},\\
&\max\{d(x_{2n+1},x_{2n+1}),d(x_{2n+1},x_{2n+2})\}\}\\
\le \max\{&\frac{d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})}{2s},d(x_{2n+1},x_{2n+2})\}.
\end{aligned}
$$

If $M(x_{2n},y_{2n},x_{2n+1},y_{2n+1})=d(x_{2n+1},x_{2n+2})$ then from (4.10), we have

$$
\begin{aligned}
0\le \zeta(s^3d(x_{2n+1},x_{2n+2}),M(x_{2n},y_{2n},x_{2n+1},y_{2n+1}))&=\zeta(s^3d(x_{2n+1},x_{2n+2}),d(x_{2n+1},x_{2n+2}))\\
&<d(x_{2n+1},x_{2n+2})-s^3d(x_{2n+1},x_{2n+2}),
\end{aligned}
$$

which is a contradiction. Therefore

$$d(x_{2n+1},x_{2n+2})\le \frac{d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})}{2s} \tag{4.11}$$

for all $n=0,1,2,\ldots$.

Similarly, we can prove that

$$d(y_{2n+1},y_{2n+2})\le \frac{d(y_{2n},y_{2n+1})+d(x_{2n},x_{2n+1})}{2s} \tag{4.12}$$

for all $n=0,1,2,\ldots$. Adding the inequalities (4.11) and (4.12), we have

$$d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2})\le h[d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})],$$

where $h=\frac{1}{2s}<1$. Also, it is easy to see that

$$d(x_{2n+2},x_{2n+3})+d(y_{2n+2},y_{2n+3})\le h[d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2})].$$

Therefore

$$d(x_n,x_{n+1})+d(y_n,y_{n+1})\le h[d(x_{n-1},x_n)+d(y_{n-1},y_n)],$$

for all $n=1,2,3,\ldots$. Continuing in the same way, we get that

$$d(x_n,x_{n+1})+d(y_n,y_{n+1})\le h^n[d(x_0,x_1)+d(y_0,y_1)].$$

For $m>n, m,n\in\mathbb{N}$, we have

$$
\begin{aligned}
d(x_n,x_m)+d(y_n,y_m)&\le s[d(x_n,x_{n+1})+d(x_{n+1},x_m)]+s[d(y_n,y_{n+1})+d(y_{n+1},y_m)]\\
&\le s[d(x_n,x_{n+1})+d(y_n,y_{n+1})]+s^2[d(x_{n+1},x_{n+2})+d(x_{n+2},x_m)]\\
&\quad +s^2[d(y_{n+1},y_{n+2})+d(y_{n+2},y_m)]\\
&= s[d(x_n,x_{n+1})+d(y_n,y_{n+1})]+s^2[d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})]\\
&\quad +s^2[d(x_{n+2},x_m)+d(y_{n+2},y_m)]\ldots+s^{m-n}[d(x_{m-1},x_m)+d(y_{m-1},y_m)]\\
&\le [sh^n+s^2h^{n+1}+\ldots+s^{m-n}h^{m-1}][d(x_0,x_1)+d(y_0,y_1)]\\
&\le sh^n[1+sh+(sh)^2\ldots+(sh)^{m-1}+\ldots][d(x_0,x_1)+d(y_0,y_1)]\\
&= sh^n(\tfrac{1}{1-sh})[d(x_0,x_1)+d(y_0,y_1)]\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

Therefore $\{x_n\}$ and $\{y_n\}$ are b-Cauchy sequences in X. Since X is b-complete, there exist $x,y\in X$ such that $x_n\to x$ and $y_n\to y$ as $n\to\infty$. We now prove that $x=f(x,y)$ and $y=f(y,x)$. On the contrary suppose that $x\neq f(x,y)$ and $y\neq f(y,x)$. We now consider

$$
\begin{aligned}
\zeta(s^3d(f(x,y),x_{2n+2}),M(x,y,x_{2n+1},y_{2n+1}))=\zeta(s^3d(f(x,y),g(x_{2n+1},y_{2n+1})),&\\
M(x,y,x_{2n+1},y_{2n+1}))\ge 0,&
\end{aligned} \tag{4.13}
$$

where

$$\begin{aligned}M(x,y,x_{2n+1},y_{2n+1}) = \max\{&\tfrac{d(x,x_{2n+1})+d(y,y_{2n+1})}{2s}, \tfrac{d(x,f(x,y))d(x_{2n+1},g(x_{2n+1},y_{2n+1}))}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))},\\ &\tfrac{d(x_{2n+1},f(x,y))d(x,g(x_{2n+1},y_{2n+1}))}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))}, \tfrac{d(f(x,y),g(x_{2n+1},y_{2n+1}))d(x,x_{2n+1})}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))},\\ &\tfrac{d(f(x,y),g(x_{2n+1},y_{2n+1}))d(y,y_{2n+1})}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))}, \tfrac{d(x_{2n+1},g(x_{2n+1},y_{2n+1}))d(y,y_{2n+1})}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))},\\ &\tfrac{d(x_{2n+1},f(x,y))d(x,x_{2n+1})}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))}, \tfrac{d(x_{2n+1},f(x,y))d(y,y_{2n+1})}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))},\\ &\max\{d(x_{2n+1},f(x,y)), d(f(x,y),g(x_{2n+1},y_{2n+1}))\}\}\\ = \max\{&\tfrac{d(x,x_{2n+1})+d(y,y_{2n+1})}{2s}, \tfrac{d(x,f(x,y))d(x_{2n+1},x_{2n+2})}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))},\\ &\tfrac{d(x_{2n+1},f(x,y))d(x,x_{2n+2})}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))}, \tfrac{d(f(x,y),x_{2n+2})d(x,x_{2n+1})}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))},\\ &\tfrac{d(f(x,y),x_{2n+2})d(y,y_{2n+1})}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))}, \tfrac{d(x_{2n+1},x_{2n+2})d(y,y_{2n+1})}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))},\\ &\tfrac{d(x_{2n+1},f(x,y))d(x,x_{2n+1})}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))}, \tfrac{d(x_{2n+1},f(x,y))d(y,y_{2n+1})}{1+d(x,x_{2n+1})+d(y,y_{2n+1})+d(x_{2n+1},f(x,y))},\\ &\max\{d(x_{2n+1},f(x,y)), d(f(x,y),x_{2n+2})\}\}.\end{aligned}$$

On taking limit superior as $n \to \infty$ in $M(x,y,x_n,y_n)$ and using Lemma 2.7, we have

$$\limsup_{n\to\infty} M(x,y,x_n,y_n) \leq sd(x,f(x,y)).$$

On letting limit superior as $n \to \infty$ in (4.13) and using the Lemma 2.7, we have

$$\begin{aligned}0 &\leq \limsup_{n\to\infty} \zeta(s^3 d(f(x,y),x_{2n+2}), M(x,y,x_{2n+1},y_{2n+1}))\\ &= \limsup_{n\to\infty} M(x,y,x_{2n+1},y_{2n+1}) - \liminf_{n\to\infty} s^3 d(f(x,y),x_{2n+2})\\ &\leq sd(x,f(x,y)) - s^3 \tfrac{d(x,f(x,y))}{s},\end{aligned}$$

a contradiction. Therefore $x = f(x,y)$. Similarly we can prove that $y = f(y,x)$. Therefore (x,y) is a coupled fixed point of f. By Proposition 4.2, we have (x,y) is a unique common coupled fixed point of f and g in X. □

5 Corollaries and examples

Corollary 5.1. Let (X,d) be a complete b-metric space with coefficient $s \geq 1$. $f : X \times X \to X$ be two maps. Assume that there exist two continuous functions $\varphi, \psi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t \leq \psi(t)$ for all $t > 0$ and $\varphi(t) = \psi(t) = 0$ if and only if $t = 0$ such that

$$\psi(s^d(f(x,y),f(u,v))) \leq \varphi(M(x,y,u,v))$$

where

$$\begin{aligned}M(x,y,u,v) = \max\{&\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\tfrac{d(f(x,y),f(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\max\{d(u,f(x,y)), d(f(x,y),f(u,v))\}\}, \text{ for all } x,y,u,v \in X.\end{aligned}$$

Then f has a unique common coupled fixed point in X.

Proof . Follows from Theorem 4.1 by choosing $\zeta(s,t) = \varphi(t) - \psi(t)$ for all $t, s \in [0,\infty)$. □

Corollary 5.2. Let (X,d) be a complete b-metric space with coefficient $s \geq 1$. $f, g : X \times X \to X$ be two maps. Assume that there exist two continuous functions $\varphi, \psi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t \leq \psi(t)$, for all $t > 0$ and $\varphi(t) = \psi(t) = 0$ if and only if $t = 0$ such that

$$\psi(s^d(f(x,y),g(u,v))) \leq \varphi(M(x,y,u,v))$$

where

$$\begin{aligned}M(x,y,u,v) = \max\{&\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\tfrac{d(f(x,y),g(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))},\\ &\max\{d(u,f(x,y)), d(f(x,y),g(u,v))\}\}, \text{ for all } x,y,u,v \in X.\end{aligned}$$

Then f and g have a unique common coupled fixed point in X.

Proof . Follows by taking $g = f$ in Corollary 5.1. □

The following is an example in support of Theorem 4.1.

Example 5.3. Let $X = [0,1]$ and let $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

Then clearly (X,d) is a b-metric space with coefficient $s = 2$. We define $f, g : X \times X \to X$ by

$$f(x,y) = \begin{cases} \frac{x^2+y^2}{16} & \text{if } x,y \in [0,\frac{1}{2}) \\ \frac{1}{32} & \text{if } x,y \in [\frac{1}{2},1] \\ 0 & \text{otherwise} \end{cases}$$

$\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty,\infty)$ by $\zeta(t,s) = \frac{99}{100}s - t, t \geq 0, s \geq 0$.

Case (i). $x,y,u,v \in [0,\frac{1}{2})$.

$$\begin{aligned} s^3 d(f(x,y),g(u,v)) &= 8[\tfrac{x^2+y^2}{16} + \tfrac{u^2+v^2}{16}]^2 \\ &\leq \tfrac{99}{400}[(x+u)^2 + (y+v)^2] \\ &= \tfrac{99}{100}(\tfrac{d(x,u)+d(y,v)}{2s}) \\ &\leq \tfrac{99}{100}(\max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(f(x,y),f(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \max\{d(u,f(x,y)), d(f(x,y),f(u,v))\}\}). \end{aligned}$$

Case (ii). $x,y,u,v \in [\frac{1}{2},1]$.

$$\begin{aligned} s^3 d(f(x,y),g(u,v)) &= 8[\tfrac{1}{32} + \tfrac{1}{32}]^2 \\ &\leq \tfrac{99}{400}[(x+u)^2 + (y+v)^2] \\ &= \tfrac{99}{100}(\tfrac{d(x,u)+d(y,v)}{2s}) \\ &\leq \tfrac{99}{100}(\max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(f(x,y),f(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \max\{d(u,f(x,y)), d(f(x,y),f(u,v))\}\}). \end{aligned}$$

Case (iii). $x,y \in [\frac{1}{2},1], u,v \in [0,\frac{1}{2})$.

$$\begin{aligned} s^3 d(f(x,y),g(u,v)) &= 8[\tfrac{1}{32} + \tfrac{u^2+v^2}{16}]^2 \\ &\leq \tfrac{99}{400}[(x+u)^2 + (y+v)^2] \\ &= \tfrac{99}{100}(\tfrac{d(x,u)+d(y,v)}{2s}) \\ &\leq \tfrac{99}{100}(\max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(f(x,y),f(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \max\{d(u,f(x,y)), d(f(x,y),f(u,v))\}\}). \end{aligned}$$

Case (iv). $x,y \in [0,\frac{1}{2}), u,v \in [\frac{1}{2},1]$.

$$\begin{aligned} s^3 d(f(x,y),g(u,v)) &= 8[\tfrac{x^2+y^2}{16} + \tfrac{1}{32}]^2 \\ &\leq \tfrac{99}{400}[(x+u)^2 + (y+v)^2] \\ &= \tfrac{99}{100}(\tfrac{d(x,u)+d(y,v)}{2s}) \\ &\leq \tfrac{99}{100}(\max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,f(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(f(x,y),f(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \max\{d(u,f(x,y)), d(f(x,y),f(u,v))\}\}). \end{aligned}$$

Therefore f satisfy all the hypothesis of Theorem 4.1 and $(0,0)$ is a unique coupled fixed point of f.

The following is an example in support of Theorem 4.3.

Example 5.4. Let $X = [0,1]$ and let $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

Then clearly (X,d) is a b-metric space with coefficient $s = 2$. We define $f, g : X \times X \to X$ by

$$f(x,y) = \begin{cases} \frac{\log(1+x+y)}{16} & \text{if } x,y \in [0,\frac{1}{2}) \\ \frac{1}{32} & \text{if } x,y \in [\frac{1}{2},1] \end{cases}$$

and

$$g(x,y) = \begin{cases} \frac{xye^{xy}}{8} & \text{if } x,y \in [0,\frac{1}{2}) \\ \log(x+y) & \text{if } x,y \in [\frac{1}{2},1]. \end{cases}$$

$\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty,\infty)$ by $\zeta(t,s) = \frac{99}{100}s - t, t \geq 0, s \geq 0$.

Case (i). $x,y,u,v \in [0,\frac{1}{2})$.

$$\begin{aligned} s^3 d(f(x,y),g(u,v)) &= 8[\tfrac{\log(1+x+y)}{16} + \tfrac{uve^{uv}}{8}]^2 \\ &\leq \tfrac{99}{400}[(x+u)^2+(y+v)^2] \\ &= \tfrac{99}{100}(\tfrac{d(x,u)+d(y,v)}{2s}) \\ &\leq \tfrac{99}{100}(\max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(f(x,y),g(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \max\{d(u,f(x,y)),d(f(x,y),g(u,v))\}\}). \end{aligned}$$

Case (ii). $x,y,u,v \in [\frac{1}{2},1]$.

$$\begin{aligned} s^3 d(f(x,y),g(u,v)) &= 8[\tfrac{1}{32} + \log(u+v)]^2 \\ &\leq \tfrac{99}{400}[(x+u)^2+(y+v)^2] \\ &= \tfrac{99}{100}(\tfrac{d(x,u)+d(y,v)}{2s}) \\ &\leq \tfrac{99}{100}(\max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(f(x,y),g(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \max\{d(u,f(x,y)),d(f(x,y),g(u,v))\}\}) \end{aligned}$$

Case (iii). $x,y \in [\frac{1}{2},1], u,v \in [0,\frac{1}{2})$.

$$\begin{aligned} s^3 d(f(x,y),g(u,v)) &= 8[\tfrac{1}{16} + \tfrac{uve^{uv}}{8}]^2 \\ &\leq \tfrac{99}{400}[(x+u)^2+(y+v)^2] \\ &= \tfrac{99}{100}(\tfrac{d(x,u)+d(y,v)}{2s}) \\ &\leq \tfrac{99}{100}(\max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(f(x,y),g(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \max\{d(u,f(x,y)),d(f(x,y),g(u,v))\}\}) \end{aligned}$$

Case (iv). $x,y \in [0,\frac{1}{2}), u,v \in [\frac{1}{2},1]$.

$$\begin{aligned} s^3 d(f(x,y),g(u,v)) &= 8[\tfrac{\log(1+x+y)}{8} + \log(x+y)]^2 \\ &\leq \tfrac{99}{400}[(x+u)^2+(y+v)^2] \\ &= \tfrac{99}{100}(\tfrac{d(x,u)+d(y,v)}{2s}) \\ &\leq \tfrac{99}{100}(\max\{\tfrac{d(x,u)+d(y,v)}{2s}, \tfrac{d(x,f(x,y))d(u,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(x,g(u,v))}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(f(x,y),g(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(f(x,y),g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,g(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \tfrac{d(u,f(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \tfrac{d(u,f(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,f(x,y))}, \\ &\quad \max\{d(u,f(x,y)),d(f(x,y),g(u,v))\}\}) \end{aligned}$$

Therefore the pair (f,g) satisfy all the hypotheses of Theorem 4.3 and $(0,0)$ is a unique common coupled fixed point of f and g.

References

[1] A. Aghajani, M. Abbas and J. R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces*, Math. Slovaca **64** (2014), no. 4, 941–960.

[2] H. Aydi, M-F. Bota, E. Karapinar and S. Mitroviç, *A fixed point theorem for set-valued quasi contractions in b-metric spaces*, Fixed Point Theory Appl. **88** (2012), 8 pages.

[3] H. Aydi, M-F. Bota, E. Karapinar and S. Moradi, *A common fixed point for weak ϕ-contractions on b-metric spaces*, Fixed Point Theory **13** (2012), no. 2, 337–346.

[4] G.V.R. Babu, T.M. Dula and P.S. Kumar, *A common fixed point theorem in b-metric spaces via simulation function*, J. Fixed Point Theory **12** (2018), 15 pages.

[5] I.A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal. Gos. Ped. Inst. Unianowsk **30** (1989), 26–37.

[6] T.G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65** (2006), 1379–1393.

[7] B. Bindu and N. Malhotra, *Common coupled fixed point for generalized rational type contractions in b-metric spaces*, J. Nonlinear Analy. Appl. **2018** (2018), no. 2, 201–211.

[8] M. Boriceanu, *Strict fixed point theorems for multivalued operators in b-metric spaces*, Int. J. Mod. Math. **4** (2009), no. 3, 285–301.

[9] M. Boriceanu, M.-F. Bota and A. Petrusel, *Multivalued fractals in b-metric spaces*, Cent. Eur. J. Math. **8** (2010), no. 2, 367-377.

[10] N. Bourbaki, *Topologie generale*, Herman: Paris, France, 1974.

[11] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis **1** (1993), 5–11.

[12] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena **46** (1998), 263–276.

[13] B.K. Dass and S. Gupta, *An extension of Banach contraction principle through rational expressions*, Indian J. Pure Appl. Math. **6** (1975), 1455–1458.

[14] H. Huang, L. Paunoviç and S. Radenoviç, *On some fixed point results for rational Geraghty contractive mappings in ordered b-metric spaces*, J. Nonlinear Sci. Appl. **8** (2015), 800–807.

[15] N. Hussain, V. Paraneh, J.R. Roshan and Z. Kadelburg, *Fixed points of cycle weakly (ψ, φ, L, A, B)-contractive mappings in ordered b-metric spaces with applications*, Fixed Point Theory Appl. **2013** (2013), 256, 18 pages.

[16] N. Hussain, J.R. Roshan, V. Parvaneh and M. Abbas, *Common fixed point results for weak contractive mappings in ordered b-dislocated metric spaces with applications*, J. Inequal. Appl. **2013** (2013), 486, 21 pages.

[17] F. Khojasteh, S. Shukla and S. Redenović, *A new approach to the study fixed point theorems via simulation functions*, Filomat **29** (2015), no. 6, 1189–1194.

[18] P. Kumam and W. Sintunavarat, *The existence of fixed point theorems for partial q-set valued quasi-contractions in b-metric spaces and related results*, Fixed Point Theory Appl. **2014** (2014), 226, 20 pages.

[19] V. Lakshmikantham and L. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70** (2009), 4341–4349.

[20] N. Malhotra and B. Bansal, *Some common coupled fixed point theorems for generalized contraction in b-metric spaces*, J. Nonlinear Sci. Appl. **8** (2015), 8–16.

[21] M. Olgun, O. Bicer and T. Alyildiz, *A new aspect to Picard operators with simulation functions*, Turk. J. Math. **40** (2016), 832–837.

[22] N.S. Prasad, D.R. Babu and V.A. Babu, *Common coupled fixed points of generalized contraction maps in b-metric spaces*, Electron. J. Math. Anal. Appl. 9 (2021), no. 1, 131–150.

[23] R.J. Shahkoohi and A. Razani, *Some fixed point theorems for rational Geraghty contractive mappings in ordered*

b-metric spaces, J. Inequal. Appl. **2014** (2014), no. 1, 1–23.

[24] W. Shatanawi, *Fixed and common fixed point for mappings satisfying some nonlinear contractions in b-metric spaces*, J. Math. Anal. **7** (2016), no. 4, 1–12.

[25] W. Shatanawi and M.B. Hani, *A coupled fixed point theorem in b-metric spaces*, Int. J. Pure Appl. Math. **109** (2016), no. 4, 889–897.

[26] W. Shatanawi, B. Samet and M. Abbas, *Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces*, Math. Comp. Model. **55** (2012), 680–687.