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Monotone α-nonexpansive mapping in ordered Banach space by AU-iteration algorithm with application to delay differential equation

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Abstract

In this paper, we adopt the AU-iteration scheme introduced by Udofia et. al. [25] (U. E. Udofia, A. E. Ofem, and D. I. Igbokwe, Convergence Analysis for a New Faster Four Steps Iterative Algorithm with an Application, Open J. Math. Anal. **5** (2021), no. 2, 95–112) to approximate the fixed point of monotone α-nonexpansive mappings in ordered Banach space. Analytically and with a numerical example we show that this iteration process converges faster than some well known existing iteration processes in literature. Further, we apply the AU-iteration process to find the unique solutions of a delayed differential equation.

Keywords: Monotone, α-Nonexpansive mappings, Ordered Banach Space, Fixed point, Contraction map, Delay Differential Equation
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1 Introduction

Throughout this paper, let ϖ be an ordered Banach space with the norm $\|\cdot\|$ and the partial order $\leq$. Let $F(\Gamma) = \{u \in \varpi : \Gamma u = u\}$ denote the set of all fixed points of a mapping $\Gamma : \varpi \longrightarrow \varpi$.

Let Γ be a mapping with domain $D(\Gamma)$ and range $R(\Gamma)$ in an ordered Banach space ϖ endowed with the partial order $\leq$, and ζ a nonempty closed convex subset of ϖ. Then, $\Gamma : D(\Gamma) \longrightarrow R(\Gamma)$ is said to be:

(1) monotone [6] if

$$\Gamma u \leq \Gamma v \ \ \forall\, u, v \in D(\Gamma) \ \text{ with } \ u \leq v, \tag{1.1}$$

(2) monotone nonexpansive [6], if Γ is monotone and

$$\|\Gamma u - \Gamma v\| \leq \|u - v\| \ \ \forall\, u, v \in D(\Gamma) \ \text{ with } \ u \leq v. \tag{1.2}$$

Remark 1.1. If Γ is does not satisfy the monotone condition, then Γ is said to be nonexpansive [14].

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(3) monotone quasi-nonexpansive [23] if Γ is monotone and there exists $F(\Gamma) \neq \emptyset$ such that

$$\|\Gamma u - q\| \leq \|u - q\|, \tag{1.3}$$

$\forall q \in F(\Gamma)$ and $u \in \zeta$, with $u \leq q$ or $u \geq q$.

(4) α-nonexpansive [5] if for some $\alpha < 1$

$$\|\Gamma u - \Gamma v\|^2 \leq \alpha\|\Gamma u - v\|^2 + \alpha\|\Gamma v - u\|^2 + (1 - 2\alpha)\|u - v\|, \forall u, v \in \zeta. \tag{1.4}$$

Remark 1.2. If Γ is monotone in (1.4), then Γ is said to be monotone α-nonexpansive [23].

In 2011, Aoyama and Kohsaka [5] introduced the class of α-nonexpansive mappings in Banach spaces (see definition 4) and obtained fixed point theorems for such mappings with a non constructive argument. However, Ariza-Riuz *et al.* [4] showed that the concept of α-nonexpansive is trivial for $\alpha < 0$.

In 2015, Bachar and Khamsi [6] introduced monotone nonexpansive mapping and studied common approximate fixed points of a monotone nonexpansive semigroup.

In 2016, Song *et al.* [23] introduced the concept of monotone α-nonexpansive mappings in ordered Banach space and extended the notion of α-nonexpansive mapping to monotone α-nonexpansive mapping in ordered Banach spaces, obtained some existence and convergence theorems for the Mann iteration under some suitable conditions.

The Mann iteration is a fundamental iteration method used in approximating fixed points of nonexpansive mappings introduced by Mann [14] in 1953 and is defined by:

$$s_{n+1} = (1 - a_n)s_n + a_n\Gamma s_n, n \geq 1 \tag{1.5}$$

where $\{a_n\} \subset (0, 1)$ and Γ a nonexpansive mapping.

Another widely used iteration method in approximating fixed point of nonexpansive mapping is the Ishikawa iteration scheme introduced by Ishikawa [13] in 1976 and defined by:

$$\begin{aligned} s_{n+1} &= (1 - a_n)s_n + a_n\Gamma u_n \\ u_n &= (1 - b_n)s_n + b_n\Gamma s_n, \end{aligned} \tag{1.6}$$

where $\{a_n\}, \{b_n\} \subset (0, 1)$, $n \in N$. In 2000, Noor [16] modified (1.6) and further introduced a three-step iteration process to solve the general variational inequalities problem and is defined by:
For an arbitrary $s_1 \in \zeta$, define a sequence $\{s_n\}$ by

$$\begin{aligned} s_{n+1} &= (1 - a_n)s_n + a_n\Gamma u_n, \\ u_n &= (1 - b_n)s_n + b_n\Gamma v_n, \\ v_n &= (1 - c_n)s_n + c_n\Gamma s_n \end{aligned} \tag{1.7}$$

where $\{a_n\}, \{b_n\}, \{c_n\} \subset (0, 1)$, $n \in N$. In 2007, Agarwal et al. [2] modified (1.6) to introduce the following two-step iteration process called the S-iteration:
For an arbitrary $s_1 \in K$, define a sequence $\{s_n\}$ by

$$\begin{aligned} s_{n+1} &= (1 - a_n)\Gamma s_n + a_n\Gamma u_n, \\ u_n &= (1 - b_n)s_n + b_n\Gamma s_n, \end{aligned} \tag{1.8}$$

where $\{a_n\}, \{b_n\} \subset (0, 1)$, $n \in N$. They claimed that the iteration (1.8) converges faster than the Mann iteration for some contractions.

In 2014, Abbas and Nazir [1] introduced the following three-step iteration process:
For an arbitrary $s_1 \in \zeta$, define a sequence $\{s_n\}$ by

$$\begin{aligned} s_{n+1} &= (1 - a_n)\Gamma v_n + a_n\Gamma u_n, \\ u_n &= (1 - b_n)\Gamma s_n + b_n\Gamma v_n \\ v_n &= (1 - c_n)s_n + c_n\Gamma s_n \end{aligned} \tag{1.9}$$

where $\{a_n\}, \{b_n\}, \{c_n\} subset(0,1)$, $n \in N$. Shahin et al. [22] in 2015 showed that the coefficients $\{a_n\}, \{b_n\}, \{c_n\}$ in iteration process (1.9) plays an essential role in the convergence rate. Infact, if $1-a_n < a_n$, $1-b_n < b_n$, $1-c_n < c_n$, for all $n \in N$, then iteration process (1.9) converges faster than (1.5), (1.6), (1.7), (1.8), for contractive mappings.

Sequel to this, Thakur et al. [24] in 2016 introduced the following three step iteration process:
For an arbitrary $s_1 \in \zeta$, define a sequence $\{s_n\}$ by

$$\begin{aligned} s_{n+1} &= \Gamma u_n, \\ u_n &= \Gamma((1-a_n)s_n + a_n v_n), \\ v_n &= (1-b_n)s_n + b_n \Gamma s_n \end{aligned} \tag{1.10}$$

where $\{a_n\}, \{b_n\} \subset (0,1)$, $n \in N$. It was asserted that the iteration process (1.10) converges faster than (1.5), (1.6), (1.7), (1.8) and (1.9), for contractive mappings.

In 2018, Piri *et al.* [19] posed the following question:
Is it possible to develop an iteration process which rate of convergence for contractive maps is faster than the iteration process (1.10) and the other iteration processes?

In the affirmative, Piri *et al.* [19] introduced the following three-step iteration process:
For an arbitrary $s_1 \in \zeta$, define a sequence $\{s_n\}$ by

$$\begin{aligned} s_{n+1} &= (1-a_n)\Gamma v_n + a_n \Gamma u_n, \\ u_n &= \Gamma v_n, \\ v_n &= \Gamma((1-b_n)s_n + b_n \Gamma s_n) \end{aligned} \tag{1.11}$$

where $\{a_n\}, \{b_n\} \subset (0,1)$, $n \in N$. They proved that the iteration process (1.11) converges faster than iteration processes (1.9) and (1.10) for contractive mappings when $1-a_n < a_n$, $1-b_n < b_n$, and $1-c_n < c_n$, for all $n \in N$, with numerical example to support the proof.

Recently, Udofia et. al. [25] introduced a four-step iteration algorithm called the AU-iteration process. They showed that the AU-iteration process converges faster than a number of well known iteration schemes in literature for contraction mappings. The AU-itration is defined for an arbitrary $s_0 \in \zeta$, define a sequence $\{s_n\}$ by

$$\begin{aligned} s_{n+1} &= \Gamma u_n, \\ u_n &= \Gamma v_n, \\ v_n &= \Gamma w_n, \\ w_n &= \Gamma((1-a_n)s_n + a_n \Gamma s_n) \end{aligned} \tag{1.12}$$

where $\{a_n\} \subset [0,1]$, $n \geq 1$.

Motivated and inspired by the work of [25, 23], we show that the four-step AU iteration process (1.12) converges faster than iteration process (1.11) for contraction mappings. Also, using AU-iteration scheme (1.12), we approximate the fixed point of monotone α-nonexpansive mappings and prove some weak and strong convergence results. Further, we provide numerical examples to show that the iteration process (1.12) converges faster than iteration process (1.11) and some existing well known iteration processes in literature. Finally, we apply our iteration process (1.12) to find the unique solution of a delayed differential equation in Banach spaces.

2 Preliminaries

Definition 2.1. A Banach space ϖ is said to be:

(i) Strictly convex if $\frac{1}{2}\|u+v\| < 1$, for all $u, v \in \varpi$ with $\|u\| = \|v\| = 1$ and $u \neq v$.
(ii) Uniformly convex if, for all $\epsilon \in (0,2]$, there exists $\delta > 0$ such that $\frac{1}{2}\|u+v\| \leq 1-\delta$, for all $u, v \in \varpi$ with $\|u\| \leq 1$, $\|v\| \leq 1$ and $\|u-v\| \geq \epsilon$.

Definition 2.2. (See [18]) A Banach space ϖ is said to satisfy the Opial's condition if for each weakly convergent sequence $\{s_n\}$ in ϖ, $\{s_n\}$ converges weakly to a point $u \in \varpi$, implies $\limsup_{n\to\infty} \|s_n - u\| < \limsup_{n\to\infty} \|s_n - v\|$, for all $v \in \varpi$ with $u \neq v$.

Definition 2.3. Let ζ be a nonempty subset of a Banach space ϖ and $\{s_n\}$ be a bounded sequence in ϖ. For each $s \in \varpi$, we define the following:

(i) Asymptotic radius of $\{s_n\}$ at s by

$$r(s, \{s_n\}) := \limsup_{n\to\infty} \|s_n - s\|$$

(ii) Asymptotic radius of $\{s_n\}$ relative to ζ by

$$r(\zeta, \{s_n\}) := \inf\{r(s, s_n) : s \in \zeta\}$$

(iii) Asymptotic center of $\{s_n\}$ relative to ζ by

$$A(\zeta, \{s_n\}) := \{s \in \zeta : r(s, \{s_n\}) = r(\zeta, \{s_n\})\}$$

It is known that in a uniformly convex Banach space, $A(\zeta, \{s_n\})$ consists of exactly one point. Also, $A(\zeta, \{s_n\})$ is nonempty and convex when ζ is weakly compact and convex.

Definition 2.4. (See [7]) Let $\{a_n\}, \{b_n\}$ be two sequences of real numbers that converge to a and b respectively. Then, $\{a_n\}$ converges faster to a than $\{b_n\}$ does to b if

$$\lim_{n\to\infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0. \tag{2.1}$$

Lemma 2.5. (See [23, Lemma 2.2]) Let ζ be a nonempty closed convex subset of an ordered Banach space $(\varpi, \leq)$ and $\Gamma : \zeta \longrightarrow \zeta$ be a monotone α-nonexpansive mapping. Then we have

(1) Γ is monotone quasi-nonexpansive;

(2) for all $u, v \in \zeta$, with $u \leq v$

$$\|\Gamma u - \Gamma v\|^2 \leq \|u - v\|^2 + \frac{2\alpha}{1-\alpha}\|\Gamma u - u\|^2 + \frac{2|\alpha|}{1-\alpha}\|\Gamma u - u\|(\|u - v\| + \|\Gamma u - \Gamma v\|).$$

Lemma 2.6. (see [26, Theorem 2]) For any real numbers $q > 1$ and $r > 0$, a Banach space ϖ is uniformly convex if and only if there exists a continuous strictly increasing convex function $f : [0, +\infty) \longrightarrow [0, +\infty)$ with $f(0) = 0$ such that

$$\|tu + (1-t)v\|^q \leq t\|u\|^q + (1-t)\|v\|^q - \omega(q, t)f(\|u - v\|), \tag{2.2}$$

for all $u, v \in B_r(0) = \{u \in \varpi : \|u\| \leq r\}$ and $t \in [0, 1]$, where, $\omega(q, t) = t^q(1-t) + t(1-t)^q$. In particular, taking $q = 2$ and $t = \frac{1}{2}$

$$\|\frac{u+v}{2}\|^2 \quad \leq \quad \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{4}f(\|u - v\|). \tag{2.3}$$

3 Main Result

3.1 Convergence Result

. In this section, we show that the AU iteration process (1.12) converges faster than iteration process (1.11) for contraction mappings.

Theorem 3.1. Let Γ be a contraction mapping defined on a nonempty closed convex subset ζ of a Banach space ϖ with a contraction factor $\delta \in (0, 1)$ and $F(\Gamma) \neq \phi$. If $\{s_n\}$ is a sequence defined by (1.12), then $\{s_n\}$ converges faster than iteration process (1.11).

Proof . Let $q \in F(T)$. From (1.12) and Lemma 2.5(1), we have

$$\begin{aligned}
\|w_n - q\| &= \|\Gamma((1-a_n)s_n + a_n\Gamma s_n) - q\| \\
&\leq \delta((1-a_n)\|s_n - q\| + a_n\|\Gamma s_n - q\|) \\
&\leq \delta((1-a_n) + a_n\delta)\|s_n - q\| \\
&= \delta(1-(1-\delta)a_n)\|s_n - q\| \\
&\leq \delta\|s_n - q\| \qquad (3.1)
\end{aligned}$$

From (1.12) and (3.1), we have

$$\begin{aligned}
\|v_n - q\| &= \|\Gamma w_n - q\| \\
&\leq \delta\|w_n - q\| \\
&\leq \delta^2\|s_n - q\| \qquad (3.2)
\end{aligned}$$

From (1.12) and (3.2), we have

$$\begin{aligned}
\|u_n - q\| &= \|\Gamma v_n - q\| \\
&\leq \delta\|v_n - q\| \\
&\leq \delta^3\|s_n - q\| \qquad (3.3)
\end{aligned}$$

From (1.12) and (3.3), we have

$$\begin{aligned}
\|s_{n+1} - q\| &= \|\Gamma u_n - q\| \\
&\leq \delta\|u_n - q\| \\
&\leq \delta(\delta^3\|s_n - q\|) \\
&= \delta^4\|s_n - q\| \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
&\leq \delta^{4n}\|s_1 - q\| \qquad (3.4)
\end{aligned}$$

$$Let \quad p_n = \delta^{4n}\|s_1 - q\| \qquad (3.5)$$

Also, from (1.11), we have

$$\begin{aligned}
\|v_n - q\| &= \|\Gamma((1-b_n)s_n + b_n\Gamma s_n) - q\| \\
&\leq \delta((1-b_n)\|s_n - q\| + b_n\|\Gamma s_n - q\|) \\
&\leq \delta(1-(1-\delta)b_n)\|s_n - q\| \\
&\leq \delta\|s_n - q\| \qquad (3.6)
\end{aligned}$$

Using (1.11) and (3.6), we have

$$\begin{aligned}
\|u_n - q\| &= \|\Gamma v_n - q\| \\
&\leq \delta\|v_n - q\| \\
&\leq \delta^2\|s_n - q\| \qquad (3.7)
\end{aligned}$$

Using (1.11) and (3.7), we have

$$\begin{aligned}
\|s_{n+1}-q\| &= \|((1-a_n)\Gamma v_n + a_n\Gamma u_n) - q\| \\
&\leq (1-a_n)\|\Gamma v_n - q\| + a_n\|\Gamma u_n - q\| \\
&\leq \delta(1-a_n)\|v_n - q\| + a_n\delta\|u_n - q\| \\
&\leq \delta^2(1-(1-\delta)a_n)\|s_n - q\| \\
&\leq \delta^2\|s_n - q\| \\
&. \\
&. \\
&. \\
&\leq \delta^{2n}\|s_1 - q\|. \qquad (3.8)
\end{aligned}$$

Let

$$r_n = \delta^{2n}\|s_1 - q\|. \qquad (3.9)$$

So from (3.5) and (3.9), we have that

$$\frac{p_n}{r_n} = \frac{\delta^{4n}\|s_1-q\|}{\delta^{2n}\|s_1-q\|} = \delta^{2n} \longrightarrow 0, \quad as \;\; n \to \infty$$

Hence (1.12) converges faster than (1.11). □

3.2 Numerical Example

We now show the comparison between the rate of convergence of AU iteration process (1.12) and other well known iteration algorithms in literature.

Example 3.2. Let $\zeta = [1, 50]$ and $\Gamma : [1, 50] \longrightarrow [1, 50]$ defined by $\Gamma(s) = (2s + 4)^{1/3}$. For Table 1 and Figure 1, we use the following parameters:

Choose $\alpha_n = \frac{n}{n+1}$, $\beta_n = \frac{1}{\sqrt{n+7}}$, $\gamma_n = \frac{2n}{5n+2}$, and the initial value $s_1 = 10$.

Obviously, the fixed point of Γ is $p = 2.0$, with a contraction constant $\delta = \frac{1}{4^{\frac{1}{3}}}$. Table 1 and Figure 1 show the behavior of AU iteration process (1.12) in comparison with the iteration processes of Noor [16], Agarwal et al. [2] (S-iteration), Abbas and Nazir [1], Thakur et al. [24], Piri et al. [19] to the fixed point of Γ in 30-iterations with $\|s_n - p\| < 10^{-15}$ as the stop criterion.

For Table 2 and Figure 2, we use the following parameters:

Choose $\alpha_n = \frac{2n}{3n+2}$, $\beta_n = \frac{n}{\sqrt{49n^2+1}}$, $\gamma_n = \frac{2n}{(3n+5)}$, and the initial value $s_1 = 5$. Table 2 and Figure 2 show the behavior of AU iteration process (1.12) in comparison with the iteration processes of Noor [16], Agarwal et al. [2] (S-iteration), Abbas and Nazir [1], Thakur et al. [24], M-iterations [**?**] and Piri et al. [19] to the fixed point of Γ in 30-iterations with $\|s_n - p\| < 10^{-15}$ as the stop criterion.

3.3 Convergence of The Iteration Process

In this section, we consider the convergence of the four-step AU iteration process defined in (1.12) for a monotone α-nonexpansive mapping Γ in an ordered Banach space $(\varpi, \leq)$.

In the sequel, we denote $F_{\leq}(\Gamma) = \{q \in F(\Gamma) : q \leq x_1\}$, $F_{\geq} = \{q \in F(T) : x_1 \leq q\}$ and $x_1 \in \zeta \subset \varpi$.

Theorem 3.3. Let ζ be a nonempty closed convex subset of uniformly convex ordered Banach space $(\varpi, \leq)$ and $\Gamma : \zeta \to \zeta$ be a monotone α-nonexpansive mapping. Assume that the sequence $\{s_n\}$ defined by the iteration process (1.12) is bounded and $s_1 \leq \Gamma s_1$ and $F_{\geq}(\Gamma) \neq 0$. Then we have

(1) $\|s_{n+1} - q\| \leq \|s_n - q\|$ and the limit $\limsup\limits_{n\to\infty} \|s_n - q\|$ exists for all $q \in F_{\geq}(\Gamma)$;

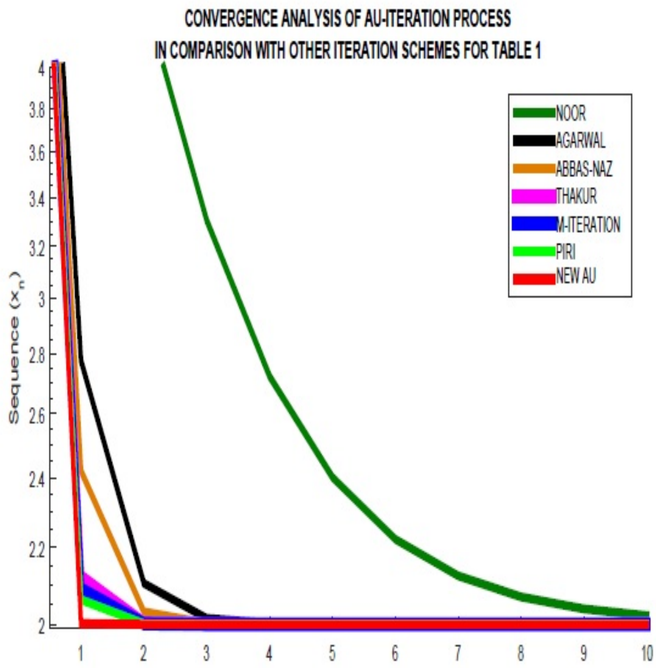

FIGURE 1

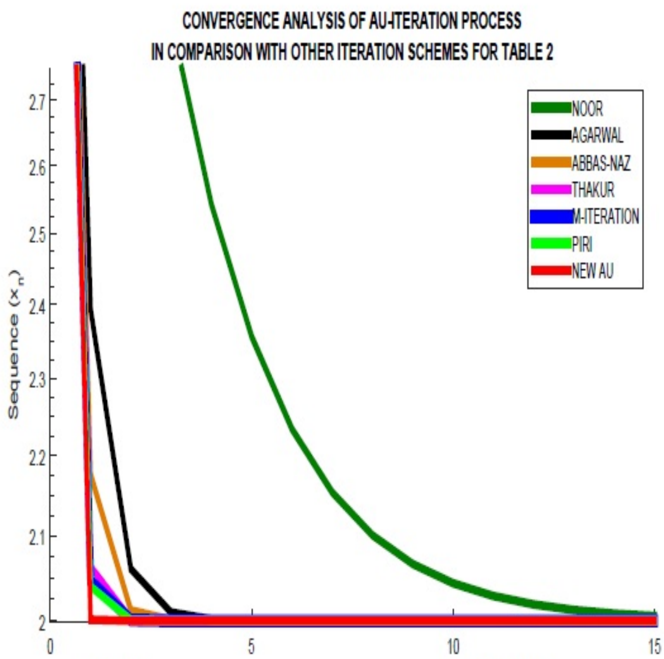

FIGURE 2

Table 1: AU iteration process in comparison with the iteration processes of Noor , Agarwal *et al.* (S-iteration), Abbas and Nazir, Thakur *et al.*, M-iteration and Piri *et al.*

n	NOOR	AGARWAL	ABBAS-NAZIR	NEW AU
1	10.000000000	10.000000000	10.000000000	10.000000000
2	6.3305751135	2.7756990223	2.4200895561	2.0024816608
3	4.3677362428	2.1043176345	2.0310872863	2.0000011168
4	3.3040966659	2.0147086469	2.0023638901	2.0000000005
5	2.7218144144	2.0020879283	2.0001801251	2.0000000000
6	2.4007571207	2.0002966707	2.0000137274	2.0000000000
7	2.2229144415	2.0000421593	2.0000010462	2.0000000000
8	2.1241253717	2.0000059913	2.0000000797	2.0000000000
9	2.0691589780	2.0000008514	2.0000000061	2.0000000000
10	2.0385466528	2.0000001210	2.0000000005	2.0000000000
11	2.0214886484	2.0000000172	2.0000000000	2.0000000000
12	2.0119806048	2.0000000024	2.0000000000	2.0000000000
13	2.0066799741	2.0000000003	2.0000000000	2.0000000000
14	2.0037246507	2.0000000000	2.0000000000	2.0000000000
15	2.0020768471	2.0000000000	2.0000000000	2.0000000000
16	2.0011580521	2.0000000000	2.0000000000	2.0000000000
17	2.0006457349	2.0000000000	2.0000000000	2.0000000000
18	2.0003600657	2.0000000000	2.0000000000	2.0000000000
19	2.0002007752	2.0000000000	2.0000000000	2.0000000000
20	2.0001119538	2.0000000000	2.0000000000	2.0000000000
21	2.0000624264	2.0000000000	2.0000000000	2.0000000000
22	2.0000348095	2.0000000000	2.0000000000	2.0000000000
23	2.0000194100	2.0000000000	2.0000000000	2.0000000000
24	2.0000108232	2.0000000000	2.0000000000	2.0000000000
25	2.0000060351	2.0000000000	2.0000000000	2.0000000000
26	2.0000033652	2.0000000000	2.0000000000	2.0000000000
27	2.0000018765	2.0000000000	2.0000000000	2.0000000000
28	2.0000010463	2.0000000000	2.0000000000	2.0000000000
29	2.0000005834	2.0000000000	2.0000000000	2.0000000000
30	2.0000003253	2.0000000000	2.0000000000	2.0000000000

(2) $\liminf_{n\to\infty} \|s_n - \Gamma s_n\| = 0$, provided $\limsup_{n\to\infty} a_n(1-a_n) > 0$.

(3) $\lim_{n\to\infty} \|s_n - \Gamma s_n\| = 0$, provided $\liminf_{n\to\infty} a_n(1-a_n) > 0$.

Proof .

(1) By Lemma (2.5)(i), Γ is quasi nonexpansive and from (1.12), we have

$$\begin{aligned} \|w_n - q\| &= \|\Gamma((1-a_n)s_n + a_n\Gamma s_n) - q\| \\ &\leq (1-a_n)\|s_n - q\| + a_n\|\Gamma s_n - q\| \\ &\leq \|s_n - q\|. \end{aligned} \tag{3.10}$$

Also from Lemma (2.5)(i), (1.12) and (3.10) we have

$$\begin{aligned} \|v_n - q\| &= \|\Gamma w_n - q\| \\ &\leq \|w_n - q\| \\ &\leq \|s_n - q\|. \end{aligned} \tag{3.11}$$

Again from Lemma (2.5)(i) , (1.12) and (3.11) we have

$$\begin{aligned} \|u_n - q\| &= \|\Gamma v_n - q\| \\ &\leq \|v_n - q\| \\ &\leq \|s_n - q\|. \end{aligned} \tag{3.12}$$

CONTINUATION OF TABLE 1

n	THAKUR	M-ITERATION	PIRI	NEW AU
1	10.000000000	10.000000000	10.000000000	10.000000000
2	2.1223548601	2.0901191316	2.0614585761	2.0024816608
3	2.0028680593	2.0014513256	2.0006991636	2.0000011168
4	2.0000679138	2.0000235145	2.0000079903	2.0000000005
5	2.0000016085	2.0000003810	2.0000000913	2.0000000000
6	2.0000000381	2.0000000062	2.0000000010	2.0000000000
7	2.0000000009	2.0000000001	2.0000000000	2.0000000000
8	2.0000000000	2.0000000000	2.0000000000	2.0000000000
9	2.0000000000	2.0000000000	2.0000000000	2.0000000000
10	2.0000000000	2.0000000000	2.0000000000	2.0000000000
11	2.0000000000	2.0000000000	2.0000000000	2.0000000000
12	2.0000000000	2.0000000000	2.0000000000	2.0000000000
13	2.0000000000	2.0000000000	2.0000000000	2.0000000000
14	2.0000000000	2.0000000000	2.0000000000	2.0000000000
15	2.0000000000	2.0000000000	2.0000000000	2.0000000000
16	2.0000000000	2.0000000000	2.0000000000	2.0000000000
17	2.0000000000	2.0000000000	2.0000000000	2.0000000000
18	2.0000000000	2.0000000000	2.0000000000	2.0000000000
19	2.0000000000	2.0000000000	2.0000000000	2.0000000000
20	2.0000000000	2.0000000000	2.0000000000	2.0000000000
21	2.0000000000	2.0000000000	2.0000000000	2.0000000000
22	2.0000000000	2.0000000000	2.0000000000	2.0000000000
23	2.0000000000	2.0000000000	2.0000000000	2.0000000000
24	2.0000000000	2.0000000000	2.0000000000	2.0000000000
25	2.0000000000	2.0000000000	2.0000000000	2.0000000000
26	2.0000000000	2.0000000000	2.0000000000	2.0000000000
27	2.0000000000	2.0000000000	2.0000000000	2.0000000000
28	2.0000000000	2.0000000000	2.0000000000	2.0000000000
29	2.0000000000	2.0000000000	2.0000000000	2.0000000000
30	2.0000000000	2.0000000000	2.0000000000	2.0000000000

Furthermore, from Lemma (2.5)(i), (1.12), (3.11) and (3.12) we have

$$\begin{aligned} \|s_{n+1}-q\| &= \|\Gamma u_n - q\| \\ &\leq \|w_n - q\| && (3.13) \\ &\leq \|s_n - q\|. && (3.14) \end{aligned}$$

Thus the sequence $\{\|s_n - q\|\}$ is bounded and nonincreasing which implies that $\lim_{n\to\infty} \|s_n - q\|$ exists, hence condition (1) holds.

By (3.13) and Lemma (2.6), we have

$$\begin{aligned} \|s_{n+1}-q\|^2 &= \|\Gamma u_n - q\|^2 \\ &\leq \|w_n - q\|^2 \\ &= \|\Gamma((1-a_n)s_n + a_n\Gamma s_n) - q\|^2 \\ &\leq \|(1-a_n)(s_n - q) + a_n(\Gamma s_n - q)\|^2 \\ &\leq (1-a_n)\|s_n - q\|^2 + a_n\|\Gamma s_n - q\|^2 \\ &\quad -a_n(1-a_n)f(\|s_n - \Gamma s_n\|) \\ &\leq \|s_n - q\|^2 - a_n(1-a_n)f(\|s_n - \Gamma s_n\|) && (3.15) \end{aligned}$$

which implies that

$$a_n(1-a_n)f(\|s_n - \Gamma s_n\|) \leq \|s_n - q\|^2 - \|s_{n+1} - q\|^2. \tag{3.16}$$

Table 2: AU iteration process in comparison with the iteration processes of Noor , Agarwal *et al.* (S-iteration), Abbas and Nazir, Thakur *et al.*, M-iteration and Piri *et al.*

n	NOOR	AGARWAL *et al.*	ABBAS-NAZIR	NEW AU
1	5.0000000000	5.0000000000	5.0000000000	5.0000000000
2	3.9458733081	2.3930286815	2.1758231998	2.0012828458
3	3.2675004502	2.0605404435	2.0119580368	2.0000006598
4	2.8282378761	2.0095680053	2.0008219461	2.0000000003
5	2.5424388125	2.0015183364	2.0000565385	2.0000000000
6	2.3558271801	2.0002410992	2.0000038893	2.0000000000
7	2.2336701993	2.0000382885	2.0000002675	2.0000000000
8	2.1535641809	2.0000060806	2.0000000184	2.0000000000
9	2.1009701195	2.0000009657	2.0000000013	2.0000000000
10	2.0664110057	2.0000001534	2.0000000001	2.0000000000
11	2.0436900959	2.0000000244	2.0000000000	2.0000000000
12	2.0287467876	2.0000000039	2.0000000000	2.0000000000
13	2.0189163575	2.0000000006	2.0000000000	2.0000000000
14	2.0124483942	2.0000000001	2.0000000000	2.0000000000
15	2.0081923286	2.0000000000	2.0000000000	2.0000000000
16	2.0053915470	2.0000000000	2.0000000000	2.0000000000
17	2.0035483572	2.0000000000	2.0000000000	2.0000000000
18	2.0023353204	2.0000000000	2.0000000000	2.0000000000
19	2.0015369829	2.0000000000	2.0000000000	2.0000000000
20	2.0010115651	2.0000000000	2.0000000000	2.0000000000
21	2.0006657638	2.0000000000	2.0000000000	2.0000000000
22	2.0004381748	2.0000000000	2.0000000000	2.0000000000
23	2.0002883868	2.0000000000	2.0000000000	2.0000000000
24	2.0001898033	2.0000000000	2.0000000000	2.0000000000
25	2.0001249201	2.0000000000	2.0000000000	2.0000000000
26	2.0000822169	2.0000000000	2.0000000000	2.0000000000
27	2.0000541116	2.0000000000	2.0000000000	2.0000000000
28	2.0000356139	2.0000000000	2.0000000000	2.0000000000
29	2.0000234395	2.0000000000	2.0000000000	2.0000000000
30	2.0000154268	2.0000000000	2.0000000000	2.0000000000

Letting $n \to \infty$, it follows from condition (1) that

$$a_n(1-a_n)f(\|s_n - \Gamma s_n\|) = 0 \tag{3.17}$$

(2) By condition (2) $\limsup_{n\to\infty} a_n(1-a_n) > 0$, and since

$$(\limsup_{n\to\infty} a_n(1-a_n))(\liminf_{n\to\infty} f(\|s_n - \Gamma s_n\|)) \le \limsup_{n\to\infty} a_n(1-a_n)f(\|s_n - \Gamma s_n\|),$$

by (3.17), we have $\liminf_{n\to\infty} f(\|s_n - \Gamma s_n\|) = 0$, and by the property of f, $\liminf_{n\to\infty} \|s_n - \Gamma s_n\| = 0$.

(3) Again by the assumption of condition (3), $\liminf_{n\to\infty} a_n(1-a_n) > 0$, and since

$$(\liminf_{n\to\infty} a_n(1-a_n))(\limsup_{n\to\infty} f(\|s_n - \Gamma s_n\|)) \le \limsup_{n\to\infty} a_n(1-a_n)f(\|s_n - \Gamma s_n\|),$$

by (3.17), we have

$$\lim_{n\to\infty} f(\|s_n - \Gamma s_n\|) = \limsup_{n\to\infty} f(\|s_n - \Gamma s_n\|) = 0$$

and by property of f,

$$\lim_{n\to\infty} \|s_n - \Gamma s_n\| = 0.$$

CONTINUATION OF TABLE 3

n	THAKUR	M-ITERATION	PIRI	NEW AU
1	5.0000000000	5.0000000000	5.0000000000	5.0000000000
2	2.0634980045	2.0463902697	2.0396086604	2.0012828458
3	2.0016708118	2.0008561796	2.0006448152	2.0000006598
4	2.0000442170	2.0000158542	2.0000105334	2.0000000003
5	2.0000011704	2.0000002936	2.0000001721	2.0000000000
6	2.0000000310	2.0000000054	2.0000000028	2.0000000000
7	2.0000000008	2.0000000001	2.0000000000	2.0000000000
8	2.0000000000	2.0000000000	2.0000000000	2.0000000000
9	2.0000000000	2.0000000000	2.0000000000	2.0000000000
10	2.0000000000	2.0000000000	2.0000000000	2.0000000000
11	2.0000000000	2.0000000000	2.0000000000	2.0000000000
12	2.0000000000	2.0000000000	2.0000000000	2.0000000000
13	2.0000000000	2.0000000000	2.0000000000	2.0000000000
14	2.0000000000	2.0000000000	2.0000000000	2.0000000000
15	2.0000000000	2.0000000000	2.0000000000	2.0000000000
16	2.0000000000	2.0000000000	2.0000000000	2.0000000000
17	2.0000000000	2.0000000000	2.0000000000	2.0000000000
18	2.0000000000	2.0000000000	2.0000000000	2.0000000000
19	2.0000000000	2.0000000000	2.0000000000	2.0000000000
20	2.0000000000	2.0000000000	2.0000000000	2.0000000000
21	2.0000000000	2.0000000000	2.0000000000	2.0000000000
22	2.0000000000	2.0000000000	2.0000000000	2.0000000000
23	2.0000000000	2.0000000000	2.0000000000	2.0000000000
24	2.0000000000	2.0000000000	2.0000000000	2.0000000000
25	2.0000000000	2.0000000000	2.0000000000	2.0000000000
26	2.0000000000	2.0000000000	2.0000000000	2.0000000000
27	2.0000000000	2.0000000000	2.0000000000	2.0000000000
28	2.0000000000	2.0000000000	2.0000000000	2.0000000000
29	2.0000000000	2.0000000000	2.0000000000	2.0000000000
30	2.0000000000	2.0000000000	2.0000000000	2.0000000000

This completes the proof. □

Theorem 3.4. Let ζ be a nonempty closed convex subset of a uniformly convex ordered Banach space $(\varpi, \leq)$ and $\Gamma : \zeta \to \zeta$ be a monotone α-nonexpansive mapping. Assume that E satisfies Opial's condition and the sequence $\{s_n\}$ is defined by the iteration process (1.12) with $s_1 \leq \Gamma s_1$ (or $\Gamma s_1 \leq s_1$). If $F_{\geq}(\Gamma) \neq 0$ (or $F_{\leq}(\Gamma) \neq 0$ and $\limsup_{n\to\infty} a_n(1-a_n) > 0$, then the sequence $\{s_n\}$ converges weakly to a fixed point w of Γ.

Proof . It follows from Theorem (3.3) that $\{s_n\}$ is bounded and $\lim_{n\to\infty} \|s_n - \Gamma s_n\| = 0$. Then there exists a subsequence $\{s_{n_k}\} \subset \{s_n\}$ such that $\{s_{n_k}\}$ converges weakly to a point $w \in \zeta$. Thus, it follows that $s_1 \leq s_{n_k} \leq w$ (or $w \leq s_{n_k} \leq s_1$) for all $k \geq 1$. On the other hand, Lemma (2.5)(2) means that

$$\begin{aligned} \|\Gamma s_{n_k} - \Gamma w\|^2 \quad &\leq \quad \|s_{n_k} - w\|^2 + \frac{2\alpha}{1-\alpha}\|\Gamma s_{n_k} - s_{n_k}\|^2 \\ &+ \frac{2|\alpha|}{1-\alpha}\|\Gamma s_{n_k} - s_{n_k}\|(\|s_{n_k} - w\| + \|\Gamma s_{n_k} - \Gamma w\|). \end{aligned} \tag{3.18}$$

By the boundedness of the sequence $\{s_n\}$ and $\lim_{n\to\infty} \|s_{n_k} - \Gamma s_{n_k}\| = 0$, we have, $\limsup_{n\to\infty} \|\Gamma s_{n_k} - \Gamma w\|^2 \leq \limsup_{n\to\infty} \|s_{n_k} - w\|^2$, and hence

$$\limsup_{n\to\infty} \|\Gamma s_{n_k} - \Gamma w\| \leq \limsup_{n\to\infty} \|s_{n_k} - w\|. \tag{3.19}$$

Now, we prove $w = \Gamma w$. In fact, suppose that $w \neq \Gamma w$. Then, by (3.19) and Opial's condition, we have

$$\begin{aligned}
\limsup_{n\to\infty} \|s_{n_k} - w\| &\leq \limsup_{n\to\infty} \|s_{n_k} - \Gamma w\| \\
&= \limsup_{n\to\infty} \|(s_{n_k} - \Gamma s_{n_k}) + (\Gamma s_{n_k} - \Gamma w)\| \\
&\leq \limsup_{n\to\infty} (\|s_{n_k} - \Gamma s_{n_k}\| + \|\Gamma s_{n_k} - \Gamma w\|) \\
&\leq \limsup_{n\to\infty} \|\Gamma s_{n_k} - \Gamma w\| \\
&\leq \limsup_{n\to\infty} \|s_{n_k} - w\| \qquad (3.20)
\end{aligned}$$

which is a contradiction. This implies that $w \in F_{\geq}(\Gamma)$ (or $w \in F_{\leq}(\Gamma)$) . Using Theorem (3.3)(2), the limit $\lim_{n\to\infty} \|s_n - w\|$ exists. Now, we show that the sequence $\{s_n\}$ converges weakly to the point w. Suppose that this does not hold. Then there exists a subsequence $\{s_{n_j}\}$ which converges weakly to a point $c \in \zeta$ and $w = c$. Similarly, we must have $c = \Gamma c$ and $\lim_{n\to\infty} \|s_n - c\|$ exists. It follows from Opial's condition that

$$\lim_{n\to\infty} \|s_n - w\| < \lim_{n\to\infty} \|s_n - c\| = \limsup_{n\to\infty} \|s_{n_j} - c\| < \lim_{n\to\infty} \|s_n - w\|.$$

This is a contradiction and hence $c = w$. This completes the proof. □

Theorem 3.5. Let ζ be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(\varpi, \leq)$ and $\Gamma : \zeta \to \zeta$ be a monotone α-nonexpansive mapping. Assume that the sequence $\{s_n\}$ is defined by the iteration process (1.12) with $s_1 \leq \Gamma s_1$. If $\limsup_{n\to\infty} a_n(1 - a_n) > 0$, then the sequence $\{s_n\}$ converges strongly to a fixed point $p \in F_{\geq}(\Gamma)$.

Proof . Following the compactness of ζ , there exists a subsequence $\{s_{n_k}\} \subset \{s_n\}$ such that $\{s_{n_k}\}$ converges strongly to a point $p \in \zeta$. Then it follows that $s_1 \leq s_{n_k} \leq p$ for all $k \geq 1$. It follows from Theorem (3.3) that $\{s_n\}$ is bounded and $\lim_{n\to\infty} \|s_n - \Gamma s_n\| = 0$. Without loss of generality, we can assume that $\lim_{k\to\infty} \|s_{n_k} - \Gamma s_{n_k}\| = 0$. On the other hand, the Lemma (2.5)(2) guarantees that

$$\begin{aligned}
\|\Gamma s_{n_k} - \Gamma p\|^2 \leq \|s_{n_k} - p\|^2 \quad &+ \quad \frac{2\alpha}{1-\alpha}\|\Gamma s_{n_k} - s_{n_k}\|^2 \\
&+ \frac{2|\alpha|}{1-\alpha}\|\Gamma s_{n_k} - s_{n_k}\|(\|s_{n_k} - p\| + \|\Gamma s_{n_k} - \Gamma p\|)
\end{aligned}$$

By the boundedness of the sequence $\{s_{n_k}\}$, $\liminf_{n\to\infty} \|s_{n_k} - p\| = 0$ and $\lim_{k\to\infty} \|s_{n_k} - \Gamma s_{n_k}\| = 0$, we have $\limsup_{k\to\infty} \|\Gamma s_{n_k} - \Gamma p\|^2 \leq 0$ and hence

$$\lim_{k\to\infty} \|\Gamma s_{n_k} - \Gamma p\| = 0 \qquad (3.21)$$

Therefore, we have

$$\begin{aligned}
\limsup_{k\to\infty} \|s_{n_k} - \Gamma p\| &\leq \limsup_{k\to\infty} \|(s_{n_k} - \Gamma s_{n_k}) + (\Gamma s_{n_k} - \Gamma p)\| \\
&\leq \limsup_{k\to\infty} (\|s_{n_k} - \Gamma s_{n_k}\| + \|\Gamma s_{n_k} - \Gamma p\|) \\
&= 0 \qquad (3.22)
\end{aligned}$$

and so $\lim_{k\to\infty} \|s_{n_k} - \Gamma p\| = 0$, which implies that $p \in F_{\geq}(\Gamma)$. Using Theorem (3.3)(1), the limit $\lim_{n\to\infty} \|s_n - p\|$ exists and so $\lim_{n\to\infty} \|s_n - p\| = 0$.
This completes the proof. □

Theorem 3.6. Let ζ be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(\varpi, \leq)$ and $\Gamma : \zeta \to \zeta$ be a monotone α-nonexpansive mapping. Assume that the sequence $\{s_n\}$ is defined by the iteration process (1.12) with $s_1 \leq \Gamma s_1$. If $\liminf_{n\to\infty} a_n(1 - a_n) > 0$, then the sequence $\{s_n\}$ converges strongly to a fixed point $p \in F_{\geq}(\Gamma)$.

Theorem 3.7. Let ζ be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(\varpi, \leq)$ and $\Gamma : \zeta \to \zeta$ be a monotone α-nonexpansive mapping. Assume that the sequence $\{s_n\}$ is defined by the iteration process (1.12) with $\Gamma s_1 \leq s_1$. If either $\liminf_{n\to\infty} a_n(1-a_n) > 0$ or $\limsup_{n\to\infty} a_n(1-a_n) > 0$, then the sequence $\{s_n\}$ converges strongly to a fixed point $p \in F_{\leq}(\Gamma)$.

4 Application to a Delay Differential Equation

Mathematical modeling with delay differential equations (DDEs) is widely used for analysis and predictions in various areas of life sciences, for example, population dynamics, epidemiology, immunology, physiology, and neural networks (see [8, 21, 20] and the references therein). Several iterative methods have been construct recently for approximating the unique solution of a delay differential equation (see for example [3, 9, 10, 11, 17] and the references therein).

To demonstrate the validity and more applicability of AU iterative scheme (1.12), we will prove its strong convergence to the following delay differential equation:

$$s'(\nu) = f(\nu, s(\nu), s(\nu-\tau), \nu \in [\nu_0, e] \tag{4.1}$$

with initial condition

$$s(\nu) = \zeta(\nu), \nu \in [\nu_0 - \tau, \nu_0] \tag{4.2}$$

The space $C([d,e])$ endowed with Chebyshev norm

$$\|s-u\|_\infty = \max_{\nu\in[d,e]} \|s(\nu) - u(\nu)\|_\infty, \forall s, u \in C([d,e]), \tag{4.3}$$

is known to be a Banach space (see [12]). We opine that the following conditions hold:

(M_1) $\nu_0, e \in \Re, \tau > 0$;
(M_2) $f \in C([\nu_0, e] \times \mathbb{R}^2, \Re)$;
(M_3) $\Psi \in C([\nu_0 - \tau, e], \Re)$;
(M_4) there exists $L_f > 0$ such that

$$|f(\nu, a_1, a_2) - f(\nu, b_1, b_2)| \leq L_f(|a_1 - b_1| + |a_2 - b_2|), \tag{4.4}$$

for all $a_1, a_2, b_1, b_2 \Re$ and $\nu \in [\nu_0, e]$;
(M_5) $2L_f(e - \nu_0) < 1$.

A function $s \in C([\nu_0 - \tau, e], \Re) \cap C^1([\nu_0, e], \Re)$ satisfying (4.1)-(4.2) is known as a solution of (4.1)-(4.2). The problem (4.1)– (4.2) can be reformulated in the following integral equation:

$$s(\nu) = \begin{cases} \Psi(\nu), & \nu \in [\nu_0 - \tau, \nu_0], \\ \Psi(\nu_o) + \int_{\nu_0}^{\nu} f(\rho, s(\rho), s(\rho - \tau))d\rho, & \nu \in [\nu_0, e]. \end{cases} \tag{4.5}$$

In 1976, Coman et al. [9] established the following result which will be useful in proving our main result.

Theorem 4.1. Assume that the conditions $(M_1) - (M_5)$ are fulfilled. Then the problem (4.1)–(4.2) has a unique solution, say $z \in C([\nu_0 - \tau, e], \Re) \cap C^1([\nu_0, e], \mathbb{R})$ and

$$z = \lim_{n\to\infty} \Gamma^n(\psi) \text{for any} s \in ([\nu_0 - \tau, e], \Re). \tag{4.6}$$

We now ready to give our main result.

Theorem 4.2. Assume that the conditions $(M_1) - (M_5)$ are fulfilled. Then the problem (4.1)–(4.2) has a unique solution, say $z \in C([\nu_0 - \tau, e], \Re) \cap C^1([\nu_0, e], \mathbb{R})$ and the AU iterative scheme (1.12) converges strongly to z.

Proof . Let $\{s_n\}$ be a sequence iteratively defined by AU iterative scheme (1.12) for the operator

$$\Gamma s(\nu) = \begin{cases} \Psi(\nu), & \nu \in [\nu_0 - \tau, \nu_0], \\ \Psi(\nu_0) + \int_{\nu_0}^{\nu} f(\rho, s(\rho), s(\rho - \tau))d\rho, & \nu \in [\nu_0, e]. \end{cases} \tag{4.7}$$

Let z stand for the fixed point of Γ. We will prove that $\lim_{n\to\infty} s_n = z$. For $\nu \in [\nu_0 - \tau, \nu_0]$, clearly, $\lim_{n\to\infty} s_n = z$. For $\nu \in [\nu_0, e]$, we have

$$\begin{aligned}
\|s_{n+1} - z\|_\infty &= \|\Gamma u_n - z\|_\infty \\
&= \|\Gamma u_n - \Gamma z\|_\infty \\
&= \max_{\nu\in[\nu_0-\tau,e]} |\Gamma u_n(\nu) - \Gamma z(\nu)| \\
&= \max_{\nu\in[\nu_0-\tau,e]} |\Psi(\nu_0) + \int_{\nu_0}^{\nu} f(\rho, u_n(\rho), u_n(\rho-\tau))d\rho - \Psi(\nu_0) \\
&\quad - \int_{\nu_0}^{\nu} f(\rho, z(\rho), z(\rho-\tau))d\rho| \\
&= \max_{\nu\in[\nu_0-\tau,e]} \left|\int_{\nu_0}^{\nu} f(\rho, u_n(\rho), u_n(\rho-\tau))d\rho - \int_{\nu_0}^{\nu} f(\rho, z(\rho), z(\rho-\tau))d\rho\right| \\
&\le \max_{\nu\in[\nu_0-\tau,e]} \int_{\nu_0}^{\nu} |f(\rho, u_n(\rho), u_n(\rho-\tau)) - f(\rho, z(\rho), z(\rho-\tau))|d\rho \\
&\le \max_{\nu\in[\nu_0-\tau,e]} \int_{\nu_0}^{\nu} L_f(|u_n(\rho) - z(\rho)| + |u_n(\rho-\tau) - z(\rho-\tau)|)d\rho \\
&\le \int_{\nu_0}^{\nu} L_f(\max_{\nu\in[\nu_0-\tau,e]} |u_n(\rho) - z(\rho)| + \max_{\nu\in[\nu_0-\tau,e]} |u_n(\rho-\tau) - z(\rho-\tau)|)d\rho \\
&\le \int_{\nu_0}^{\nu} L_f(\|u_n - z\|_\infty + \|u_n - z\|_\infty)d\rho \\
&\le 2L_f(\nu - \nu_0)\|u_n - z\|_\infty \\
&\le 2L_f(e - \nu_0)\|u_n - z\|_\infty
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
\|u_n - z\|_\infty &= \|\Gamma v_n - z\|_\infty \\
&= \|\Gamma v_n - \Gamma z\|_\infty \\
&= \max_{\nu\in[\nu_0-\tau,e]} |\Gamma v_n(\nu) - \Gamma z(\nu)| \\
&= \max_{\nu\in[\nu_0-\tau,e]} |\Psi(\nu_0) \\
&\quad + \int_{\nu_0}^{\nu} f(\rho, v_n(\rho), v_n(\rho-\tau))d\rho - \Psi(\nu_0) - \int_{\nu_0}^{\nu} f(\rho, z(\rho), z(\rho-\tau))d\rho\Big| \\
&= \max_{\nu\in[\nu_0-\tau,e]} \left|\int_{\nu_0}^{\nu} f(\rho, v_n(\rho), v_n(\rho-\tau))d\rho - \int_{\nu_0}^{\nu} f(\rho, z(\rho), z(\rho-\tau))d\rho\right| \\
&\le \max_{\nu\in[\nu_0-\tau,e]} \int_{\nu_0}^{\nu} |f(\rho, v_n(\rho), v_n(\rho-\tau)) - f(\rho, z(\rho), z(\rho-\tau))|d\rho \\
&\le \max_{\nu\in[\nu_0-\tau,e]} \int_{\nu_0}^{\nu} L_f(|v_n(\rho) - z(\rho)| + |v_n(\rho-\tau) - z(\rho-\tau)|)d\rho \\
&\le \int_{\nu_0}^{\nu} L_f(\max_{\nu\in[\nu_0-\tau,e]} |v_n(\rho) - z(\rho)| + \max_{\nu\in[\nu_0-\tau,e]} |v_n(\rho-\tau) - z(\rho-\tau)|)d\rho \\
&\le \int_{\nu_0}^{\nu} L_f(\|v_n - z\|_\infty + \|v_n - z\|_\infty)d\rho \\
&\le 2L_f(\nu - \nu_0)\|v_n - z\|_\infty \\
&\le 2L_f(e - \nu_0)\|v_n - z\|_\infty.
\end{aligned} \tag{4.9}$$

Putting (4.9) into (4.8), we obtain

$$\|s_{n+1} - z\|_\infty \leq (2L_f(e-\nu_0))^2 \|v_n - z\|_\infty \tag{4.10}$$

and

$$\begin{aligned}
\|v_n - z\|_\infty &= \|\Gamma w_n - z\|_\infty \\
&= \|\Gamma w_n - \Gamma z\|_\infty \\
&= \max_{\nu\in[\nu_0-\tau,e]} |\Gamma w_n(\nu) - \Gamma z(\nu)| \\
&= \max_{\nu\in[\nu_0-\tau,e]} \Big|\Psi(\nu_0) \\
&\quad + \int_{\nu_0}^{\nu} f(\rho, w_n(\rho), w_n(\rho-\tau))d\rho - \Psi(\nu_0) - \int_{\nu_0}^{\nu} f(\rho, z(\rho), z(\rho-\tau))d\rho\Big| \\
&= \max_{\nu\in[\nu_0-\tau,e]} \left|\int_{\nu_0}^{\nu} f(\rho, w_n(\rho), w_n(\rho-\tau))d\rho - \int_{\nu_0}^{\nu} f(\rho, z(\rho), z(\rho-\tau))d\rho\right| \\
&\leq \max_{\nu\in[\nu_0-\tau,e]} \int_{\nu_0}^{\nu} |f(\rho, w_n(\rho), w_n(\rho-\tau)) - f(\rho, z(\rho), z(\rho-\tau))|d\rho \\
&\leq \max_{\nu\in[\nu_0-\tau,e]} \int_{\nu_0}^{\nu} L_f(|w_n(\rho) - z(\rho)| + |w_n(\rho-\tau) - z(\rho-\tau)|)d\rho \qquad (4.11) \\
&\quad \leq \int_{\nu_0}^{\nu} L_f(\max_{\nu\in[\nu_0-\tau,e]} |w_n(\rho) - z(\rho)| + \max_{\nu\in[\nu_0-\tau,e]} |w_n(\rho-\tau) - z(\rho-\tau)|)d\rho \\
&\leq \int_{\nu_0}^{\nu} L_f(\|w_n - z\|_\infty + \|w_n - z\|_\infty)d\rho \\
&\leq 2L_f(\nu - \nu_0)\|w_n - z\|_\infty \\
&\leq 2L_f(e - \nu_0)\|w_n - z\|_\infty. \qquad (4.12)
\end{aligned}$$

Substituting (4.12) into (4.10), we have

$$\|s_{n+1} - z\|_\infty \leq (2L_f(e-\nu_0))^3 \|w_n - z\|_\infty \tag{4.13}$$

and

$$\begin{aligned}
\|w_n - z\|_\infty &= \|\Gamma((1-a_n)s_n + a_n\Gamma s_n) - z\|_\infty \\
&= \|\Gamma((1-a_n)s_n + a_n\Gamma s_n) - \Gamma z\|_\infty \\
&= \max_{\nu\in[\nu_0-\tau,e]} |\Gamma((1-a_n)s_n + a_n\Gamma s_n)(\nu) - Tz(\nu)| \\
&= \max_{\nu\in[\nu_0-\tau,e]} \Big|\Psi(\nu_0) + \int_{\nu_0}^{\nu} f(\rho, ((1-a_n)s_n + a_n\Gamma s_n)(\rho), ((1-a_n)s_n + a_n\Gamma s_n)(\rho-\tau))d\rho \\
&\quad -\Psi(\nu_0) - \int_{\nu_0}^{\nu} f(\rho, z(\rho), z(\rho-\tau))d\rho\Big| \\
&= \max_{\nu\in[\nu_0-\tau,e]} \Big|\int_{\nu_0}^{\nu} f(\rho, ((1-a_n)s_n + a_n\Gamma s_n)(\rho), ((1-a_n)s_n + a_n\Gamma s_n)(\rho-\tau))d\rho \\
&\quad - \int_{\nu_0}^{\nu} f(\rho, z(\rho), z(\rho-\tau))d\rho\Big| \\
&\leq \max_{\nu\in[\nu_0-\tau,e]} \int_{\nu_0}^{\nu} |f(\rho, \Gamma((1-a_n)s_n + a_n\Gamma s_n)(\rho), ((1-a_n)s_n + a_n\Gamma s_n)(\rho-\tau)) \\
&\quad -f(\rho, z(\rho), z(\rho-\tau))|d\rho
\end{aligned}$$

$$
\begin{aligned}
&\leq \max_{\nu\in[\nu_0-\tau,e]}\int_{\nu_0}^{\nu} L_f(|((1-a_n)s_n+a_n\Gamma s_n)(\rho)-z(\rho)| \\
&\quad +|((1-a_n)s_n+a_n\Gamma s_n)(\rho-\tau)-z(\rho-\tau)|)d\rho \\
&\leq \int_{\nu_0}^{\nu} L_f(\max_{\nu\in[\nu_0-\tau,e]}|((1-a_n)s_n+a_n\Gamma s_n)(\rho)-z(\rho)| \\
&\quad +\max_{\nu\in[\nu_0-\tau,e]}|((1-a_n)s_n+a_n\Gamma s_n)(\rho-\tau)-z(\rho-\tau)|)d\rho \\
&\leq \int_{\nu_0}^{\nu} L_f(\|w_n-z\|_\infty+\|((1-a_n)s_n+a_n\Gamma s_n)-z\|_\infty)d\rho \\
&\leq 2L_f(\nu-\nu_0)\|((1-a_n)s_n+a_n\Gamma s_n)-z\|_\infty \\
&\leq 2L_f(e-\nu_0)\|((1-a_n)s_n+a_n\Gamma s_n)-z\|_\infty. \qquad (4.14)
\end{aligned}
$$

Putting (4.14) into (4.13), we obtain

$$\|s_{n+1}-z\|_\infty \leq (2L_f(e-\nu_0))^4\|((1-a_n)s_n+a_n\Gamma s_n)-z\|_\infty. \qquad (4.15)$$

and

$$
\begin{aligned}
\|(1-a_n)s_n &+ a_n\Gamma s_n-z\|_\infty \\
&= \|(1-a_n)(s_n-z)+a_n(a_n\Gamma s_n-z)\|_\infty \\
&\leq (1-a_n)\|s_n-z\|_\infty+a_n\|\Gamma s_n-\Gamma z\|_\infty \\
&= (1-a_n)\|s_n-z\|_\infty+a_n\max_{\nu\in[\nu_0-\tau,v]}|\Gamma s_n(\nu)-\Gamma z(\nu)| \\
&= (1-a_n)\|s_n-z\|_\infty+a_n\max_{\nu\in[\nu_0-\tau,e]}\Big|\Psi(\nu_0) \\
&\quad +\int_{\nu_0}^{\nu} f(s,s_n(\rho),s_n(\rho-\tau))d\rho-\Psi(\nu_0)-\int_{\nu_0}^{\nu} f(\rho,z(\rho),z(\rho-\tau))d\rho\Big| \\
&= (1-a_n)\|s_n-z\|_\infty+a_n\max_{\nu\in[\nu_0-\tau,e]}\Big|\int_{\nu_0}^{\nu} f(\rho,s_n(\rho),s_n(\rho-\tau))d\rho \\
&\quad -\int_{\nu_0}^{\nu} f(\rho,z(\rho),z(\rho-\tau))d\rho\Big| \\
&\leq (1-a_n)\|s_n-z\|_\infty+a_n\max_{\nu\in[nu_0-\tau,e]}\int_{\nu_0}^{\nu}|f(s,s_n(\rho),s_n(\rho-\tau)) \\
&\quad -f(\rho,z(\rho),z(\rho-\tau))|d\rho \\
&\leq (1-a_n)\|s_n-z\|_\infty+a_n\max_{\ell\in[\nu_0-\tau,e]}\int_{\nu_0}^{\nu} L_f(|s_n(\rho)-z(\rho)| \\
&\quad +|s_n(\rho-\tau)-z(\rho-\tau)|)d\rho \\
&\leq (1-a_n)\|s_n-z\|_\infty+a_n\int_{\nu_0}^{\nu} L_f(\max_{\nu\in[\nu_0-\tau,e]}|s_n(\rho)-z(\rho)| \\
&\quad +\max_{\nu\in[\nu_0-\tau,e]}|s_n(\rho-\tau)-z(\rho-\tau)|)d\rho \\
&\leq (1-a_n)\|s_n-z\|_\infty+a_n\int_{\nu_0}^{\nu} L_f(\|s_n-z\|_\infty+\|s_n-z\|_\infty)d\rho \\
&\leq (1-a_n)\|s_n-z\|_\infty+2a_nL_f(\nu-\nu_0)\|s_n-z\|_\infty \\
&\leq [1-a_n(1-2L_f(e-\nu_0))]\|s_n-z\|_\infty. \qquad (4.16)
\end{aligned}
$$

Substituting (4.16) into (4.15), we have

$$\|s_{n+1}-z\| \leq (2L_f(e-\nu_0))^4[1-a_n(1-2L_f(e-\nu_0))]\|s_n-z\|_\infty. \qquad (4.17)$$

Recalling from assumption (M_5) that $2L_f(e-\nu_0)<1$, it follows that $(2L_f(e-\nu_0))^4<1$. Thus from (4.17), we have

$$\|s_{n+1}-z\| \leq [1-a_n(1-2L_f(e-\nu_0))]\|s_n-z\|_\infty. \qquad (4.18)$$

Inductively, from (4.18) we have

$$\|s_{n+1} - z\|_\infty \leq \prod_{k=0}^{n} [1 - a_k(1 - 2L_f(e - \nu_0))] \|s_0 - z\|_\infty. \tag{4.19}$$

Since $a_k \in [0, 1]$, for all $k \in \mathbb{N}$, from assumption (M_5), we get

$$1 - a_k(1 - 2L_f(e - \nu_0)) < 1. \tag{4.20}$$

From classical analysis, it is well known that $\exp^{-s} \geq 1 - s$, for all $s \in [0, 1]$, thus from (4.19) we get

$$\|s_{n+1} - z\|_\infty \leq \|s_0 - z\|_\infty \exp^{-(1 - a_k(1 - 2L_f(e - \nu_0))) \sum_{s=0}^{n} a_k}, \tag{4.21}$$

which yields $\lim_{n\to\infty} \|s_n - z\|_\infty = 0$. □

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