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Toeplitz-plus-Hankel matrices with perturbed corners

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Abstract

This paper examines suitable borderings and modification techniques for finding some special properties of a class of real heptadiagonal symmetric Toeplitz matrices and anti-heptadiagonal persymmetric Hankel matrices with perturbed corners as the zeros of explicit rational functions. An orthogonal diagonalization, inverse and determinant, and a formula to compute its integer powers for these matrices are shown. Then, these results are expanded for the corresponding Toeplitz-plus-Hankel matrices with perturbed corners.

Keywords: Heptadiagonal symmetric matrix, Anti-heptadiagonal persymmetric matrices, Toeplitz-plus-Hankel matrix
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1 Introduction and preliminaries

Band matrices arise in a wide variety of applications such as the finite difference approximation to ordinary differential equations and in certain statistical problems. The integer powers of these matrices are required in different fields such as numerical analysis, differential equations, linear dynamical systems or graph theory. Spectral and computational properties of symmetric Toeplitz matrices and persymmetric Hankel matrices in special cases have been studied by several authors such as Bini, Fasino, Silva and Shams Solary, in [1, 4, 10, 12]. In brief, spectral and computational properties of symmetric Toeplitz matrices and persymmetric Hankel matrices have been studied by several authors such as Bini, Fasino and Lita da Silva in Bini and Capovani (1983), Fasino (1988), Lita da Silva (2016) and Solary (2013). This motivates us to derive an orthogonal diagonalization for heptadiagonal symmetric Toeplitz matrices and anti-heptadiagonal persymmetric Hankel matrices with perturbed corners. We say that a, b, c, d, $e \in \mathbb{R}$, $\mathbf{T}_n$ is an $n \times n$ heptadiagonal symmetric Toeplitz matrices, $\mathbf{H}_n$ is an $n \times n$ anti-heptadiagonal persymmetric Hankel

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matrices with perturbed corners

$$\mathbf{T}_n = \begin{pmatrix} e & b & c & d & & & & & & \\ b & a & b & c & d & & & & & \\ c & b & a & b & c & d & & & & \\ d & c & b & a & b & c & d & & & \\ & d & c & b & a & b & c & d & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & d & c & b & a & b & c & d \\ & & & & d & c & b & a & b & c \\ & & & & & d & c & b & a & b \\ & & & & & & d & c & b & e \end{pmatrix} \tag{1.1}$$

and

$$\mathbf{H}_n = \begin{pmatrix} & & & & & & d & c & b & e \\ & & & & & d & c & b & a & b \\ & & & & d & c & b & a & b & c \\ & & & d & c & b & a & b & c & d \\ & & d & c & b & a & b & c & d & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ d & c & b & e & b & c & d & & & \\ c & b & a & b & c & d & & & & \\ b & a & b & c & d & & & & & \\ e & b & c & d & & & & & & \end{pmatrix}. \tag{1.2}$$

By using the technique of bordering for a class of simultaneously diagonalizable matrices, we made suitable submatrices of the heptadiagonal symmetric Toeplitz matrices and anti-heptadiagonal persymmetric Hankel matrices. These submatrices help us to find the integer powers, the determinant and the inverse of heptadiagonal symmetric Toeplitz matrices and anti-heptadiagonal persymmetric Hankel matrices with perturbed corners. We generalize these results for the following Toeplitz-plus-Hankel matrices with perturbed corners. Here we use the following notations:

$$\lambda_k = 2d\cos\left(\frac{3k\pi}{n+1}\right) + 2c\cos\left(\frac{2k\pi}{n+1}\right) + 2b\cos\left(\frac{k\pi}{n+1}\right) + a, \tag{1.3}$$

$$\mu_k = -2d\cos\left[\frac{(n-2)k\pi}{n+1}\right] - 2c\cos\left[\frac{(n-1)k\pi}{n+1}\right] - 2b\cos\left(\frac{nk\pi}{n+1}\right) - a\cos(k\pi) \tag{1.4}$$

$k = 1, \dots, n$.

(a) If n is even,

$\mathbf{D}_1 = diag(\lambda_1, \lambda_3, \dots, \lambda_{n-1})$, $\mathbf{D}_2 = diag(\lambda_2, \lambda_4, \dots, \lambda_n)$,
$\mathbf{D}_3 = diag(\mu_1, \mu_3, \dots, \mu_{n-1})$, $\mathbf{D}_4 = diag(\mu_2, \mu_4, \dots, \mu_n)$,
$\mathbf{P}$ is the $n \times n$ permutation matrix defined by:

$$[\mathbf{P}]_{i,j} = \begin{cases} 1 & if \quad i = 2j-1 \quad or \quad i = 2j-n \\ 0 & otherwise. \end{cases} \tag{1.5}$$

$$\mathbf{U} = \left(\begin{array}{c|c|c} \theta & {\mathbf{w}^{(1)}}^T & \theta \\ \hline \mathbf{w}^{(1)} & \mathbf{V} & \mathbf{w}^{(2)} \\ \hline \theta & {\mathbf{w}^{(2)}}^T & -\theta \end{array}\right) \tag{1.6}$$

$\mathbf{U}$ is an orthogonal and symmetric matrix $n \times n$, since $\mathbf{U}^T = \mathbf{U} = \mathbf{U}^{-1}$ where

$$\theta = \sqrt{\frac{2}{n+1}}\sin\left(\frac{\pi}{n+1}\right),$$

$$\mathbf{w}_i^{(1)} = (-1)^i \mathbf{w}_i^{(2)} = \sqrt{\frac{2}{n+1}}\sin\left[\frac{\pi(i+1)}{n+1}\right]$$

$$[\mathbf{V}]_{ij} = \sqrt{\frac{2}{n+1}} \sin\left[\frac{\pi(i+1)(j+1)}{n+1}\right].$$

$$\mathbf{u} = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin\left(\frac{\pi}{n+1}\right) \\ \sin\left(\frac{3\pi}{n+1}\right) \\ \vdots \\ \sin\left[\frac{(n-1)\pi}{n+1}\right] \end{pmatrix}, \quad \mathbf{v} = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin\left(\frac{2\pi}{n+1}\right) \\ \sin\left(\frac{4\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{n\pi}{n+1}\right) \end{pmatrix} \tag{1.7}$$

(b) If n is odd,

$\mathbf{D}_1 = diag(\lambda_1, \lambda_3, \ldots, \lambda_n)$, $\mathbf{D}_2 = diag(\lambda_2, \lambda_4, \ldots, \lambda_{n-1})$,
$\mathbf{D}_3 = diag(\mu_1, \mu_3, \ldots, \mu_n)$, $\mathbf{D}_4 = diag(\mu_2, \mu_4, \ldots, \mu_{n-1})$,
$\mathbf{P}$ is the $n \times n$ permutation matrix defined by:

$$[\mathbf{P}]_{i,j} = \begin{cases} 1 & if \quad i = 2j-1 \quad or \quad i = 2j-n-1 \\ 0 & otherwise. \end{cases} \tag{1.8}$$

$$\mathbf{U} = \left(\begin{array}{c|c|c} \theta & \mathbf{w}^{(1)^T} & \theta \\ \hline \mathbf{w}^{(1)} & \mathbf{V} & \mathbf{w}^{(2)} \\ \hline \theta & \mathbf{w}^{(2)^T} & \theta \end{array}\right) \tag{1.9}$$

$$\mathbf{u} = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin\left(\frac{\pi}{n+1}\right) \\ \sin\left(\frac{3\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{n\pi}{n+1}\right) \end{pmatrix}, \quad \mathbf{v} = \sqrt{\frac{2}{n+1}} \begin{pmatrix} \sin\left(\frac{2\pi}{n+1}\right) \\ \sin\left(\frac{4\pi}{n+1}\right) \\ \vdots \\ \sin\left[\frac{(n-1)\pi}{n+1}\right] \end{pmatrix}. \tag{1.10}$$

Also we have

$$[\xi]_{ij} = [1 + (-1)^{i+j}] \left[c - a + e + 2d\cos\left(\frac{i\pi}{n+1}\right) + 2d\cos\left(\frac{j\pi}{n+1}\right)\right], \quad i,j = 1, \ldots, n, \tag{1.11}$$

where $\xi^{(1)}$ is an order-preserving submatrix of ξ corresponds to $i,\ j$ are odd, $\xi^{(2)}$ is an order-preserving submatrix of ξ corresponds to $i,\ j$ are even and $[\xi]_{ij} = 0$ otherwise.
$\xi^{(1)} \circ \mathbf{u}\mathbf{u}^T$ is shown the Hadamard product of $\xi^{(1)}$ and $\mathbf{u}\mathbf{u}^T$, $\xi^{(2)} \circ \mathbf{v}\mathbf{v}^T$ is shown the Hadamard product of $\xi^{(2)}$ and $\mathbf{v}\mathbf{v}^T$[8].

2 An orthogonal diagonalization of the matrix $\mathbf{T}_n$

Let an $n \times n$ Toeplitz matrix similar (1.1). The basic tool of our analysis in this paper is the bordering technique, see [1, 2]. This technique helps us to find a class of simultaneously diagonalizable matrices which have a suitable submatrix generating by band symmetric Toeplits matrices. Setting

$$\hat{\mathbf{T}}_n = \begin{pmatrix} a-c & b-d & c & d & & & & & \\ b-d & a & b & c & d & & & & \\ c & b & a & b & c & d & & & \\ d & c & b & a & b & c & d & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & d & c & b & a & b & c & d \\ & & & d & c & b & a & b & c \\ & & & & d & c & b & a & b-d \\ & & & & & d & c & b-d & a-c \end{pmatrix}$$

and

$$\hat{\mathbf{E}}_T = \begin{pmatrix} c-a+e & d & 0 & & & \\ d & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & d \\ & & 0 & d & c-a+e \end{pmatrix}.$$

Then $\mathbf{T}_n = \hat{\mathbf{T}}_n + \hat{\mathbf{E}}_T$, by Proposition 3.1 in [1] and a theoretical and computational analysis of $\mathbf{U}\hat{\mathbf{E}}_T\mathbf{U}$, we deduce

$$\mathbf{U}\mathbf{T}_n\mathbf{U} = \mathbf{U}(\hat{\mathbf{T}}_n + \hat{\mathbf{E}}_T)\mathbf{U} = diag(\lambda_1, \lambda_2, \ldots, \lambda_n) + \mathbf{E}_T$$

where

$$[\mathbf{E}_T]_{ij} = \frac{2[1+(-1)^{i+j}]}{n+1} \sin\left(\frac{i\pi}{n+1}\right) \sin\left(\frac{j\pi}{n+1}\right) \left[c-a+e+2d\left[\cos\left(\frac{i\pi}{n+1}\right)+\cos\left(\frac{j\pi}{n+1}\right)\right]\right]$$

for $i, j = 1, 2, \ldots, n$, since $[\mathbf{E}_T]_{ij} = 0$ whenever $i+j$ is odd, we can permute rows and columns of $diag(\lambda_1, \lambda_2, \ldots, \lambda_n) + \mathbf{E}_T$ according to the permutation matrices (1.5) and (1.8) for n even or n odd and other convenient relations of Theorem 12 respectively, which derives:

$$\mathbf{T}_n = \mathbf{U}\mathbf{P}\left(\begin{array}{c|c} \mathbf{D}_1 + \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{D}_2 + \xi^{(2)} \circ \mathbf{v}\mathbf{v}^T \end{array}\right)\mathbf{P}^\mathbf{T}\mathbf{U}. \quad (2.1)$$

Then we deduce

Theorem 1. Let $\mathbf{T}_n$ be an $n \times n$ matrix similar (1.1) and λ_k, $k = 1, \ldots, n$ be given in (1.3),
(a) If n is even then

$$\mathbf{T}_n = \mathbf{U}\mathbf{P}\left(\begin{array}{c|c} diag(\lambda_1, \lambda_3, \ldots, \lambda_{n-1}) + \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T & \mathbf{0} \\ \hline \mathbf{0} & diag(\lambda_2, \lambda_4, \ldots, \lambda_n) + \xi^{(2)} \circ \mathbf{v}\mathbf{v}^T \end{array}\right)\mathbf{P}^\mathbf{T}\mathbf{U}. \quad (2.2)$$

where $\mathbf{P}$ is the $n \times n$ permutation matrix defined by (1.5), $\mathbf{U}$ in (1.6) is orthogonal matrix and $\mathbf{u}$, $\mathbf{v}$ are given by (1.7).
(b) If n is odd then

$$\mathbf{T}_n = \mathbf{U}\mathbf{P}\left(\begin{array}{c|c} diag(\lambda_1, \lambda_3, \ldots, \lambda_n) + \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T & \mathbf{0} \\ \hline \mathbf{0} & diag(\lambda_2, \lambda_4, \ldots, \lambda_{n-1}) + \xi^{(2)} \circ \mathbf{v}\mathbf{v}^T \end{array}\right)\mathbf{P}^\mathbf{T}\mathbf{U}. \quad (2.3)$$

where $\mathbf{P}$ is the $n \times n$ permutation matrix defined by (1.8), $\mathbf{U}$ in (1.9) and $\mathbf{u}$, $\mathbf{v}$ are given by (1.10).

The decomposition obtained in the above theorem can be used for finding the inverse and the spectral properties of the matrix $\mathbf{T}_n$.
Let $\mathbf{M} = \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T$ and $\mathbf{N} = \xi^{(2)} \circ \mathbf{v}\mathbf{v}^T$, $\mathbf{I}$ is the identity matrix (with convenient dimension).
According to Sherman-Morrison-Woodbury formula [7, 9], we have

$$(\mathbf{D}_1 + \mathbf{M})^{-1} = \mathbf{D}_1^{-1} - \mathbf{D}_1^{-1}(\mathbf{I} + \mathbf{M}\mathbf{D}_1^{-1})^{-1}\mathbf{M}\mathbf{D}_1^{-1}$$

and

$$(\mathbf{D}_2 + \mathbf{N})^{-1} = \mathbf{D}_2^{-1} - \mathbf{D}_2^{-1}(\mathbf{I} + \mathbf{N}\mathbf{D}_2^{-1})^{-1}\mathbf{N}\mathbf{D}_2^{-1},$$

then we have

$$\mathbf{T}_n^{-1} = \mathbf{U}\mathbf{P}\left(\begin{array}{c|c} \mathbf{D}_1^{-1}[\mathbf{I} - (\mathbf{I} + \mathbf{M}\mathbf{D}_1^{-1})^{-1}\mathbf{M}\mathbf{D}_1^{-1}] & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{D}_2^{-1}[\mathbf{I} - (\mathbf{I} + \mathbf{N}\mathbf{D}_2^{-1})^{-1}\mathbf{M}\mathbf{D}_2^{-1}] \end{array}\right)\mathbf{P}^\mathbf{T}\mathbf{U}. \quad (2.4)$$

By taking the determinant of both sides of (2.1) and using [5], we have

$$det(\mathbf{D}_1 + \epsilon\mathbf{M}) = det(\mathbf{D}_1)(1 + \epsilon \;\; trace(\mathbf{D}_1^{-1}\mathbf{M})) + O(\epsilon^2),$$

for any complex ϵ sufficiently small in modulus.
Then for n even we have

$$det\left(\mathbf{T}_n\right) = \left\{1+\frac{4}{n+1}\sum_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{2d\cos\left[\frac{3(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2i-1)\pi}{n+1}\right]+a}\right\}$$
$$\times\left\{1+\frac{4}{n+1}\sum_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)\sin^2\left[\frac{2i\pi}{n+1}\right]}{2d\cos\left[\frac{3(2i)i\pi}{n+1}\right]+2c\cos\left[\frac{2(2i)\pi}{n+1}\right]+2b\cos\left[\frac{2i\pi}{n+1}\right]+a}\right\}$$
$$\times\prod_{i=1}^{n}\left\{2d\cos\left(\frac{3i\pi}{n+1}\right)+2c\cos\left(\frac{2i\pi}{n+1}\right)+2b\cos\left(\frac{i\pi}{n+1}\right)+a\right\} \tag{2.5}$$

and for n odd, we deduce

$$det\left(\mathbf{T}_n\right) = \left\{1+\frac{4}{n+1}\sum_{i=1}^{\frac{n+1}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)pi}{n+1}\right]\right)\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{2d\cos\left[\frac{3(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2i-1)\pi}{n+1}\right]+a}\right\}$$
$$\times\left\{1+\frac{4}{n+1}\sum_{i=1}^{\frac{n-1}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)\sin^2\left[\frac{2i\pi}{n+1}\right]}{2d\cos\left[\frac{3(2i)i\pi}{n+1}\right]+2c\cos\left[\frac{2(2i)\pi}{n+1}\right]+2b\cos\left[\frac{2i\pi}{n+1}\right]+a}\right\}$$
$$\times\prod_{i=1}^{n}\left\{2d\cos\left(\frac{3i\pi}{n+1}\right)+2c\cos\left(\frac{2i\pi}{n+1}\right)+2b\cos\left(\frac{i\pi}{n+1}\right)+a\right\}. \tag{2.6}$$

2.1 Spectral properties for T_N

The results of the preceding section allow us to find straightforward spectral properties of heptadiagonal symmetric Toeplitz matrices with perturbed corners. In this section we are concerned with separation properties of the eigenvalues, structure of the eigenvectors and finally the development of efficient methods for finding eigenvalues and eigenvectors of integer powers of these matrices. Let us point out that this problem was studied where the eigenvalues are simple if the diagonal matrix has multiple eigenvalues then deflation can be used just as in [3, 6] to eliminate them converting the original problem into another one where the eigenvalues are simple, thus ensuring that the hypothesis holds.

Lemma 2. Let $\mathbf{T}_n$ in (1.1) be an $n\times n$ heptadiagonal symmetric Toeplitz matrices with perturbed corners and

$$\lambda_k = 2d\cos\left(\frac{3k\pi}{n+1}\right)+2c\cos\left(\frac{2k\pi}{n+1}\right)+2b\cos\left(\frac{k\pi}{n+1}\right)+a,\quad k=1,\ldots,n.$$

(a) If n is even, $\mathbf{u}$, $\mathbf{v}$ and $\xi^{(1)}$ are defined by (1.7) and (1.11) respectively,

i. $\lambda_1,\lambda_3,\ldots,\lambda_{n-1}$ are all distinct then the eigenvalues of
$diag(\lambda_1,\lambda_3,\ldots,\lambda_{n-1})+\xi^{(1)}\circ\mathbf{u}\mathbf{u}^T$ are the zeros of the rational function

$$f(t) = 1+\frac{4}{n+1}\sum_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{t-2d\cos\left[\frac{3(2i-1)\pi}{n+1}\right]-2c\cos\left[\frac{2(2i-1)\pi}{n+1}\right]-2b\cos\left[\frac{(2i-1)\pi}{n+1}\right]-a}. \tag{2.7}$$

Moreover, the eigenvalues $\beta_1,\ \beta_2,\ \ldots,\ \beta_{\frac{n}{2}}$ of $diag(\lambda_1, \lambda_3, \ldots, \lambda_{n-1}) + \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T$ are all simple and

$$\mathbf{f_j} = \begin{pmatrix} \dfrac{\left(c-a+e+4d\cos\left(\frac{\pi}{n+1}\right)\right)\sin\left(\frac{\pi}{n+1}\right)}{\left\{2d\cos\left(\frac{3\pi}{n+1}\right)+2c\cos\left(\frac{2\pi}{n+1}\right)+2b\cos\left(\frac{\pi}{n+1}\right)+a-\beta_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2i-1)\pi}{n+1}\right]+a-\beta_j\right\}^2}}} \\ \dfrac{\left(c-a+e+4d\cos\left(\frac{3\pi}{n+1}\right)\right)\sin\left(\frac{3\pi}{n+1}\right)}{\left\{2d\cos\left(\frac{9\pi}{n+1}\right)+2c\cos\left(\frac{6\pi}{n+1}\right)+2b\cos\left(\frac{3\pi}{n+1}\right)+a-\beta_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2i-1)\pi}{n+1}\right]+a-\beta_j\right\}^2}}} \\ \vdots \\ \dfrac{\left(c-a+e+4d\cos\left[\frac{(n-1)\pi}{n+1}\right]\right)\sin\left[\frac{(n-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(n-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(n-1)\pi}{n+1}\right]+2b\cos\left[\frac{(n-1)\pi}{n+1}\right]+a-\beta_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2i-1)\pi}{n+1}\right]+a-\beta_j\right\}^2}}} \end{pmatrix} \tag{2.8}$$

is an eigenvector associated to β_j, $j = 1, \ldots,\ \frac{n}{2}$.

ii. $\lambda_2, \lambda_4, \ldots, \lambda_n$ are all distinct then the eigenvalues of
$diag(\lambda_2, \lambda_4, \ldots, \lambda_n) + \xi^{(1)} \circ \mathbf{v}\mathbf{v}^T$ are the zeros of the rational function

$$g(t) = 1 + \frac{4}{n+1}\sum_{i=1}^{\frac{n}{2}} \frac{\left(c - a + e + 4d\cos\left(\frac{2i\pi}{n+1}\right)\right)\sin^2\left(\frac{2i\pi}{n+1}\right)}{t - 2d\cos\left[\frac{3(2i)\pi}{n+1}\right] - 2c\cos\left[\frac{2(2i)\pi}{n+1}\right] - 2b\cos\left[\frac{2i\pi}{n+1}\right] - a}. \tag{2.9}$$

Moreover, the eigenvalues $\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_{\frac{n}{2}}$ of $diag(\lambda_2, \lambda_4, \ldots, \lambda_n) + \xi^{(2)} \circ \mathbf{v}\mathbf{v}^T$ are all simple and

$$\mathbf{g_j} = \begin{pmatrix} \dfrac{\left(c-a+e+4d\cos\left(\frac{2\pi}{n+1}\right)\right)\sin\left(\frac{2\pi}{n+1}\right)}{\left\{2d\cos\left(\frac{3\pi}{n+1}\right)+2c\cos\left(\frac{2\pi}{n+1}\right)+2b\cos\left(\frac{\pi}{n+1}\right)+a-\gamma_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{3(2i)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i)\pi}{n+1}\right]+2b\cos\left[\frac{(2i)\pi}{n+1}\right]+a-\gamma_j\right\}^2}}} \\ \dfrac{\left(c-a+e+4d\cos\left(\frac{4\pi}{n+1}\right)\right)\sin\left(\frac{4\pi}{n+1}\right)}{\left\{2d\cos\left(\frac{6\pi}{n+1}\right)+2c\cos\left(\frac{4\pi}{n+1}\right)+2b\cos\left(\frac{2\pi}{n+1}\right)+a-\gamma_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{3(2i)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i)\pi}{n+1}\right]+2b\cos\left(\frac{2i\pi}{n+1}\right)+a-\gamma_j\right\}^2}}} \\ \vdots \\ \dfrac{\left(c-a+e+4d\cos\left(\frac{n\pi}{n+1}\right)\right)\sin\left(\frac{n\pi}{n+1}\right)}{\left\{2d\cos\left(\frac{3n\pi}{n+1}\right)+2c\cos\left(\frac{2n\pi}{n+1}\right)+2b\cos\left(\frac{n\pi}{n+1}\right)+a-\gamma_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left[\frac{2i\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(2i)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i)\pi}{n+1}\right]+2b\cos\left(\frac{2i\pi}{n+1}\right)+a-\gamma_j\right\}^2}}} \end{pmatrix} \tag{2.10}$$

is an eigenvector associated to γ_j, $j = 1, \ldots,\ \frac{n}{2}$.

(b) If n is odd, $\mathbf{u}$, $\mathbf{v}$ and $\xi^{(2)}$ are defined by (1.10) and (1.11) respectively,

i. $\lambda_1, \lambda_3, \ldots, \lambda_n$ are all distinct then the eigenvalues of
$diag(\lambda_1, \lambda_3, \ldots, \lambda_n) + \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T$ are the zeros of the rational function

$$f(t) = 1 + \frac{4}{n+1}\sum_{i=1}^{\frac{n+1}{2}} \frac{\left(c - a + e + 4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{t - 2d\cos\left[\frac{3(2i-1)\pi}{n+1}\right] - 2c\cos\left[\frac{2(2i-1)\pi}{n+1}\right] - 2b\cos\left[\frac{(2i-1)\pi}{n+1}\right] - a}. \tag{2.11}$$

Moreover, the eigenvalues $\beta_1,\ \beta_2,\ \ldots,\ \beta_{\frac{n+1}{2}}$ of $diag(\lambda_1, \lambda_3, \ldots, \lambda_n) + \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T$ are all simple and

$$\mathbf{f_j} = \begin{pmatrix} \dfrac{\left(c-a+e+4d\cos\left(\frac{\pi}{n+1}\right)\right)\sin\left(\frac{\pi}{n+1}\right)}{\left\{2d\cos\left(\frac{3\pi}{n+1}\right)+2c\cos\left(\frac{2\pi}{n+1}\right)+2b\cos\left(\frac{\pi}{n+1}\right)+a-\beta_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2i-1)\pi}{n+1}\right]+a-\beta_j\right\}^2}}} \\ \dfrac{\left(c-a+e+4d\cos\left(\frac{3\pi}{n+1}\right)\right)\sin\left(\frac{3\pi}{n+1}\right)}{\left\{2d\cos\left(\frac{9\pi}{n+1}\right)+2c\cos\left(\frac{6\pi}{n+1}\right)+2b\cos\left(\frac{3\pi}{n+1}\right)+a-\beta_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2i-1)\pi}{n+1}\right]+a-\beta_j\right\}^2}}} \\ \vdots \\ \dfrac{\left(c-a+e+4d\cos\left(\frac{n\pi}{n+1}\right)\right)\sin\left(\frac{n\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{3n\pi}{n+1}\right]+2c\cos\left[\frac{2n\pi}{n+1}\right]+2b\cos\left[\frac{n\pi}{n+1}\right]+a-\beta_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2i-1)\pi}{n+1}\right]+a-\beta_j\right\}^2}}} \end{pmatrix} \tag{2.12}$$

is an eigenvector associated to $\beta_j,\ j=1,\dots,\ \frac{n+1}{2}$.

ii. $\lambda_2,\lambda_4,\dots,\lambda_{n-1}$ are all distinct then the eigenvalues of $diag(\lambda_2,\lambda_4,\dots,\lambda_{n-1})+\xi^{(1)}\circ\mathbf{v}\mathbf{v}^T$ are the zeros of the rational function

$$g(t)=1+\frac{4}{(n+1)}\sum_{i=1}^{\frac{n-1}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)\sin^2\left(\frac{2i\pi}{n+1}\right)}{t-2d\cos\left[\frac{3(2i)\pi}{n+1}\right]-2c\cos\left[\frac{2(2i)\pi}{n+1}\right]-2b\cos\left(\frac{2i\pi}{n+1}\right)-a}. \tag{2.13}$$

Moreover, the eigenvalues $\gamma_1,\ \gamma_2,\ \dots,\ \gamma_{\frac{n-1}{2}}$ of $diag(\lambda_2,\lambda_4,\dots,\lambda_{n-1})+\xi^{(2)}\circ\mathbf{v}\mathbf{v}^T$ are all simple and

$$\mathbf{g_j}=\begin{pmatrix}\dfrac{\left(c-a+e+4d\cos\left(\frac{2\pi}{n+1}\right)\right)\sin\left(\frac{2\pi}{n+1}\right)}{\left\{2d\cos\left(\frac{3\pi}{n+1}\right)+2c\cos\left(\frac{2\pi}{n+1}\right)+2b\cos\left(\frac{\pi}{n+1}\right)+a-\gamma_j\right\}\sqrt{\sum_{i=1}^{\frac{n-1}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{3(2i)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i)\pi}{n+1}\right]+2b\cos\left(\frac{2i\pi}{n+1}\right)+a-\gamma_j\right\}^2}}}\\ \dfrac{\left(c-a+e+4d\cos\left(\frac{4\pi}{n+1}\right)\right)\sin\left(\frac{4\pi}{n+1}\right)}{\left\{2d\cos\left(\frac{6\pi}{n+1}\right)+2c\cos\left(\frac{4\pi}{n+1}\right)+2b\cos\left(\frac{2\pi}{n+1}\right)+a-\gamma_j\right\}\sqrt{\sum_{i=1}^{\frac{n-1}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left(\frac{3(2i)\pi}{n+1}\right)+2c\cos\left(\frac{2(2i)\pi}{n+1}\right)+2b\cos\left(\frac{2i\pi}{n+1}\right)+a-\gamma_j\right\}^2}}}\\ \vdots\\ \dfrac{\left(c-a+e+4d\cos\left[\frac{(n-1)\pi}{n+1}\right]\right)\sin\left[\frac{(n-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(n-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(n-1)\pi}{n+1}\right]+2b\cos\left[\frac{(n-1)\pi}{n+1}\right]+a-\gamma_j\right\}\sqrt{\sum_{i=1}^{\frac{n-1}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{3(2i)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i)\pi}{n+1}\right]+2b\cos\left[\frac{2i\pi}{n+1}\right]+a-\gamma_j\right\}^2}}}\end{pmatrix} \tag{2.14}$$

is an eigenvector associated to $\gamma_j,\ j=1,\dots,\ \frac{n-1}{2}$.

Proof . (a) and (b) can be proven in the same way, then we only prove (a). Let $n\in\mathbb{N}$ be even.
i. Suppose $\mathbf{u}$ given by (1.7), $\sqrt{\frac{2}{n+1}}\sin\left[\frac{(2i-1)\pi}{n+1}\right]\neq 0$ for $i=1,\dots,\frac{n}{2}$ and $\lambda_1,\lambda_3,\dots,\lambda_{n-1}$ in (1.1) all distinct, then

$$det\left[diag(\lambda_1,\lambda_3,\dots,\lambda_{n-1})+\xi^{(1)}\circ\mathbf{u}\mathbf{u}^T-tI_{\frac{n}{2}}\right]=f(t)\prod_{i=1}^{\frac{n}{2}}(\lambda_{2i-1}-t)$$

for this work, we use Equation (3) in [5] and [6].
The eigenvalues of $diag(\lambda_1,\lambda_3,\dots,\lambda_{n-1})+\xi^{(1)}\circ\mathbf{u}\mathbf{u}^T$ can be computed by finding the zeros of $f(t)$.

ii. Consider $\mathbf{v}$ given by (1.7), $\sqrt{\frac{2}{n+1}}\sin\left(\frac{2i\pi}{n+1}\right)\neq 0$ for $i=1,\dots,\frac{n}{2}$ and $\lambda_2,\lambda_4,\dots,\lambda_n$ in (1.1) all distinct, then

$$det\left[diag(\lambda_2,\lambda_4,\dots,\lambda_n)+\xi^{(1)}\circ\mathbf{v}\mathbf{v}^T-tI_{\frac{n}{2}}\right]=g(t)\prod_{i=1}^{\frac{n}{2}}(\lambda_{2i}-t).$$

The eigenvalues of $diag(\lambda_2,\lambda_4,\dots,\lambda_n)+\xi^{(2)}\circ\mathbf{v}\mathbf{v}^T$ can be computed by finding the zeros of $g(t)$. □ Now we suppose that eigenvectors corresponding to distinct eigenvalues $\beta_j,\ j=1,\dots,\ \frac{n}{2}$ in Lemma 2 for (a) n even are such that $\parallel\mathbf{f_j}\parallel=1$ for $j=1,\dots,\ \frac{n}{2}$, we can deduce that $\{\mathbf{f_1},...,\mathbf{f_{\frac{n}{2}}}\}$ is an orthonormal set, then we have an $\frac{n}{2}\times\frac{n}{2}$ orthogonal matrix

$$\mathbf{F}_{\frac{\mathbf{n}}{\mathbf{2}}}=\left(\frac{\left(c-a+e+4d\cos\left[\frac{(2k-1)\pi}{n+1}\right]\right)\sin\left[\frac{(2k-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(2k-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(2k-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2k-1)\pi}{n+1}\right]+a-\beta_j\right\}\sqrt{\sum_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2i-1)\pi}{n+1}\right]+a-\beta_j\right\}^2}}}\right)_{k,j} \tag{2.15}$$

Analogously, we have the $\frac{n}{2}\times\frac{n}{2}$ orthogonal matrix

$$\mathbf{G}_{\frac{\mathbf{n}}{\mathbf{2}}}=\left(\frac{\left(c-a+e+4d\cos\left(\frac{2k\pi}{n+1}\right)\right)\sin\left(\frac{2k\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{3(2k)\pi}{n+1}\right]+2c\cos\left[\frac{2(2k)\pi}{n+1}\right]+2b\cos\left(\frac{2k\pi}{n+1}\right)+a-\gamma_j\right\}\sqrt{\sum_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left(\frac{3(2i)\pi}{n+1}\right)+2c\cos\left(\frac{2(2i)\pi}{n+1}\right)+2b\cos\left(\frac{2i\pi}{n+1}\right)+a-\gamma_j\right\}^2}}}\right)_{k,j}. \tag{2.16}$$

We repeated the simulations above with sample (b) n odd, so that for i. we have an orthogonal matrix

$$\mathbf{F}_{\frac{\mathbf{n+1}}{\mathbf{2}}}=\left(\frac{\left(c-a+e+4d\cos\left[\frac{(2k-1)\pi}{n+1}\right]\right)\sin\left[\frac{(2k-1)\pi}{n+1}\right]}{\left\{2d\cos\left(\frac{3(2k-1)\pi}{n+1}\right)+2c\cos\left[\frac{2(2k-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2k-1)\pi}{n+1}\right]+a-\beta_j\right\}\sqrt{\sum_{i=1}^{\frac{n+1}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2i-1)\pi}{n+1}\right]+a-\beta_j\right\}^2}}}\right)_{k,j} \tag{2.17}$$

analogously, for ii. we deduce

$$\mathbf{G}_{\frac{\mathbf{n-1}}{\mathbf{2}}} = \left(\frac{\left(c-a+e+4d\cos\left[\frac{2k\pi}{n+1}\right]\right)\sin\left[\frac{2k\pi}{n+1}\right]}{\left\{2d\cos\left(\frac{3(2k)\pi}{n+1}\right)+2c\cos\left[\frac{2(2k)\pi}{n+1}\right]+2b\cos\left[\frac{2k\pi}{n+1}\right]+a-\gamma_j\right\}\sqrt{\sum_{i=1}^{\frac{n-1}{2}} \frac{\left(c-a+e+4d\cos\left[\frac{2i\pi}{n+1}\right)\right]^2 \sin^2\left[\frac{2i\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(2i)\pi}{n+1}\right]+2c\cos\left[\frac{2(2i)\pi}{n+1}\right]+2b\cos\left(\frac{2i\pi}{n+1}\right]+a-\gamma_j\right\}^2}}} \right)_{k,j} \tag{2.18}$$

by the column vectors (2.12) and (2.14).

Theorem 3. Let $\mathbf{T}_n$ in (1.1) be an $n \times n$ heptadiagonal symmetric Toeplitz matrices with perturbed corners,
(a) If n is even, $\lambda_1, \lambda_3, \ldots, \lambda_{n-1}$ are all distinct and $\lambda_2, \lambda_4, \ldots, \lambda_n$ are all distinct, $\beta_1,\ \beta_2,\ \ldots,\ \beta_{\frac{n}{2}}$ are the zeros of (2.7), $\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_{\frac{n}{2}}$ are the zeros of (2.9), $\mathbf{P}$ is the $n \times n$ permutation matrix (1.5), $\mathbf{U}$ is the $n \times n$ matrix in (1.6), $\mathbf{F}_{\frac{n}{2}}$ and $\mathbf{G}_{\frac{n}{2}}$ are the matrices (2.15) and (2.16) respectively, then

$$\mathbf{T}_n = \mathbf{UP} \begin{pmatrix} \mathbf{F}_{\frac{n}{2}} & 0 \\ 0 & \mathbf{G}_{\frac{n}{2}} \end{pmatrix} diag\left(\beta_1,\ \beta_2,\ \ldots,\ \beta_{\frac{n}{2}}, \gamma_1,\ \gamma_2,\ \ldots,\ \gamma_{\frac{n}{2}}\right) \begin{pmatrix} \mathbf{F}^T_{\frac{n}{2}} & 0 \\ 0 & \mathbf{G}^T_{\frac{n}{2}} \end{pmatrix} \mathbf{P^T U}.$$

(b) If n is odd, $\lambda_1, \lambda_3, \ldots, \lambda_n$ are all distinct and $\lambda_2, \lambda_4, \ldots, \lambda_{n-1}$ are all distinct, $\beta_1,\ \beta_2,\ \ldots,\ \beta_{\frac{n+1}{2}}$ are the zeros of (2.11), $\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_{\frac{n-1}{2}}$ are the zeros of (2.13), $\mathbf{P}$ is the $n \times n$ permutation matrix (1.8),
$\mathbf{U}$ is the $n \times n$ matrix in (1.9), $\mathbf{F}_{\frac{n+1}{2}}$ and $\mathbf{G}_{\frac{n-1}{2}}$ are the matrices (2.17) and (2.18) respectively, then

$$\mathbf{T}_n = \mathbf{UP} \begin{pmatrix} \mathbf{F}_{\frac{n+1}{2}} & 0 \\ 0 & \mathbf{G}_{\frac{n-1}{2}} \end{pmatrix} diag\left(\beta_1,\ \beta_2,\ \ldots,\ \beta_{\frac{n+1}{2}}, \gamma_1,\ \gamma_2,\ \ldots,\ \gamma_{\frac{n-1}{2}}\right) \begin{pmatrix} \mathbf{F}^T_{\frac{n+1}{2}} & 0 \\ 0 & \mathbf{G}^T_{\frac{n-1}{2}} \end{pmatrix} \mathbf{P^T U}.$$

Proof . (a) According to (a) of Lemma 2 and (2.15), we have

$$diag(\lambda_1, \lambda_3, \ldots, \lambda_{n-1}) + \xi^{(1)} \circ \mathbf{uu}^T = \mathbf{F}_{\frac{n}{2}} diag\left(\beta_1,\ \beta_2,\ \ldots,\ \beta_{\frac{n}{2}}\right) \mathbf{F}^T_{\frac{n}{2}} \tag{2.19}$$

where $\left(\beta_1,\ \beta_2,\ \ldots,\ \beta_{\frac{n}{2}}\right)$ are the zeros of (2.7) and the matrix $\mathbf{F}_{\frac{n}{2}}$ is orthogonal. We can also show

$$diag(\lambda_2, \lambda_4, \ldots, \lambda_n) + \xi^{(1)} \circ \mathbf{vv}^T = \mathbf{G}_{\frac{n}{2}} diag\left(\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_{\frac{n}{2}}\right) \mathbf{G}^T_{\frac{n}{2}} \tag{2.20}$$

where $\left(\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_{\frac{n}{2}}\right)$ are the zeros of (2.9) and the matrix $\mathbf{G}_{\frac{n}{2}}$ is orthogonal. Therefore, from (a) of Theorem 1 and Equations (2.19) and (2.20) we obtain

$$\mathbf{T}_n = \mathbf{UP} \begin{pmatrix} \mathbf{F}_{\frac{n}{2}} & 0 \\ 0 & \mathbf{G}_{\frac{n}{2}} \end{pmatrix} diag\left(\beta_1,\ \beta_2,\ \ldots,\ \beta_{\frac{n}{2}}, \gamma_1,\ \gamma_2,\ \ldots,\ \gamma_{\frac{n}{2}}\right) \begin{pmatrix} \mathbf{F}^T_{\frac{n}{2}} & 0 \\ 0 & \mathbf{G}^T_{\frac{n}{2}} \end{pmatrix} \mathbf{P^T U}. \tag{2.21}$$

(b) The proof of (b) follows the same steps of (a). According to (b) of Lemma 2, for n odd and $\lambda_1, \lambda_3, \ldots, \lambda_n$ are all distinct and $\lambda_2, \lambda_4, \ldots, \lambda_{n-1}$ are all distinct. In the same way, we have

$$diag(\lambda_1, \lambda_3, \ldots, \lambda_{n-1}) + \xi^{(1)} \circ \mathbf{uu}^T = \mathbf{F}_{\frac{n+1}{2}} diag\left(\beta_1,\ \beta_2,\ \ldots,\ \beta_{\frac{n+1}{2}}\right) \mathbf{F}^T_{\frac{n+1}{2}} \tag{2.22}$$

where $\left(\beta_1,\ \beta_2,\ \ldots,\ \beta_{\frac{n+1}{2}}\right)$ are the zeros of (2.11) and the matrix $\mathbf{F}_{\frac{n+1}{2}}$ is orthogonal. We can also write

$$diag(\lambda_2, \lambda_4, \ldots, \lambda_{n-1}) + \xi^{(1)} \circ \mathbf{vv}^T = \mathbf{G}_{\frac{n-1}{2}} diag\left(\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_{\frac{n-1}{2}}\right) \mathbf{G}^T_{\frac{n-1}{2}} \tag{2.23}$$

where $\left(\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_{\frac{n-1}{2}}\right)$ are the zeros of (2.13) and the matrix $\mathbf{G}_{\frac{n-1}{2}}$ is orthogonal. Therefore, from (b) of Theorem 1 and Equations (2.22) and (2.23) we obtain

$$\mathbf{T}_n = \mathbf{UP} \begin{pmatrix} \mathbf{F}_{\frac{n+1}{2}} & 0 \\ 0 & \mathbf{G}_{\frac{n-1}{2}} \end{pmatrix} diag\left(\beta_1,\ \ldots,\ \beta_{\frac{n+1}{2}}, \gamma_1,\ \ldots,\ \gamma_{\frac{n-1}{2}}\right) \begin{pmatrix} \mathbf{F}^T_{\frac{n+1}{2}} & 0 \\ 0 & \mathbf{G}^T_{\frac{n-1}{2}} \end{pmatrix} \mathbf{P^T U}. \tag{2.24}$$

□

We finish this section with more general results on spectral properties of heptadiagonal symmetric Toeplitz matrices with perturbed corners $\mathbf{T}_n$.
We need the following theorem of [13]:

Theorem 4. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be $n\times n$ symmetric matrices with eigenvalues x_i, y_i, z_i, respectively, arranged in increasing order, and $\mathbf{A} = \mathbf{B} + \mathbf{C}$; then $y_i + z_n \le x_i \le y_i + z_i$.

Moreover, we observe that if $\mathbf{D} = diag(\lambda_1, \ldots, \lambda_n)$ and $\mathbf{u} \in \mathbb{R}^n$, $u_i \neq 0$, $i = 1, 2, \ldots, n$ where $\lambda_1 \ge \lambda_2 \ldots \ge \lambda_n$, the eigenvalues τ_i, $i = 1, 2, \ldots, n$ of $\mathbf{D} + \mathbf{u}\mathbf{u}^T$ arranged in non-increasing order, are such that:

$$\lambda_1 + \parallel \mathbf{u} \parallel^2 \ge \tau_1 \ge \lambda_1, \quad \lambda_{i-1} \ge \tau_i \ge \lambda_i, \quad i = 2, 3, \ldots, \ n.$$

Furthermore, interchanging the roles of $\mathbf{D}$ and $\mathbf{u}\mathbf{u}^T$ we have $\mu_1 \ge \lambda_n + \parallel \mathbf{u} \parallel^2$.

Applying these bounds to the submatrices $\mathbf{D}_1 + \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T$ and $\mathbf{D}_2 + \xi^{(2)} \circ \mathbf{v}\mathbf{v}^T$ from decomposition (2.2) and (2.3), then we obtain

Lemma 5. Let

$$T_n = \begin{pmatrix} e & b & c & d & & & & & & \\ b & a & b & c & d & & & & & \\ c & b & a & b & c & d & & & & \\ d & c & b & a & b & c & d & & & \\ & d & c & b & a & b & c & d & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & d & c & b & a & b & c & d \\ & & & & d & c & b & a & b & c \\ & & & & & d & c & b & a & b \\ & & & & & & d & c & b & e \end{pmatrix}$$

its eigenvalues are partitionable into two subsets $\left(\beta_1, \ \beta_2, \ \ldots, \ \beta_{\frac{n}{2}}\right)$, $\left(\gamma_1, \ \gamma_2, \ \ldots, \ \gamma_{\frac{n}{2}}\right)$, for n even, $\left(\beta_1, \ \beta_2, \ \ldots, \ \beta_{\frac{n+1}{2}}\right)$, $\left(\gamma_1, \ \gamma_2, \ \ldots, \ \gamma_{\frac{n-1}{2}}\right)$, for n odd.
We only give the results for n *even*, the same way is shown for n odd.
Considering the polynomial $\eta(t) = dt^3 + ct^2 + (b - 3d)t + (a - 2c)$ [1, 11], and assuming that $\eta\left\{2\cos\left(\frac{i\pi}{n+1}\right)\right\}$ for $i = 1, 2, \ldots, n$ with λ_{2i-1} and λ_{2i}. Then we have

$$\lambda_{2i-1} \le \beta_i \le \lambda_{2i+1} \quad \lambda_{2i} \le \gamma_i \le \lambda_{2i+2}, \quad i = 1, 2, \ldots, \frac{n}{2}$$

$$\lambda_1 + \parallel \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T \parallel^2 \ge \beta_i \ge \max\{\lambda_3, \quad \lambda_{n-1} + \parallel \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T \parallel^2\}$$

$$\lambda_2 + \parallel \xi^{(2)} \circ \mathbf{v}\mathbf{v}^T \parallel^2 \ge \gamma_i \ge \max\{\lambda_4, \quad \lambda_n + \parallel \xi^{(2)} \circ \mathbf{v}\mathbf{v}^T \parallel^2\}$$

from decomposition (2.2) and (2.3).

Theorem 6. Let n, $m \in \mathbb{N}$ and $\mathbf{T}_n$ in (1.1) be an $n \times n$ heptadiagonal symmetric Toeplitz matrices with perturbed corners

(a) If n is even, $\lambda_1, \lambda_3, \ldots, \lambda_{n-1}$ are all distinct and $\lambda_2, \lambda_4, \ldots, \lambda_n$ are all distinct, $\beta_1, \ \beta_2, \ \ldots, \ \beta_{\frac{n}{2}}$ are the zeros of (2.7), $\gamma_1, \ \gamma_2, \ \ldots, \ \gamma_{\frac{n}{2}}$ are the zeros of (2.9), $\mathbf{P}$ is the $n \times n$ permutation matrix (1.5), $\mathbf{U}$ is the $n \times n$ matrix in (1.6), $\mathbf{F}_{\frac{n}{2}}$ and $\mathbf{G}_{\frac{n}{2}}$ are the matrices (2.15) and (2.16) respectively, $\mathbf{T}_n$ is nonsingular then for every integer m, we have

$$\mathbf{T}_n^m = \mathbf{UP}\begin{pmatrix} \mathbf{F}_{\frac{n}{2}} & 0 \\ 0 & \mathbf{G}_{\frac{n}{2}} \end{pmatrix} diag\left(\beta_1^m, \ \beta_2^m, \ \ldots, \ \beta_{\frac{n}{2}}^m, \gamma_1^m, \ \gamma_2^m, \ \ldots, \ \gamma_{\frac{n}{2}}^m\right) \begin{pmatrix} \mathbf{F}_{\frac{n}{2}}^T & 0 \\ 0 & \mathbf{G}_{\frac{n}{2}}^T \end{pmatrix} \mathbf{P}^{\mathbf{T}}\mathbf{U}.$$

(b) If n is odd, $\lambda_1, \lambda_3, \ldots, \lambda_n$ are all distinct and $\lambda_2, \lambda_4, \ldots, \lambda_{n-1}$ are all distinct, $\beta_1, \ \beta_2, \ \ldots, \ \beta_{\frac{n+1}{2}}$ are the zeros of (2.11), $\gamma_1, \ \gamma_2, \ \ldots, \ \gamma_{\frac{n-1}{2}}$ are the zeros of (2.13), $\mathbf{P}$ is the $n \times n$ permutation matrix (1.8),

$\mathbf{U}$ is the $n\times n$ matrix in (1.9), $\mathbf{F}_{\frac{n+1}{2}}$ and $\mathbf{G}_{\frac{n-1}{2}}$ are the matrices (2.17) and (2.18) respectively, $\mathbf{T}_n$ is nonsingular then for every integer m, we deduce

$$\mathbf{T}_n^m = \mathbf{UP}\begin{pmatrix} \mathbf{F}_{\frac{n+1}{2}} & 0 \\ 0 & \mathbf{G}_{\frac{n-1}{2}} \end{pmatrix} diag\left(\beta_1^m, \ \ldots, \ \beta^m_{\frac{n+1}{2}}, \gamma_1^m, \ \ldots, \ \gamma^m_{\frac{n-1}{2}}\right) \begin{pmatrix} \mathbf{F}^T_{\frac{n+1}{2}} & 0 \\ 0 & \mathbf{G}^T_{\frac{n-1}{2}} \end{pmatrix} \mathbf{P^T U}.$$

Proof . We can prove these results by applying (a) and (b) of Theorem 3. □

3 An orthogonal diagonalization of the matrix $\mathbf{H}_n$

Let an $n \times n$ Hankel matrix similar (1.2). The algorithm obtained in the Section 2 can be extended to anti-heptadiagonal persymmetric Hankel matrices with perturbed corners. This algorithm helps us to find a class of simultaneously diagonalizable matrices which have a suitable submatrix generating by band symmetric Hankel matrices. Let

$$\hat{\mathbf{H}}_n = \begin{pmatrix} & & & & & d & c & b-d & a-c \\ & & & & d & c & b & a & b-d \\ & & & d & c & b & a & b & c \\ & & d & c & b & a & b & c & d \\ & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \\ d & c & b & a & b & c & d & & \\ c & b & a & b & c & d & & & \\ b-d & a & b & c & d & & & & \\ a-c & b-d & c & d & & & & & \end{pmatrix}$$

and

$$\hat{\mathbf{E}}_H = \begin{pmatrix} & & & 0 & d & c-a+e \\ & & & 0 & 0 & d \\ & & & 0 & 0 & 0 \\ & & \iddots & \iddots & \iddots & \\ 0 & 0 & 0 & & & \\ d & 0 & 0 & & & \\ c-a+e & d & 0 & & & \end{pmatrix}.$$

Then $\mathbf{H}_n = \hat{\mathbf{H}}_n + \hat{\mathbf{E}}_H$, by some theoretical aspects in computational analysis of $\mathbf{U}\hat{\mathbf{E}}_H\mathbf{U}$ and

$$\mathbf{UH}_n\mathbf{U} = \mathbf{U}(\hat{\mathbf{H}}_n + \hat{\mathbf{E}}_H)\mathbf{U} = diag(\mu_1, \mu_2, \ldots, \mu_n) + \mathbf{E}_H$$

where

$$[\mathbf{E}_H]_{i,j} = \frac{2[1+(-1)^{i+j}]}{n+1} \sin\left(\frac{i\pi}{n+1}\right) \sin\left(\frac{j\pi}{n+1}\right) \left[c-a+e+4d\left(\cos\left[\frac{(i+j)\pi}{2n+2}\right] \cos\left[\frac{(i-j)\pi}{2n+2}\right]\right)\right]$$

for $i, j = 1, 2, \ldots, n$, since $[\mathbf{E}_H]_{ij} = 0$ whenever $i+j$ is odd, we can permute rows and columns of $diag(\mu_1, \mu_2, \ldots, \mu_n)+\mathbf{E}_H$ according to the permutation matrices (1.5) and (1.8) for n even or n odd and other convenien relations of Theorem 12 respectively, we have

$$\mathbf{H}_n = \mathbf{UP}\left(\begin{array}{c|c} \mathbf{D}_3 + \xi^{(1)} \circ \mathbf{uu}^T & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{D}_4 - \xi^{(2)} \circ \mathbf{vv}^T \end{array}\right)\mathbf{P^T U}. \tag{3.1}$$

Thus we have the following theorem:

Theorem 7. Let $\mathbf{H}_n$ be an $n \times n$ matrix similar (1.2) and μ_k, $k = 1, \ldots, n$ be given in (1.4),
(a) If n is even then

$$\mathbf{H}_n = \mathbf{UP}\left(\begin{array}{c|c} diag(\mu_1, \mu_3, \ldots, \mu_{n-1}) + \xi^{(1)} \circ \mathbf{uu}^T & \mathbf{0} \\ \hline \mathbf{0} & diag(\mu_2, \mu_4, \ldots, \mu_n) - \xi^{(2)} \circ \mathbf{vv}^T \end{array}\right)\mathbf{P^T U}. \tag{3.2}$$

where $\mathbf{P}$ is the $n \times n$ permutation matrix defined by (1.5), $\mathbf{U}$ in (1.6) is orthogonal matrix and $\mathbf{u}$, $\mathbf{v}$ are given by (1.7).
(b) If n is odd then

$$\mathbf{H}_n = \mathbf{UP}\left(\begin{array}{c|c} diag(\mu_1, \mu_3, \ldots, \mu_n) + \xi^{(1)} \circ \mathbf{uu}^T & \mathbf{0} \\ \hline \mathbf{0} & diag(\mu_2, \mu_4, \ldots, \mu_{n-1}) - \xi^{(2)} \circ \mathbf{vv}^T \end{array}\right)\mathbf{P^T U}. \tag{3.3}$$

where $\mathbf{P}$ is the $n \times n$ permutation matrix defined by (1.8), $\mathbf{U}$ in (1.9) and $\mathbf{u}$, $\mathbf{v}$ are given by (1.10).

The above theorem can be used for finding the inverse and the spectral properties of the matrix $\mathbf{H}_n$.
By $\mathbf{M} = \xi^{(1)} \circ \mathbf{uu}^T$ and $\mathbf{N} = -\xi^{(2)} \circ \mathbf{vv}^T$, $\mathbf{I}$ is the identity matrix (with convenient dimension) and Sherman-Morrison-Woodbury formula:

$$(\mathbf{D}_3 + \mathbf{M})^{-1} = \mathbf{D}_3^{-1} - \mathbf{D}_3^{-1}(\mathbf{I} + \mathbf{MD}_3^{-1})^{-1}\mathbf{MD}_3^{-1}$$

and

$$(\mathbf{D}_4 + \mathbf{N})^{-1} = \mathbf{D}_4^{-1} - \mathbf{D}_4^{-1}(\mathbf{I} + \mathbf{ND}_4^{-1})^{-1}\mathbf{ND}_4^{-1},$$

then we dedude

$$\mathbf{H}_n^{-1} = \mathbf{UP}\left(\begin{array}{c|c} \mathbf{D}_3^{-1}(\mathbf{I} - (\mathbf{I} + \mathbf{MD}_3^{-1})^{-1}\mathbf{MD}_3^{-1}) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{D}_4^{-1}(\mathbf{I} - (\mathbf{I} + \mathbf{ND}_4^{-1})^{-1}\mathbf{ND}_4^{-1}) \end{array}\right)\mathbf{P^T U}. \tag{3.4}$$

Also from (3.1) and using [5], for n even we deduce

$$\begin{aligned} det\,(\mathbf{H}_n) = &\left\{1 - \frac{4}{n+1}\sum_{i=1}^{\frac{n}{2}} \frac{\left[c - a + e + 4d\cos\left(\frac{(2i-1)\pi}{n+1}\right)\right]\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{2d\cos\left[\frac{(n-2)(2i-1)\pi}{n+1}\right] + 2c\cos\left[\frac{(n-1)(2i-1)\pi}{n+1}\right] + 2b\cos\left[\frac{n(2i-1)\pi}{n+1}\right] - a}\right\} \\ &\times\left\{1 + \frac{4}{n+1}\sum_{i=1}^{\frac{n}{2}} \frac{\left[c - a + e + 4d\cos\left(\frac{2i\pi}{n+1}\right)\right]\sin^2\left[\frac{2i\pi}{n+1}\right]}{2d\cos\left[\frac{2i(n-2)\pi}{n+1}\right] + 2c\cos\left[\frac{2i(n-1)\pi}{n+1}\right] + 2b\cos\left[\frac{2in\pi}{n+1}\right] + a}\right\} \\ &\times\prod_{i=1}^{n}\left\{2d\cos\left[\frac{(n-2)i\pi}{n+1}\right] + 2c\cos\left[\frac{(n-1)i\pi}{n+1}\right] + 2b\cos\left(\frac{ni\pi}{n+1}\right) + (-1)^i a\right\} \end{aligned} \tag{3.5}$$

and for n odd, we deduce

$$\begin{aligned} det\,(\mathbf{H}_n) = &\left\{\frac{4}{n+1}\sum_{i=1}^{\frac{n+1}{2}} \frac{\left[c - a + e + 4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right]\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{2d\cos\left[\frac{(n-2)(2i-1)\pi}{n+1}\right] + 2c\cos\left[\frac{(n-1)(2i-1)\pi}{n+1}\right] + 2b\cos\left[\frac{(2i-1)n\pi}{n+1}\right] - a} - 1\right\} \\ &\times\left\{1 + \frac{4}{n+1}\sum_{i=1}^{\frac{n-1}{2}} \frac{\left[c - a + e + 4d\cos\left(\frac{2i\pi}{n+1}\right)\right]\sin^2\left[\frac{2i\pi}{n+1}\right]}{2d\cos\left[\frac{2i(n-2)\pi}{n+1}\right] + 2c\cos\left[\frac{2i(n-1)\pi}{n+1}\right] + 2b\cos\left[\frac{2in\pi}{n+1}\right] + a}\right\} \\ &\times\prod_{i=1}^{n}\left\{2d\cos\left[\frac{(n-2)i\pi}{n+1}\right] + 2c\cos\left[\frac{(n-1)i\pi}{n+1}\right] + 2b\cos\left(\frac{ni\pi}{n+1}\right) + (-1)^i a\right\}. \end{aligned} \tag{3.6}$$

3.1 Spectral properties for H_N

We consider the spectral properties of anti-heptadiagonal persymmetric Hankel matrices with perturbed corners. Let us point out that this problem was considered where the eigenvalues are simple if the diagonal matrix has multiple eigenvalues then deflation can be used just as in [3, 6] to eliminate them converting the original problem into another one where the eigenvalues are simple, thus ensuring that the hypothesis holds.

Lemma 8. Let $\mathbf{H}_n$ in (1.2) be an $n \times n$ anti-heptadiagonal persymmetric Hankel matrices with perturbed corners, for $k = 1, \ldots, n$, we have

$$\mu_k = -2d\cos\left[\frac{(n-2)k\pi}{n+1}\right] - 2c\cos\left[\frac{(n-1)k\pi}{n+1}\right] - 2b\cos\left(\frac{nk\pi}{n+1}\right) - a\cos(k\pi).$$

(a) If n is even, $\mathbf{u}$, $\mathbf{v}$ and $\xi^{(1)}$ are defined by (1.7) and (1.11) respectively,

i. $\mu_1, \mu_3, \ldots, \mu_{n-1}$ are all distinct then the eigenvalues of $diag(\mu_1, \mu_3, \ldots, \mu_{n-1}) + \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T$ are the zeros of the rational function

$$r(t) = 1 - \frac{4}{n+1} \sum_{i=1}^{\frac{n}{2}} \frac{\left(c - a + e + 4d \cos\left[\frac{(2i-1)\pi}{n+1}\right]\right) \sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{2d\cos\left[\frac{(n-2)(2i-1)\pi}{n+1}\right] + 2c\cos\left[\frac{(n-1)(2i-1)\pi}{n+1}\right] + 2b\cos\left[\frac{n(2i-1)\pi}{n+1}\right] - a + t}. \tag{3.7}$$

Moreover, the eigenvalues $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{\frac{n}{2}}$ of $diag(\mu_1, \mu_3, \ldots, \mu_{n-1}) + \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T$ are all simple and

$$\mathbf{r_j} = \begin{pmatrix} \dfrac{\left(c-a+e+4d\cos\left(\frac{\pi}{n+1}\right)\right)\sin\left(\frac{\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{(n-1)\pi}{n+1}\right]+2b\cos\left[\frac{n\pi}{n+1}\right]-a+\alpha_j\right\}\sqrt{\sum_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{(n-2)(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{(n-1)(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{n(2i-1)\pi}{n+1}\right]-a+\alpha_j\right\}^2}}} \\ \dfrac{\left(c-a+e+4d\cos\left(\frac{3\pi}{n+1}\right)\right)\sin\left(\frac{3\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{3(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{3(n-1)\pi}{n+1}\right]+2b\cos\left(\frac{3n\pi}{n+1}\right)-a+\alpha_j\right\}\sqrt{\sum_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{(n-2)(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{(n-1)(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{n(2i-1)\pi}{n+1}\right])-a+\alpha_j\right\}^2}}} \\ \vdots \\ \dfrac{\left(c-a+e+4d\cos\left[\frac{(n-1)\pi}{n+1}\right]\right)\sin\left[\frac{(n-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{(n-2)(n-1)\pi}{n+1}\right]+2c\cos\left[\frac{(n-1)^2\pi}{n+1}\right]+2b\cos\left[\frac{(n-1)n\pi}{n+1}\right]-a+\alpha_j\right\}\sqrt{\sum_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{(n-2)(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{(n-1)(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{n(2i-1)\pi}{n+1}\right]-a+\alpha_j\right\}^2}}} \end{pmatrix} \tag{3.8}$$

is an eigenvector associated to α_j, $j = 1, \ldots,\ \frac{n}{2}$.

ii. $\mu_2, \mu_4, \ldots, \mu_n$ are all distinct then the eigenvalues of $diag(\mu_2, \mu_4, \ldots, \mu_n) - \xi^{(1)} \circ \mathbf{v}\mathbf{v}^T$ are the zeros of the rational function

$$s(t) = 1 + \frac{4}{n+1} \sum_{i=1}^{\frac{n}{2}} \frac{\left(c - a + e + 4d \cos\left(\frac{2i\pi}{n+1}\right)\right) \sin^2\left(\frac{2i\pi}{n+1}\right)}{2d\cos\left[\frac{2i(n-2)\pi}{n+1}\right] + 2c\cos\left[\frac{2i(n-1)\pi}{n+1}\right] + 2b\cos\left(\frac{2in\pi}{n+1}\right) + a + t}. \tag{3.9}$$

Moreover, the eigenvalues $\theta_1,\ \theta_2,\ \ldots,\ \theta_{\frac{n}{2}}$ of $diag(\mu_2, \mu_4, \ldots, \mu_n) - \xi^{(2)} \circ \mathbf{v}\mathbf{v}^T$ are all simple and

$$\mathbf{s_j} = \begin{pmatrix} \dfrac{\left(c-a+e+4d\cos\left(\frac{2\pi}{n+1}\right)\right)\sin\left(\frac{2\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{2(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{2(n-1)\pi}{n+1}\right]+2b\cos\left(\frac{2n\pi}{n+1}\right)+a+\theta_j\right\}\sqrt{\sum_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{2i(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{2i(n-1)\pi}{n+1}\right]+2b\cos\left(\frac{2in\pi}{n+1}\right)+a+\theta_j\right\}^2}}} \\ \dfrac{\left(c-a+e+4d\cos\left(\frac{4\pi}{n+1}\right)\right)\sin\left(\frac{4\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{4(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{4(n-1)\pi}{n+1}\right]+2b\cos\left(\frac{4n\pi}{n+1}\right)+a+\theta_j\right\}\sqrt{\sum_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{2i(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{2i(n-1)\pi}{n+1}\right]+2b\cos\left(\frac{2in\pi}{n+1}\right)+a+\theta_j\right\}^2}}} \\ \vdots \\ \dfrac{\left(c-a+e+4d\cos\left(\frac{n\pi}{n+1}\right)\right)\sin\left(\frac{n\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{(n-2)n\pi}{n+1}\right]+2c\cos\left[\frac{(n-1)n\pi}{n+1}\right]+2b\cos\left(\frac{n^2\pi}{n+1}\right)+a+\theta_j\right\}\sqrt{\sum_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{2i(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{2i(n-1)\pi}{n+1}\right]+2b\cos\left(\frac{2in\pi}{n+1}\right)+a+\theta_j\right\}^2}}} \end{pmatrix} \tag{3.10}$$

is an eigenvector associated to θ_j, $j = 1, \ldots,\ \frac{n}{2}$.

(b) If n is odd, $\mathbf{u}$, $\mathbf{v}$ and $\xi^{(2)}$ are defined by (1.10) and (1.11) respectively,

i. $\mu_1, \mu_3, \ldots, \mu_n$ are all distinct then the eigenvalues of $diag(\mu_1, \mu_3, \ldots, \mu_n) + \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T$ are the zeros of the rational function

$$r(t) = 1 - \frac{4}{n+1} \sum_{i=1}^{\frac{n+1}{2}} \frac{\left(c - a + e + 4d \cos\left[\frac{(2i-1)\pi}{n+1}\right]\right) \sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{2d\cos\left[\frac{(n-2)(2i-1)\pi}{n+1}\right] + 2c\cos\left[\frac{(n-1)(2i-1)\pi}{n+1}\right] + 2b\cos\left[\frac{n(2i-1)\pi}{n+1}\right] - a + t}. \tag{3.11}$$

Moreover, the eigenvalues $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{\frac{n+1}{2}}$ of $diag(\mu_1,\mu_3,\ldots,\mu_n)+\xi^{(1)}\circ \mathbf{u}\mathbf{u}^T$ are all simple and

$$\mathbf{r}_j=\begin{pmatrix} \dfrac{\left(c-a+e+4d\cos\left(\frac{\pi}{n+1}\right)\right)\sin\left(\frac{\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{(n-1)\pi}{n+1}\right]+2b\cos\left(\frac{n\pi}{n+1}\right)-a+\alpha_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n+1}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{(n-2)(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{(n-1)(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{n(2i-1)\pi}{n+1}\right]-a+\alpha_j\right\}^2}}} \\ \dfrac{\left(c-a+e+4d\cos\left(\frac{3\pi}{n+1}\right)\right)\sin\left(\frac{3\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{3(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{3(n-1)\pi}{n+1}\right]+2b\cos\left(\frac{3n\pi}{n+1}\right)-a+\alpha_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n+1}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{(n-2)(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{(n-1)(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{n(2i-1)\pi}{n+1}\right]-a+\alpha_j\right\}^2}}} \\ \vdots \\ \dfrac{\left(c-a+e+4d\cos\left(\frac{n\pi}{n+1}\right)\right)\sin\left(\frac{n\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{(n-2)n\pi}{n+1}\right]+2c\cos\left[\frac{(n-1)n\pi}{n+1}\right]+2b\cos\left[\frac{n^2\pi}{n+1}\right]-a+\alpha_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n+1}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{(n-2)(2i-1)\pi}{n+1}\right]+2c\cos\left[\frac{(n-1)(2i-1)\pi}{n+1}\right]+2b\cos\left[\frac{n(2i-1)\pi}{n+1}\right]-a+\alpha_j\right\}^2}}} \end{pmatrix} \tag{3.12}$$

is an eigenvector associated to α_j, $j=1,\ldots,\ \frac{n+1}{2}$.

ii. $\mu_2,\mu_4,\ldots,\mu_{n-1}$ are all distinct then the eigenvalues of
$diag(\mu_2,\mu_4,\ldots,\mu_{n-1})-\xi^{(1)}\circ \mathbf{v}\mathbf{v}^T$ are the zeros of the rational function

$$s(t)=1+\frac{4}{n+1}\sum_{i=1}^{\frac{n-1}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)\sin^2\left(\frac{2i\pi}{n+1}\right)}{2d\cos\left[\frac{2i(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{2i(n-1)\pi}{n+1}\right]+2b\cos\left(\frac{2ni\pi}{n+1}\right)+a+t}. \tag{3.13}$$

Moreover, the eigenvalues $\theta_1,\ \theta_2,\ \ldots,\ \theta_{\frac{n-1}{2}}$ of $diag(\mu_2,\mu_4,\ldots,\mu_{n-1})-\xi^{(2)}\circ \mathbf{v}\mathbf{v}^T$ are all simple and

$$\mathbf{s_j}=\begin{pmatrix} \dfrac{\left(c-a+e+4d\cos\left(\frac{2\pi}{n+1}\right)\right)\sin\left(\frac{2\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{2(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{2(n-1)\pi}{n+1}\right]+2b\cos\left[\frac{2n\pi}{n+1}\right]+a+\theta_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n-1}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{2i(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{2i(n-1)\pi}{n+1}\right]+2b\cos\left(\frac{2in\pi}{n+1}\right)+a+\theta_j\right\}^2}}} \\ \dfrac{\left(c-a+e+4d\cos\left(\frac{4\pi}{n+1}\right)\right)\sin\left(\frac{4\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{4(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{4(n-1)\pi}{n+1}\right]+2b\cos\left(\frac{4n\pi}{n+1}\right)+a+\theta_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n-1}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{2i(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{2i(n-1)\pi}{n+1}\right]+2b\cos\left[\frac{2in\pi}{n+1}\right]+a+\theta_j\right\}^2}}} \\ \vdots \\ \dfrac{\left(c-a+e+4d\cos\left[\frac{(n-1)\pi}{n+1}\right]\right)\sin\left[\frac{(n-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{(n-2)(n-1)\pi}{n+1}\right]+2c\cos\left[\frac{(n-1)^2\pi}{n+1}\right]+2b\cos\left[\frac{(n-1)n\pi}{n+1}\right]+a+\theta_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n-1}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{2i(n-2)\pi}{n+1}\right]+2c\cos\left[\frac{2i(n-1)\pi}{n+1}\right]+2b\cos\left(\frac{2in\pi}{n+1}\right)+a+\theta_j\right\}^2}}} \end{pmatrix} \tag{3.14}$$

is an eigenvector associated to θ_j, $j=1,\ldots,\ \frac{n-1}{2}$.

Proof . Use the procedure described in Lemma 2 for proof (a) and (b). □ We consider the eigenvectors corresponding to distinct eigenvalues α_j, $j=1,\ldots,\ \frac{n}{2}$ in Lemma 8 for n even such that $\|\ \mathbf{r}_j\ \|=1$ for $j=1,\ldots,\ \frac{n}{2}$, we have $\{\mathbf{r_1},...,\mathbf{r}_{\frac{n}{2}}\}$ is an orthonormal set, then we have an $\frac{n}{2}\times\frac{n}{2}$ orthogonal matrix

$$\mathbf{R}_{\frac{n}{2}}=\left(\frac{\left(c-a+e+4d\cos\left[\frac{(2k-1)\pi}{n+1}\right]\right)\sin\left[\frac{(2k-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(2k-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(2k-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2k-1)\pi}{n+1}\right]+a-\alpha_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left(\frac{3\pi}{n+1}\right)+2c\cos\left(\frac{2\pi}{n+1}\right)+2b\cos\left(\frac{\pi}{n+1}\right)+a-\alpha_j\right\}^2}}}\right)_{k,j} \tag{3.15}$$

Analogously, we have the $\frac{n}{2}\times\frac{n}{2}$ orthogonal matrix

$$\mathbf{S}_{\frac{n}{2}}=\left(\frac{\left(c-a+e+4d\cos\left(\frac{2k\pi}{n+1}\right)\right)\sin\left(\frac{2k\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{3(2k)\pi}{n+1}\right]+2c\cos\left[\frac{2(2k)\pi}{n+1}\right]+2b\cos\left(\frac{2k\pi}{n+1}\right)+a-\theta_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left(\frac{3\pi}{n+1}\right)+2c\cos\left(\frac{2\pi}{n+1}\right)+2b\cos\left(\frac{\pi}{n+1}\right)+a-\theta_j\right\}^2}}}\right)_{k,j}. \tag{3.16}$$

We repeated the simulations above with sample n odd so that for i. we have an orthogonal matrix

$$\mathbf{R}_{\frac{n+1}{2}}=\left(\frac{\left(c-a+e+4d\cos\left[\frac{(2k-1)\pi}{n+1}\right]\right)\sin\left[\frac{(2k-1)\pi}{n+1}\right]}{\left\{2d\cos\left[\frac{3(2k-1)\pi}{n+1}\right]+2c\cos\left[\frac{2(2k-1)\pi}{n+1}\right]+2b\cos\left[\frac{(2k-1)\pi}{n+1}\right]+a-\alpha_j\right\}\sqrt{\sum\limits_{i=1}^{\frac{n+1}{2}}\frac{\left(c-a+e+4d\cos\left[\frac{(2i-1)\pi}{n+1}\right]\right)^2\sin^2\left[\frac{(2i-1)\pi}{n+1}\right]}{\left\{2d\cos\left(\frac{3\pi}{n+1}\right)+2c\cos\left(\frac{2\pi}{n+1}\right)+2b\cos\left(\frac{\pi}{n+1}\right)+a-\alpha_j\right\}^2}}}\right)_{k,j} \tag{3.17}$$

Analogously, for ii. we define

$$\mathbf{S}_{\frac{n-1}{2}} = \left(\frac{\left(c-a+e+4d\cos\left(\frac{2k\pi}{n+1}\right)\right)\sin\left(\frac{2k\pi}{n+1}\right)}{\left\{2d\cos\left[\frac{3(2k)\pi}{n+1}\right]+2c\cos\left[\frac{2(2k)\pi}{n+1}\right]+2b\cos\left(\frac{2k\pi}{n+1}\right)+a-\theta_j\right\}\sqrt{\sum_{i=1}^{\frac{n-1}{2}}\frac{\left(c-a+e+4d\cos\left(\frac{2i\pi}{n+1}\right)\right)^2\sin^2\left(\frac{2i\pi}{n+1}\right)}{\left\{2d\cos\left(\frac{3\pi}{n+1}\right)+2c\cos\left(\frac{2\pi}{n+1}\right)+2b\cos\left(\frac{\pi}{n+1}\right)+a-\theta_j\right\}^2}}} \right)_{k,j}. \tag{3.18}$$

Theorem 9. Let $\mathbf{H}_n$ in (1.2) be an $n \times n$ anti-heptadiagonal persymmetric Hankel matrices with perturbed corners,
(a) If n is even, $\mu_1, \mu_3, \ldots, \mu_{n-1}$ are all distinct and $\mu_2, \mu_4, \ldots, \mu_n$ are all distinct, $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{\frac{n}{2}}$ are the zeros of (3.7), $\theta_1,\ \theta_2,\ \ldots,\ \theta_{\frac{n}{2}}$ are the zeros of (3.9), $\mathbf{P}$ is the $n \times n$ permutation matrix (1.5), $\mathbf{U}$ is the $n \times n$ matrix in (1.6), $\mathbf{R}_{\frac{n}{2}}$ and $\mathbf{S}_{\frac{n}{2}}$ are the matrices (3.15) and (3.16) respectively, then

$$\mathbf{H}_n = \mathbf{UP}\begin{pmatrix} \mathbf{R}_{\frac{n}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{\frac{n}{2}} \end{pmatrix} diag\left(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{\frac{n}{2}}, \theta_1,\ \theta_2,\ \ldots,\ \theta_{\frac{n}{2}}\right) \begin{pmatrix} \mathbf{R}^T_{\frac{n}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^T_{\frac{n}{2}} \end{pmatrix} \mathbf{P^T U}. \tag{3.19}$$

(b) If n is odd, $\mu_1, \mu_3, \ldots, \mu_n$ are all distinct and $\mu_2, \mu_4, \ldots, \mu_{n-1}$ are all distinct, $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{\frac{n+1}{2}}$ are the zeros of (3.11), $\theta_1,\ \theta_2,\ \ldots,\ \theta_{\frac{n-1}{2}}$ are the zeros of (3.13), $\mathbf{P}$ is the $n \times n$ permutation matrix (1.8),
$\mathbf{U}$ is the $n \times n$ matrix in (1.9), $\mathbf{R}_{\frac{n+1}{2}}$ and $\mathbf{S}_{\frac{n-1}{2}}$ are the matrices (3.17) and (3.18) respectively, then

$$\mathbf{H}_n = \mathbf{UP}\begin{pmatrix} \mathbf{R}_{\frac{n+1}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{\frac{n-1}{2}} \end{pmatrix} diag\left(\alpha_1,\ \ldots,\ \alpha_{\frac{n+1}{2}}, \theta_1,\ \ldots,\ \theta_{\frac{n-1}{2}}\right) \begin{pmatrix} \mathbf{R}^T_{\frac{n+1}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^T_{\frac{n-1}{2}} \end{pmatrix} \mathbf{P^T U}. \tag{3.20}$$

Proof . (a) In Lemma 8, we have

$$diag(\mu_1, \mu_3, \ldots, \mu_{n-1}) + \xi^{(1)} \circ \mathbf{uu}^T = \mathbf{R}_{\frac{n}{2}} diag\left(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{\frac{n}{2}}\right) \mathbf{R}^T_{\frac{n}{2}} \tag{3.21}$$

where $\left(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{\frac{n}{2}}\right)$ are the zeros of (3.7) and the matrix $\mathbf{R}_{\frac{n}{2}}$ is orthogonal. We can also show

$$diag(\mu_2, \mu_4, \ldots, \mu_n) + \xi^{(1)} \circ \mathbf{vv}^T = \mathbf{S}_{\frac{n}{2}} diag\left(\theta_1,\ \theta_2,\ \ldots,\ \theta_{\frac{n}{2}}\right) \mathbf{S}^T_{\frac{n}{2}} \tag{3.22}$$

where $\left(\theta_1,\ \theta_2,\ \ldots,\ \theta_{\frac{n}{2}}\right)$ are the zeros of (3.9) and the matrix $\mathbf{S}_{\frac{n}{2}}$ is orthogonal. Therefore, from (a) of Theorem 7 and (3.21) and (3.22) we obtain

$$\mathbf{H}_n = \mathbf{UP}\begin{pmatrix} \mathbf{R}_{\frac{n}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{\frac{n}{2}} \end{pmatrix} diag\left(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{\frac{n}{2}}, \theta_1,\ \theta_2,\ \ldots,\ \theta_{\frac{n}{2}}\right) \begin{pmatrix} \mathbf{R}^T_{\frac{n}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^T_{\frac{n}{2}} \end{pmatrix} \mathbf{P^T U}.$$

The same method can also be used to prove the part (b). □ Now we are in a position to extend the results of Theorem 4 for anti-heptadiagonal persymmetric Hankel matrices with perturbed corners.

Lemma 10. Let

$$\mathbf{H}_n = \begin{pmatrix} & & & & & & d & c & b & e \\ & & & & & d & c & b & a & b \\ & & & & d & c & b & a & b & c \\ & & & d & c & b & a & b & c & d \\ & & d & c & b & a & b & c & d & \\ & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & & \\ d & c & b & e & b & c & d & & & \\ c & b & a & b & c & d & & & & \\ b & a & b & c & d & & & & & \\ e & b & c & d & & & & & & \end{pmatrix},$$

its eigenvalues are partitionable into two subsets
$\left(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{\frac{n}{2}}\right)$, $\left(\theta_1,\ \theta_2,\ \ldots,\ \theta_{\frac{n}{2}}\right)$, for n even,

$\left(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{\frac{n+1}{2}}\right)$, $\left(\theta_1,\ \theta_2,\ \ldots,\ \theta_{\frac{n-1}{2}}\right)$, for n odd.
We only extend the results for n *even*, the same way is shown for n odd.
Considering the polynomial $\phi(t) = -dt^3 - ct^2 - (b-3d)t - (a-2c)$, and assuming that $\phi\left\{2\cos\left(\frac{i\pi}{n+1}\right)\right\}$ for $i = 1, 2, \ldots, n$ with μ_{2i-1} and μ_{2i}. Then we have

$$\mu_{2i-1} \le \alpha_i \le \mu_{2i+1} \quad \mu_{2i} \le \theta_i \le \mu_{2i+2}, \quad i = 1, 2, \ldots, \frac{n}{2}$$

$$\mu_1 + \parallel \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T \parallel^2 \ge \alpha_i \ge \max\{\mu_3, \quad \mu_{n-1} + \parallel \xi^{(1)} \circ \mathbf{u}\mathbf{u}^T \parallel^2\}$$

$$\mu_2 + \parallel \xi^{(2)} \circ \mathbf{v}\mathbf{v}^T \parallel^2 \ge \theta_i \ge \max\{\mu_4, \quad \mu_n + \parallel \xi^{(2)} \circ \mathbf{v}\mathbf{v}^T \parallel^2\}.$$

Theorem 11. Let $n,\ m \in \mathbb{N}$ and $\mathbf{H}_n$ in (1.2) be an $n \times n$ anti-heptadiagonal persymmetric Hankel matrices with perturbed corners,

(a) If n is even, $\mu_1, \mu_3, \ldots, \mu_{n-1}$ are all distinct and $\mu_2, \mu_4, \ldots, \mu_n$ are all distinct, $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{\frac{n}{2}}$ are the zeros of (3.7), $\theta_1,\ \theta_2,\ \ldots,\ \theta_{\frac{n}{2}}$ are the zeros of (3.9), $\mathbf{P}$ is the $n \times n$ permutation matrix (1.5), $\mathbf{U}$ is the $n \times n$ matrix in (1.6), $\mathbf{R}_{\frac{n}{2}}$ and $\mathbf{S}_{\frac{n}{2}}$ are the matrices (3.15) and (3.16) respectively, $\mathbf{H}_n$ is nonsingular then for every integer m, we have

$$\mathbf{H}_n^m = \mathbf{UP}\begin{pmatrix} \mathbf{R}_{\frac{n}{2}} & \mathbf{0} \\ \mathbf{0} & S_{\frac{n}{2}} \end{pmatrix} diag\left(\alpha_1^m,\ \alpha_2^m,\ \ldots,\ \alpha_{\frac{n}{2}}^m, \theta_1^m,\ \theta_2^m,\ \ldots,\ \theta_{\frac{n}{2}}^m\right)\begin{pmatrix} \mathbf{R}_{\frac{n}{2}}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{\frac{n}{2}}^T \end{pmatrix}\mathbf{P^T U}.$$

(b) If n is odd, $\mu_1, \mu_3, \ldots, \mu_n$ are all distinct and $\mu_2, \mu_4, \ldots, \mu_{n-1}$ are all distinct, $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{\frac{n+1}{2}}$ are the zeros of (3.11), $\theta_1,\ \theta_2,\ \ldots,\ \theta_{\frac{n-1}{2}}$ are the zeros of (3.13), $\mathbf{P}$ is the $n \times n$ permutation matrix (1.8),
$\mathbf{U}$ is the $n \times n$ matrix in (1.9), $\mathbf{R}_{\frac{n+1}{2}}$ and $\mathbf{S}_{\frac{n-1}{2}}$ are the matrices (3.17) and (3.18) respectively, $\mathbf{H}_n$ is nonsingular then for every integer m, we deduce

$$\mathbf{H}_n^m = \mathbf{UP}\begin{pmatrix} \mathbf{R}_{\frac{n+1}{2}} & \mathbf{0} \\ \mathbf{0} & S_{\frac{n-1}{2}} \end{pmatrix} diag\left(\alpha_1^m,\ \ldots,\ \alpha_{\frac{n+1}{2}}^m, \theta_1^m,\ \ldots,\ \theta_{\frac{n-1}{2}}^m\right)\begin{pmatrix} \mathbf{R}^T{}_{\frac{n+1}{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^T{}_{\frac{n-1}{2}} \end{pmatrix}\mathbf{P^T U}.$$

Proof . We can easily deduce these results by Theorem 9. □

4 Final Comments

In this section, an orthogonal block diagonalization of the matrix $\mathbf{T}_n + \mathbf{H}_n$ is introduced by $\mathbf{T}_n$ and $\mathbf{H}_n$ that are the heptadiagonal symmetric Toeplitz matrices and anti-heptadiagonal persymmetric Hankel matrices, both having perturbed corners. The representations (2.1) and (3.1) yield results for computing the inverse, the determinant and eigenproblems for this class of matrices. Also, these relations help us to find an orthogonal block diagonalization of the matrix $\mathbf{T}_n + \mathbf{H}_n$:

Theorem 12. Let $\mathbf{T}_n + \mathbf{H}_n$ be an $n \times n$ Toeplitz-plus-Hankel matrix with perturbed corners

$$\begin{pmatrix}
e & b & c & d & & & & & & & d & c & b & e \\
b & a & b & c & d & & & & & d & c & b & a & b \\
c & b & a & b & c & d & & & d & c & b & a & b & c \\
d & c & b & a & b & c & d & d & c & b & a & b & c & d \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \\
& \iddots & \iddots & \iddots & \iddots & \iddots & \iddots \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
d & c & b & a & b & c & d & d & c & b & a & b & c & d \\
c & b & a & b & c & d & & & d & c & b & a & b & c \\
b & a & b & c & d & & & & & d & c & b & a & b \\
e & b & c & d & & & & & & & d & c & b & e
\end{pmatrix} \tag{4.1}$$

that the following Toeplitz matrix and Hankel matrix are shown in (1.1) and (1.2). Then the following relation holds:

$$\mathbf{T}_n + \mathbf{H}_n = \mathbf{UP} \left(\begin{array}{c|c} \mathbf{D}_1 + \mathbf{D}_3 + 2\ (\xi^{(1)} \circ \mathbf{uu}^T) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right) \mathbf{P}^T \mathbf{U}. \tag{4.2}$$

Proof . Here we prove this theorem using Theorem 3 and Theorem 9. We should collect Equations (2.21) and (3.19) for n even, and Equations (2.24) and (3.20) for n odd. Then

$$\mathbf{T}_n + \mathbf{H}_n = \mathbf{UP} \left(\begin{array}{c|c} \mathbf{D}_1 + \mathbf{D}_3 + 2\ (\xi^{(1)} \circ \mathbf{uu}^T) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right) \mathbf{P}^{\mathbf{T}} \mathbf{U}.$$

□

Also,

Corollary 1. Let $\mathbf{T}_n + \mathbf{H}_n$ be an $n \times n$ Toeplitz-plus-Hankel matrix with perturbed corners (4.1) and $m \in \mathbb{N}$, then by using Theorem 6 and Theorem 11 are shown:

$$[\mathbf{T}_n + \mathbf{H}_n]^m = \mathbf{UP} \left(\begin{array}{c|c} [\mathbf{D}_1 + \mathbf{D}_3 + 2\ (\xi^{(1)} \circ \mathbf{uu}^T)]^m & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right) \mathbf{P}^{\mathbf{T}} \mathbf{U}.$$

These updating methods may yield faster performance or improved accuracy, either for sequential or for parallel computations in similar cases.

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