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Hermite-Hadamard type fractional integral inequalities for strongly generalized-prequasi-invex function

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Abstract

In this research paper we studied strongly generalized-prequasi-invex function. Built on the new definition, k-Riemann–Liouville fractional integral inequalities for strongly generalized-prequasi-invex functions are estimated. A bunch of new Hermite–Hadamard type Inequalities in this direction via Katugampola fractional integrals are also derived.

Keywords: Hermite–Hadamard type Inequality, Strongly generalized-prequasi-invex functions, k-Riemann–Liouville fractional integrals, Katugampola fractional integrals
2020 MSC: 26D10; 90C26; 26A33

1 Introduction

The concept of preinvex functions was given by Weir [15, 14]. This concept provides a wide setting to investigate the optimization problems. Yang et al. [16] discussed numerous concepts of prequasiinvex functions. For more investigation on the generalized strongly convex functions. See, [1, 2, 8] and references therein.
A preeminent inequality is Hermite Hadamard inequality started by J. Hadamard in 1881.

$$\chi\left(\frac{m_1+m_2}{2}\right) \leq \frac{1}{m_2-m_1}\int_{m_1}^{m_2}\chi(z)dz \leq \frac{\chi(m_1)+\chi(m_2)}{2}$$

There are several results associated with Hermite Hadamard inequality for convex functions and their generalizations involving different kinds of fractional operators. See ([3, 7, 9, 10] and references therein).
In this research paper, we study the concept of strongly generalized-prequasi-invex functions as a generalization of strongly η-quasiconvex functions. Utilizing the concept of strongly generalized-prequasi-invex functions we establish some new k-Riemann–Liouville fractional integral inequalities and Hermite–Hadamard involving the Katugampola fractional integrals. Our outcomes are generalizations of the consequences obtained in paper [5] and [11].

2 Preliminary results

This section consist of some basic definitions and our proposed definition that will be mandatory for the establishment of results presented in next sections.

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Definition 2.1. [15] Consider $\mathcal{N} \subseteq R$ and let $\eta : \mathcal{N} \times \mathcal{N} \to R$, $\mathcal{N}$ is called to be invex function with respect to η iff

$$m_1 + \Lambda\eta(m_2, m_1) \in \mathcal{N}, \quad \forall m_1, m_2 \in \mathcal{N}, \Lambda \in [0, 1] \tag{2.1}$$

Definition 2.2. [15] Consider $\mathcal{N} \subseteq R$ be an invex subset of R. The function $\chi : \mathcal{N} \to R$ is called to be preinvex on $\mathcal{N}$ with respect to η if

$$\chi(m_1 + \Lambda\eta(m_2, m_1)) \leq \Lambda\chi(m_2) + (1 - \Lambda)\chi(m_1) \quad \forall m_1, m_2 \in \mathcal{N}, \Lambda \in [0, 1] \tag{2.2}$$

Definition 2.3. [16] Consider $\mathcal{N} \subseteq R$ be an invex subset with respect to $\eta : \mathcal{N} \times \mathcal{N} \to R$. Let $\chi : \mathcal{N} \to R$. Then χ is called to be prequasi-invex on $\mathcal{N}$ if,

$$\chi(m_1 + \Lambda\eta(m_2, m_1)) \leq \max\{\chi(m_1), \chi(m_2)\} \quad \forall m_1, m_2 \in \mathcal{N}, \Lambda \in [0, 1] \tag{2.3}$$

Now we define strongly generalized-prequasi-invex function.

Definition 2.4. Consider $\mathcal{N} \subseteq R$ be an invex set with respect to $\eta : \mathcal{N} \times \mathcal{N} \to R$ and let the function $\chi : \mathcal{N} \to R$ is strongly generalized-prequasi-invex on $\mathcal{N}$ with respect to a bifunction η and modulus $\mu \geq 0$ iff,

$$\begin{aligned} \chi(m_1 + \Lambda\eta(m_2, m_1)) \quad \leq \quad & \max\{\chi(m_1 + \eta(m_2, m_1)), \chi(m_1 + \eta(m_2, m_1)) + \eta(\chi(m_1), \chi(m_1 + \eta(m_2, m_1))\} \\ & -\mu\Lambda(1 - \Lambda)\eta^2(m_2, m_1), \end{aligned} \tag{2.4}$$

for all $m_1, m_2 \in \mathcal{N}$, for all $0 \leq \Lambda \leq 1$.

The function χ is called to be strongly generalized-prequasi-concave, iff $-\chi$ is strongly generalized-prequasi-invex.

Remark 2.5. If $\Lambda = 1$ in (2.4), then

$$0 \leq \eta(\chi(m_1), \chi(m_1 + \eta(m_2, m_1)) \tag{2.5}$$

if $\Lambda = \frac{1}{2}$ then (2.4) satisfies

$$\frac{\chi(2m_1 + \eta(m_2, m_1))}{2} \quad \leq \quad \chi(m_1 + \eta(m_2, m_1)) + \eta(\chi(m_1), \chi(m_1 + \eta(m_2, m_1))) - (\frac{\mu}{2} - \frac{\mu}{4})\eta^2(m_2, m_1). \tag{2.6}$$

This is called strongly Jensen generalized-prequasi-invex function.

Example 2.6. The function $\chi(m) = m^2$ is strongly generalized-prequasi-invex with respect to $\eta(m_2, m_1) = m_1 - m_2$ and modulus $\mu = 1$. Let $\Lambda \in [0, 1]$. Then

$$\begin{aligned} & \max\{\chi(m_1 + \eta(m_2, m_1)), \chi(m_1 + \eta(m_2, m_1)) + \eta(\chi(m_1), \chi(m_1 + \eta(m_2, m_1))\} - \mu\Lambda(1 - \Lambda)\eta^2(m_2, m_1) \\ & \geq \chi(m_1 + \eta(m_2, m_1)) + \eta(\chi(m_1), \chi(m_1 + \eta(m_2, m_1)) - \mu\Lambda(1 - \Lambda)\eta^2(m_2, m_1) \\ & \geq (m_1 + \eta(m_2, m_1))^2 + \Lambda(m_1^2 - (m_1 + \eta(m_2, m_1))^2) - \mu\Lambda(1 - \Lambda)\eta^2(m_2, m_1) \\ & = m_1^2 + 2m_1\Lambda\eta(m_2, m_1) + \Lambda^2\eta^2(m_2, m_1) \\ & \geq \chi(m_1 + \Lambda\eta(m_2, m_1)). \end{aligned}$$

Definition 2.7. [6] Let $\chi \in L[m_1, m_2]$. The left-Hand and right-Hand k-Riemann–Liouville fractional integral operators $k\boldsymbol{J}^{\alpha}_{m_1^+}$ and $k\boldsymbol{J}^{\alpha}_{m_2^-}$ of order $\alpha > 0$ with $m_2 > m_1 \geq 0$, are defined as

$$k\boldsymbol{J}^{\alpha}_{m_1^+}\chi(g) = \frac{1}{k\Gamma_k(\alpha)} \int_{m_1}^{g} (g - t)^{\frac{\alpha}{k} - 1}\chi(t)dt, \quad g > m_1$$

and

$$k\boldsymbol{J}^{\alpha}_{m_2^-}\chi(g) = \frac{1}{k\Gamma_k(\alpha)} \int_{g}^{m_2} (t - g)^{\frac{\alpha}{k} - 1}\chi(t)dt, \quad g < m_2$$

$k > 0$, and $\Gamma_k$is the k-gamma function defined as

$$\Gamma_k(g) = \int_0^{\infty} (t)^{g-1}e^{-\frac{t^k}{k}}dt, \quad Re(g) > 0$$

Definition 2.8. [4] Let $[m_1, m_2] \subseteq R$ is a finite interval. The left-Hand and right-Hand Katugampola fractional integrals for order $\alpha > 0$ of $\chi \in X_c^p(m_1, m_2)$ are given as

$$^{\rho}I^{\alpha}_{m_1^+}\chi(g) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{m_1}^{g}(g^{\rho}-t^{\rho})^{\alpha-1}t^{\rho-1}\chi(t)dt$$

and

$$^{\rho}I^{\alpha}_{m_2^-}\chi(g) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{g}^{m_2}(t^{\rho}-g^{\rho})^{\alpha-1}t^{\rho-1}\chi(t)dt$$

with $m_1 < g < m_2$ and $\rho > 0$. Here $X_c^p(m_1, m_2)(c \in \mathbb{R}, 1 \leq p \leq \infty)$ denotes a set of Lebesgue measurable functions χ of all complex values on $[m_1, m_2]$ for which $||\chi||_{X_c^p} < \infty$. Where the norm is described as

$$||\chi||_{X_c^{\infty}} = \left(\int_{m_1}^{m_2}|t^c\chi(t)|^p\frac{dt}{t}\right)^{\frac{1}{p}}\ (1 \leq p < \infty)$$

for $p = \infty$

$$||\chi||_{X_c^{\infty}} = ess \sup_{m_1 \leq t \leq m_1}[t^c|\chi(t)|].$$

3 Inequalities via k-Riemann–Liouville fractional integrals

Following notations will be used in our results throughout this section. For $\chi : \mathcal{N} \to R$ along with $\eta : \mathcal{N} \times \mathcal{N} \to R$, we define

$$\mathcal{G}(\chi;\eta) := \max\{\chi(m_1 + \eta(m_2, m_1)), \chi(m_1 + \eta(m_2, m_1)) + \eta(\chi(m_1), \chi(m_1 + \eta(m_2, m_1)))\} \tag{3.1}$$

and

$$\mathcal{H}(\chi;\eta) := \max\{\chi(m_1), \chi(m_1) + \eta(\chi(m_1 + \eta(m_2, m_1)), \chi(m_1))\} \tag{3.2}$$

Theorem 3.1. Suppose $\mathcal{N} \subseteq R$ be an invex set with respect to $\eta : \mathcal{N} \times \mathcal{N} \to R$ with $m_1, m_2 \in \mathcal{N}$ and $m_1 < m_1 + \Lambda\eta(m_2, m_1)$. Let $\alpha, k \geq 0$ and let $\chi : \mathcal{N} \to R$ is η-prequasi- invex on $[m_1, m_1 + \eta(m_2, m_1)]$ with modulus $\mu \geq 0$. If $\chi \in L_1([m_1, m_1 + \eta(m_2, m_1)])$, then we have following inequality:

$$\begin{aligned}\frac{\Gamma_k(\alpha+k)}{2\eta(m_2, m_1)^{\frac{\alpha}{k}}}&[k\boldsymbol{J}^{\alpha}_{m_1^+}\chi(m_1 + \eta(m_2, m_1)) + k\boldsymbol{J}^{\alpha}_{(m_1+\eta(m_2,m_1))^-}\chi(m_1)]\\ &\leq \frac{\mathcal{G}(\chi;\eta) + \mathcal{H}(\chi;\eta)}{2} - \alpha\mu\eta^2(m_2, m_1)\left(\frac{1}{\alpha+k} - \frac{1}{\alpha+2k}\right)\end{aligned} \tag{3.3}$$

Proof . As χ is strongly η-prequasi-invex on $[m_1, m_1 + \eta(m_2, m_1)]$ with $\mu \geq 0$. This implies that

$$\chi(m_1 + \Lambda\eta(m_2, m_1)) \leq \mathcal{G}(\chi;\eta) - \mu\Lambda(1-\Lambda)\eta^2(m_2, m_1) \tag{3.4}$$

and

$$\chi(m_1 + (1-\Lambda)\eta(m_2, m_1)) \leq \mathcal{H}(\chi;\eta) - \mu\Lambda(1-\Lambda)\eta^2(m_2, m_1) \tag{3.5}$$

for all $\Lambda \in [0,1]$. Adding (3.4) and (3.5)

$$\chi(m_1+\Lambda\eta(m_2,m_1))+\chi(m_1+(1-\Lambda)\eta(m_2,m_1)) \quad \leq \quad \mathcal{G}(\chi;\eta)+\mathcal{H}(\chi;\eta)-2\mu\Lambda(1-\Lambda)\eta^2(m_2,m_1). \tag{3.6}$$

We multiply both sides of (3.6) by $\Lambda^{\frac{\alpha}{k}-1}$ and subsequently integrate the outcome with respect to Λ over the interval $[0,1]$. It gives

$$\begin{aligned}
&\int_0^1 \Lambda^{\frac{\alpha}{k}-1}\chi(m_1+\Lambda\eta(m_2,m_1))d\Lambda+\int_0^1 \Lambda^{\frac{\alpha}{k}-1}\chi(m_1+(1-\Lambda)\eta(m_2,m_1))d\Lambda\\
&\leq \mathcal{G}(\chi;\eta)\int_0^1 \Lambda^{\frac{\alpha}{k}-1}d\Lambda+\mathcal{H}(\chi;\eta)\int_0^1 \Lambda^{\frac{\alpha}{k}-1}d\Lambda-2\mu\eta^2(m_2,m_1)\int_0^1 \Lambda^{\frac{\alpha}{k}-1}\Lambda(1-\Lambda)d\Lambda\\
&=\frac{2k}{\alpha}\left[\frac{\mathcal{G}(\chi;\eta)+\mathcal{H}(\chi;\eta)}{2}-\alpha\mu\eta^2(m_2,m_1)\left(\frac{1}{\alpha+k}-\frac{1}{\alpha+2k}\right)\right]
\end{aligned} \tag{3.7}$$

Employ the substitutions $u = m_1+\Lambda\eta(m_2,m_1)$ and $v = m_1+(1-\Lambda)\eta(m_2,m_1)$ into the definition of k- Riemann–Liouville fractional integrals, we attain

$$\begin{aligned}
\int_0^1 \Lambda^{\frac{\alpha}{k}-1}\chi(m_1+\Lambda\eta(m_2,m_1))d\Lambda &= \frac{1}{\eta(m_2,m_1)^{\frac{\alpha}{k}}}\int_{m_1}^{m_1+\eta(m_2,m_1)}(u-m_1)^{\frac{\alpha}{k}-1}\chi(u)du\\
&= \frac{k\Gamma_k(\alpha)}{\eta(m_2,m_1)^{\frac{\alpha}{k}}}\times k\boldsymbol{J}^{\alpha}_{(m_1+\eta(m_2,m_1))^-}\chi(m_1)
\end{aligned} \tag{3.8}$$

and

$$\int_0^1 \Lambda^{\frac{\alpha}{k}-1}\chi(m_1+(1-\Lambda)\eta(m_2,m_1))d\Lambda = \frac{k\Gamma_k(\alpha)}{\eta(m_2,m_1)^{\frac{\alpha}{k}}}\times k\boldsymbol{J}^{\alpha}_{m_1^+}\chi(m_1+\eta(m_2,m_1)). \tag{3.9}$$

Employing (3.8) and (3.9) in (3.7), we get

$$\begin{aligned}
&\frac{\Gamma_k(\alpha+k)}{2\eta(m_2,m_1)^{\frac{\alpha}{k}}}[k\boldsymbol{J}^{\alpha}_{m_1^+}\chi(m_1+\eta(m_2,m_1))+k\boldsymbol{J}^{\alpha}_{(m_1+\eta(m_2,m_1))^-}\chi(m_1)]\\
&\quad\leq\frac{2k}{\alpha}\left[\frac{\mathcal{G}(\chi;\eta)+\mathcal{H}(\chi;\eta)}{2}-\alpha\mu\eta^2(m_2,m_1)\left(\frac{1}{\alpha+k}-\frac{1}{\alpha+2k}\right)\right].
\end{aligned}$$

The required result is done. □

Following corollary can be obtained by letting $\mu = 0$ in Theorem 3.1.

Corollary 3.2. Let $\mathcal{N} \subseteq R$ be an invex set with respect to $\eta : \mathcal{N}\times\mathcal{N} \to R$ with $m_1, m_2 \in \mathcal{N}$ and $m_1 < m_1+\eta(m_2,m_1)$. Let $\alpha, k \geq 0$ and let $\chi : \mathcal{N} \to R$ is strongly η-prequasi-invex on $[m_1, m_1+\eta(m_2,m_1)]$ with modulus 0. If $\chi \in L_1([m_1, m_1+\eta(m_2,m_1)])$, then we have:

$$\frac{\Gamma_k(\alpha+k)}{2\eta(m_2,m_1)^{\frac{\alpha}{k}}}[k\boldsymbol{J}^{\alpha}_{m_1^+}\chi(m_1+\eta(m_2,m_1))+k\boldsymbol{J}^{\alpha}_{(m_1+\eta(m_2,m_1))^-}\chi(m_1)]\leq\frac{\mathcal{G}(\chi;\eta)+\mathcal{H}(\chi;\eta)}{2}$$

Lemma 3.3. Let $\mathcal{N} \subseteq R$ is the invex set with respect to $\eta : \mathcal{N}\times\mathcal{N} \to R$ and $m_1, m_2 \in \mathcal{N}$ with $m_1 < m_1+\eta(m_2,m_1)$. If $\chi : \mathcal{N} \to R$ is a differentiable function in such a way that $\chi^{'} \in L([m_1, m_1+\eta(m_2,m_1)])$ then we have following inequality:

$$\begin{aligned}
&\frac{\chi(m_1)+\chi(m_1+\eta(m_2,m_1))}{2}-\frac{\Gamma_k(\alpha+k)}{2\eta(m_2,m_1)^{\frac{\alpha}{k}}}[k\boldsymbol{J}^{\alpha}_{m_1^+}\chi(m_1+\eta(m_2,m_1))+k\boldsymbol{J}^{\alpha}_{(m_1+\eta(m_2,m_1))^-}\chi(m_1)]\\
&\qquad=\frac{\eta(m_2,m_1)}{2}\int_0^1[(1-\Lambda)^{\frac{\alpha}{k}}-\Lambda^{\frac{\alpha}{k}}]\chi^{'}(m_1+(1-\Lambda)\eta(m_2,m_1))d\Lambda
\end{aligned}$$

Proof . This result is achieved by integration by parts. □

Theorem 3.4. Suppose $\mathcal{N} \subseteq R$ be an invex set with respect to $\eta : \mathcal{N} \times \mathcal{N} \to R$ with $m_1, m_2 \in \mathcal{N}$ and $m_1 < m_1 + \eta(m_2, m_1)$. Let $\alpha, k \geq 0$ and suppose $\chi : [m_1, m_1+\eta(m_2, m_1)] \to R$ be differentiable function on $(m_1, m_1+\eta(m_2, m_1))$. If $|\chi^{'}|$ is strongly η-prequasi-invex on $[m_1, m_1 + \eta(m_2, m_1)]$ with modulus $\mu \geq 0$ and $\chi^{'} \in L_1([m_1, m_1 + \eta(m_2, m_1)])$, then we have following inequality:

$$\left|\frac{\chi(m_1) + \chi(m_1 + \eta(m_2, m_1))}{2} - \frac{\Gamma_k(\alpha + k)}{2\eta(m_2, m_1)^{\frac{\alpha}{k}}}\left[k\boldsymbol{J}^{\alpha}_{m_1^+}\chi(m_1 + \eta(m_2, m_1)) + k\boldsymbol{J}^{\alpha}_{(m_1+\eta(m_2,m_1))^-}\chi(m_1)\right]\right|$$
$$\leq \frac{\eta(m_2, m_1)}{2}(\mathcal{R}(\alpha; k))^{1-\frac{1}{q}}\mathcal{G}(|\chi^{'}|^q; \eta)(\mathcal{R}(\alpha; k)) - \mu\eta^2(m_2, m_1)\mathcal{S}(\alpha; k)^{\frac{1}{q}} \tag{3.10}$$

where

$$\mathcal{R}(\alpha; k) = \frac{2}{(\frac{\alpha}{k} + 1)}\left(1 - \frac{1}{2^{\frac{\alpha}{k}}}\right)$$
$$\mathcal{S}(\alpha; k) = \frac{2}{(\frac{\alpha}{k} + 2)}\left(1 - \frac{1}{2^{\frac{\alpha}{k}+1}}\right) - \frac{1}{(\frac{\alpha}{k} + 3)}\left(1 - \frac{1}{2^{\frac{\alpha}{k}+2}}\right)$$

Proof . Here we use Lemma 3.3, the powerful Hölder's inequality, and property of the absolute values, we have

$$\left|\frac{\chi(m_1) + \chi(m_1 + \eta(m_2, m_1))}{2} - \frac{\Gamma_k(\alpha + k)}{2\eta(m_2, m_1)^{\frac{\alpha}{k}}}\left[k\boldsymbol{J}^{\alpha}_{m_1^+}\chi(m_1 + \eta(m_2, m_1)) + k\boldsymbol{J}^{\alpha}_{(m_1+\eta(m_2,m_1))^-}\chi(m_1)\right]\right|$$
$$\leq \frac{\eta(m_2, m_1)}{2}\int_0^1 |(1-\Lambda)^{\frac{\alpha}{k}} - \Lambda^{\frac{\alpha}{k}}||\chi^{'}(m_1 + (1-\Lambda)\eta(m_2, m_1))|d\Lambda$$
$$\leq \frac{\eta(m_2, m_1)}{2}\left(\int_0^1 |(1-\Lambda)^{\frac{\alpha}{k}} - \Lambda^{\frac{\alpha}{k}}|d\Lambda\right)^{1-\frac{1}{q}} \times \left(\int_0^1 |(1-\Lambda)^{\frac{\alpha}{k}} - \Lambda^{\frac{\alpha}{k}}||\chi^{'}(m_1 + (1-\Lambda)\eta(m_2, m_1))|d\Lambda\right)^{\frac{1}{q}}$$
$$\leq \frac{\eta(m_2, m_1)}{2}\left(\int_0^1 |(1-\Lambda)^{\frac{\alpha}{k}} - \Lambda^{\frac{\alpha}{k}}|d\Lambda\right)^{1-\frac{1}{q}} \times$$
$$\left(\int_0^1 |(1-\Lambda)^{\frac{\alpha}{k}} - \Lambda^{\frac{\alpha}{k}}|[\max|\{\chi^{'}(m_1 + \eta(m_2, m_1))|^q, |\chi^{'}(m_1 + \eta(m_2, m_1))|^q\right.$$
$$\left. + |\eta(\chi^{'}(m_1)|^q, |\chi^{'}(m_1 + \eta(m_2, m_1))|^q - \mu\Lambda(1-\Lambda)\eta^2(m_2, m_1)\}]d\Lambda\right)^{\frac{1}{q}}$$

where

$$\int_0^1 |(1-\Lambda)^{\frac{\alpha}{k}} - \Lambda^{\frac{\alpha}{k}}|d\Lambda = \frac{2}{(\frac{\alpha}{k} + 1)}\left(1 - \frac{1}{2^{\frac{\alpha}{k}}}\right)$$
$$\int_0^1 \Lambda(1-\Lambda)|(1-\Lambda)^{\frac{\alpha}{k}} - \Lambda^{\frac{\alpha}{k}}|d\Lambda = \frac{2}{(\frac{\alpha}{k} + 2)}\left(1 - \frac{1}{2^{\frac{\alpha}{k}+1}}\right) - \frac{1}{(\frac{\alpha}{k} + 3)}\left(1 - \frac{1}{2^{\frac{\alpha}{k}+2}}\right)$$

This proves the theorem. □

Lemma 3.5. See Ref ([12, 13]) . If $\gamma \in (0, 1]$ and $0 \leq u < v$, implies

$$|u^{\gamma} - v^{\gamma}| \leq (u - v)^{\gamma}$$

Theorem 3.6. Suppose $\mathcal{N} \subseteq R$ be an invex set with respect to $\eta : \mathcal{N} \times \mathcal{N} \to R$ with $m_1, m_2 \in \mathcal{N}$ and $m_1 < m_1 + \eta(m_2, m_1)$. Let $\alpha, k \geq 0$ and suppose $\chi : [m_1, m_1+\eta(m_2, m_1)] \to R$ be differentiable function on $(m_1, m_1+\eta(m_2, m_1))$. If $|\chi^{'}|$ is strongly η-prequasi-invex on $[m_1, m_1 + \eta(m_2, m_1)]$ with modulus $\mu \geq 0$ and $\chi^{'} \in L_1([m_1, m_1 + \eta(m_2, m_1)])$, then we have following inequality:

$$\left|\frac{\chi(m_1) + \chi(m_1 + \eta(m_2, m_1))}{2} - \frac{\Gamma_k(\alpha + k)}{2\eta(m_2, m_1)^{\frac{\alpha}{k}}}[k\boldsymbol{J}^{\alpha}_{m_1^+}\chi(m_1 + \eta(m_2, m_1))\right.$$
$$\left. + k\boldsymbol{J}^{\alpha}_{(m_1+\eta(m_2,m_1))^-}\chi(m_1)]\right| \leq \frac{\eta(m_2, m_1)}{2}\left(\frac{1}{\frac{\alpha}{k}r + 1}\right)^{\frac{1}{r}}\left(\mathcal{G}(|\chi^{'}|^q; \eta) - \mu\frac{\eta(m_2, m_1)^2}{6}\right)$$

where $\frac{1}{r} + \frac{1}{q} = 1$ and $\frac{\alpha}{k} \in (0, 1]$

Proof . Utilizing the property of lemma 3.5,

$$|u^{\frac{\alpha}{k}} - v^{\frac{\alpha}{k}}| \leq |u-v|^{\frac{\alpha}{k}}$$

for all $u, v \in [0,1]$ with $\frac{\alpha}{k} \in (0,1]$. Applying this fact ,we derive;

$$\begin{aligned}
\int_0^1 \left|(1-\Lambda)^{\frac{\alpha}{k}} - \Lambda^{\frac{\alpha}{k}}\right|^r d\Lambda &\leq \int_0^1 |1-2\Lambda|^{\frac{\alpha}{k}r} d\Lambda \\
&= \int_0^{\frac{1}{2}} |1-2\Lambda|^{\frac{\alpha}{k}r} d\Lambda + \int_{\frac{1}{2}}^1 |1-2\Lambda|^{\frac{\alpha}{k}r} d\Lambda \\
&= \int_0^{\frac{1}{2}} (1-2\Lambda)^{\frac{\alpha}{k}r} d\Lambda + \int_{\frac{1}{2}}^1 (2\Lambda-1)^{\frac{\alpha}{k}r} d\Lambda \\
&= = \frac{1}{\frac{\alpha}{k}r+1}.
\end{aligned} \tag{3.11}$$

Since the function $|\chi'|$ is strongly η-prequasi-invex on $[m_1, m_1+\eta(m_2,m_1)]$ with modulus $\mu \geq 0$, we have

$$\begin{aligned}
|\chi'(m_1+(1-\Lambda)\eta(m_2,m_1))|^q \leq \max\{&|\chi'(m_1+\eta(m_2,m_1))|^q, |\chi'(m_1+\eta(m_2,m_1))|^q \\
&+\eta(|\chi'(m_1)|^q, |\chi'(m_1+\eta(m_2,m_1))|^q)\} - \mu\Lambda(1-\Lambda)\eta^2(m_2,m_1)
\end{aligned} \tag{3.12}$$

Applying Lemma 3.5, the well- known Holder's inequality, property of absolute values, along with inequalities (3.11) and (3.12), we get

$$\begin{aligned}
&\left|\frac{\chi(m_1)+\chi(m_1+\eta(m_2,m_1))}{2} - \frac{\Gamma_k(\alpha+k)}{2\eta(m_2,m_1)^{\frac{\alpha}{k}}}[k\boldsymbol{J}^{\alpha}_{m_1^+}\chi(m_1+\eta(m_2,m_1)) + k\boldsymbol{J}^{\alpha}_{(m_1+\eta(m_2,m_1))^-}\chi(m_1)]\right| \\
&\leq \frac{\eta(m_2,m_1)}{2}\int_0^1 |(1-\Lambda)^{\frac{\alpha}{k}} - \Lambda^{\frac{\alpha}{k}}||f'(m_1+(1-\Lambda)\eta(m_2,m_1))|d\Lambda \\
&\leq \frac{\eta(m_2,m_1)}{2}\left(\int_0^1 |(1-\Lambda)^{\frac{\alpha}{k}} - \Lambda^{\frac{\alpha}{k}}|^r d\Lambda\right)^{\frac{1}{r}} \left(\int_0^1 |(1-\Lambda)^{\frac{\alpha}{k}} - \Lambda^{\frac{\alpha}{k}}|^q d\Lambda\right)^{\frac{1}{q}} \\
&= \frac{\eta(m_2,m_1)}{2}\left(\frac{1}{\frac{\alpha}{k}r+1}\right)^{\frac{1}{r}} \Bigg(\max\{|\chi'(m_1+\eta(m_2,m_1))|^q, |\chi'(m_1+\eta(m_2,m_1))|^q \\
&+|\eta(\chi'(m_1)|^q, |\chi'(m_1+\eta(m_2,m_1))|^q)\} - \mu\frac{\eta(m_2,m_1)^2}{6}\Bigg)^{\frac{1}{q}}.
\end{aligned}$$

The proof is estimated. □

4 Inequalities via Katugampola fractional integrals

We established Hermite–Hadamard type inequalities by means of generalized-prequasi-invex function involving Katugampola fractional integrals.

Following notations will be used throughout the section where convenient. Let $\rho > 0$ and $0 \leq m_1 < m_1+\eta(m_2,m_1)$, for any function $\chi : [m_1^\rho, m_1^\rho+\eta(m_2^\rho, m_1^\rho)] \to R$ and a bifunction $\eta : \mathcal{N}\times\mathcal{N} \to R$, we write

$$\mathcal{G}(\chi;\eta,\rho) := \max\{\chi(m_1^\rho+\eta(m_2^\rho,m_1^\rho)), \chi(m_1^\rho+\eta(m_2^\rho,m_1^\rho))+\eta(\chi(m_1^\rho), \chi(m_1^\rho+\eta(m_2^\rho,m_1^\rho))\}$$

and

$$\mathcal{H}(\chi;\eta,\rho) := \max\{\chi(m_1^\rho), \chi(m_1^\rho)+\eta(\chi(m_1^\rho+\eta(m_2^\rho,m_1^\rho)), \chi(m_1^\rho))\}$$

Theorem 4.1. Suppose $\alpha, \rho > 0$ and suppose $\mathcal{N} \subseteq R$ be an invex set with respect to $\eta : \mathcal{N} \times \mathcal{N} \to R$ with $m_1^\rho, m_2^\rho \in \mathcal{N}$ and $m_1^\rho < m_1^\rho + \eta(m_2^\rho, m_1^\rho)$. Suppose $\chi : \mathcal{N} \to R$ is generalized-prequasi-invex on $[m_1^\rho, m_1^\rho + \eta(m_2^\rho, m_1^\rho)]$ and $\chi \in X_c^p(m_1^\rho, m_2^\rho)$ with modulus $\mu \geq 0$. If $\chi \in L_1([m_1^\rho, m_1^\rho + \eta(m_2^\rho, m_1^\rho)])$, then we have following inequality:

$$\frac{\rho^\alpha \Gamma(\alpha+1)}{2\eta(m_2^\rho, m_1^\rho)^\alpha}[{}^\rho I^\alpha_{m_1^+}\chi(m_1^\rho + \eta(m_2^\rho, m_1^\rho)) + {}^\rho I^\alpha_{(m_1^\rho+\eta(m_2^\rho,m_1^\rho))^{\frac{1}{\rho}-}}\chi(m_1^\rho)] \leq \frac{\mathcal{G}(\chi;\eta,\rho) + \mathcal{H}(\chi;\eta,\rho)}{2} - \frac{\alpha\mu\eta^2(m_2^\rho, m_1^\rho)}{(\alpha+1)(\alpha+2)} \tag{4.1}$$

Proof . Applying properties of change of variables and the Katugampola fractional integrals.

$$\int_0^1 \Lambda^{\alpha\rho-1}\chi(m_1^\rho + \Lambda^\rho\eta(m_2^\rho, m_1^\rho))d\Lambda + \int_0^1 \Lambda^{\alpha\rho-1}\chi(m_1^\rho + (1-\Lambda^\rho)\eta(m_2^\rho, m_1^\rho))d\Lambda$$
$$= \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\eta(m_2^\rho, m_1^\rho)^\alpha}[{}^\rho I^\alpha_{m_1^+}\chi(m_1^\rho + \eta(m_2^\rho, m_1^\rho)) + {}^\rho I^\alpha_{(m_1^\rho+\eta(m_2^\rho,m_1^\rho))^{\frac{1}{\rho}-}}\chi(m_1^\rho)]. \tag{4.2}$$

Since χ is strongly generalized-prequasi-invex on $[m_1^\rho, m_1^\rho + \eta(m_2^\rho, m_1^\rho)]$ with $\mu \geq 0$. This implies that

$$\chi(m_1^\rho + \Lambda^\rho\eta(m_2^\rho, m_1^\rho)) \leq \mathcal{G}(\chi;\eta,\rho) - \mu\Lambda^\rho(1-\Lambda^\rho)\eta^2(m_2^\rho, m_1^\rho) \tag{4.3}$$

and

$$\chi(m_1^\rho + (1-\Lambda^\rho)\eta(m_2^\rho, m_1^\rho)) \leq \mathcal{H}(\chi;\eta,\rho) - \mu\Lambda^\rho(1-\Lambda^\rho)\eta^2(m_2^\rho, m_1^\rho) \tag{4.4}$$

for all $\Lambda \in [0,1]$. By adding (4.2) and (4.3), we obtain

$$\chi(m_1^\rho + \Lambda^\rho\eta(m_2^\rho, m_1^\rho)) + \chi(m_1^\rho + (1-\Lambda^\rho)\eta(m_2^\rho, m_1^\rho)) \leq \mathcal{G}(\chi;\eta,\rho) + \mathcal{H}(\chi;\eta,\rho) - 2\mu\Lambda^\rho(1-\Lambda^\rho)\eta^2(m_2^\rho, m_1^\rho) \tag{4.5}$$

Multiply both sides of eq (4.5) by $\Lambda^{\alpha\rho-1}$, integrate the outcome with respect to Λ over $[0,1]$ and then use (4.2), we have

$$\frac{\rho^{\alpha-1}\Gamma(\alpha)}{\eta(m_2^\rho, m_1^\rho)^\alpha}[{}^\rho I^\alpha_{m_1^+}\chi(m_1^\rho + \eta(m_2^\rho, m_1^\rho)) + {}^\rho I^\alpha_{(m_1^\rho+\eta(m_2^\rho,m_1^\rho))^{\frac{1}{\rho}-}}\chi(m_1^\rho)]$$
$$\leq \frac{\mathcal{G}(\chi;\eta,\rho) + \mathcal{H}(\chi;\eta,\rho)}{2} - \frac{\alpha\mu\eta^2(m_2^\rho, m_1^\rho)}{\rho(\alpha+1)(\alpha+2)} \tag{4.6}$$

The intended inequality follows from (4.6) by multiplying through by $\frac{\alpha\rho}{2}$ $\square$

Lemma 4.2. Suppose $\alpha, \rho > 0$ and suppose $\mathcal{N} \subseteq R$ be an invex set with respect to $\eta :$ $mathcalN \times \mathcal{N} \to R$ with $m_1^\rho, m_2^\rho \in \mathcal{N}$ and $m_1^\rho < m_1^\rho + \eta(m_2^\rho, m_1^\rho)$. If $\chi : \mathcal{N} \to R$ is a differentiable function in such a way that $\chi' \in L([m_1^\rho, m_1^\rho + \eta(m_2^\rho, m_1^\rho)])$. We have following inequality:

$$\frac{\chi(m_1^\rho) + \chi(m_1^\rho + \eta(m_2^\rho, m_1^\rho))}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(m_2^\rho, m_1^\rho)^\alpha}[{}^\rho I^\alpha_{m_1^+}\chi(m_1^\rho + \eta(m_2^\rho, m_1^\rho)) + {}^\rho I^\alpha_{(m_1^\rho+\eta(m_2^\rho,m_1^\rho))^{\frac{1}{\rho}-}}\chi(m_1^\rho)]$$
$$= \frac{\eta(m_2^\rho, m_1^\rho)\rho}{2}\int_0^1 [(1-\Lambda^\rho)^\alpha - \Lambda^{\rho\alpha}]\Lambda^{\rho-1}\chi'(m_1^\rho + (1-\Lambda^\rho)\eta(m_2^\rho, m_1^\rho))d\Lambda. \tag{4.7}$$

Proof . We use the identical reasons as used in the establishment of Lemma 3.5 . First see

$$\int_0^1 [(1-\Lambda^\rho)^\alpha]\Lambda^{\rho-1}\chi'(m_1^\rho + (1-\Lambda^\rho)\eta(m_2^\rho, m_1^\rho))d\Lambda$$
$$= \left.\frac{(1-\Lambda^\rho)^\alpha\chi(m_1^\rho + (1-\Lambda^\rho)\eta(m_2^\rho, m_1^\rho))}{-\rho\eta(m_2^\rho, m_1^\rho))}\right|_0^1 - \frac{\alpha}{\eta(m_2^\rho, m_1^\rho))}\int_0^1 (1-\Lambda^\rho)^{\alpha-1}\Lambda^{\rho-1}\chi(m_1^\rho + (1-\Lambda^\rho)\eta(m_2^\rho, m_1^\rho))d\Lambda$$
$$= \frac{\chi(m_1^\rho + \eta(m_2^\rho, m_1^\rho)}{\rho\eta(m_2^\rho, m_1^\rho))} - \frac{\alpha}{\eta(m_2^\rho, m_1^\rho))}\int_{m_1}^{(m_1^\rho+\eta(m_2^\rho,m_1^\rho))^{\frac{1}{\rho}}} \left(\frac{x^\rho - m_1^\rho}{\eta(m_2^\rho, m_1^\rho)}\right)^{\alpha-1}\frac{x^{\rho-1}}{\eta(m_2^\rho, m_1^\rho)}\chi(x^\rho)dx$$
$$= \frac{\chi(m_1^\rho + \eta(m_2^\rho, m_1^\rho)}{\rho\eta(m_2^\rho, m_1^\rho))} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{\eta(m_2^\rho, m_1^\rho)^{\alpha+1}}{}^\rho I^\alpha_{(m_1^\rho+\eta(m_2^\rho,m_1^\rho))^{\frac{1}{\rho}-}}\chi(m_1^\rho). \tag{4.8}$$

Similarly, we can show that

$$\int_0^1 \Lambda^{\rho\alpha}\Lambda^{\rho-1}\chi^{'}(m_1^\rho+(1-\Lambda^\rho)\eta(m_2^\rho,m_1^\rho))d\Lambda = -\frac{\chi(m_1^\rho)}{\rho\eta(m_2^\rho,m_1^\rho)}+\frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{\eta(m_2^\rho,m_1^\rho)} \, {}^\rho I^\alpha_{m_1^+}\chi(m_1^\rho+\eta(m_2^\rho,m_1^\rho)) \tag{4.9}$$

Thus from (4.8) and (4.9) we get (4.7). □

Theorem 4.3. Suppose $\alpha,\rho>0$ and suppose $\mathcal{N}\subseteq R$ be an invex set with respect to $\eta:\mathcal{N}\times\mathcal{N}\to R$ with $m_1^\rho, m_2^\rho\in\mathcal{N}$ and $m_1^\rho < m_1^\rho+\eta(m_2^\rho,m_1^\rho)$. If $\chi:\mathcal{N}\to R$ is differentiable function on $(m_1^\rho, m_1^\rho+\eta(m_2^\rho,m_1^\rho)$ If $|\chi^{'}|^q$ is a strongly generalized-prequasi-invex function on$(m_1^\rho, m_1^\rho+\eta(m_2^\rho,m_1^\rho)$ for $q\geq 1$, then we have following inequality:

$$\left|\frac{\chi(m_1^\rho)+\chi(m_1^\rho+\eta(m_2^\rho,m_1^\rho))}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(m_2^\rho,m_1^\rho)^\alpha}[{}^\rho I^\alpha_{m_1^+}\chi(m_1^\rho+\eta(m_2^\rho,m_1^\rho))+{}^\rho I^\alpha_{(m_1^\rho+\eta(m_2^\rho,m_1^\rho))^{\frac{1}{\rho}-}}\chi(m_1^\rho)]\right|$$
$$\leq \frac{\eta(m_2^\rho,m_1^\rho)}{2}\left(\frac{2}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)\right)^{1-\frac{1}{q}}\left(\frac{2}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)\mathcal{G}(|\chi^{'q}|;\eta,\rho)-\frac{\mu\eta(m_2^\rho,m_1^\rho)(2^{\alpha+2}-\alpha-4)}{2^{\alpha+1}(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}. \tag{4.10}$$

Proof . Applying Lemma 4.2, the famous Holder's inequality and the strongly η-priquasi-invex function, we obtain

$$\left|\frac{\chi(m_1^\rho)+\chi(m_1^\rho+\eta(m_2^\rho,m_1^\rho))}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(m_2^\rho,m_1^\rho)^\alpha}[{}^\rho I^\alpha_{m_1^+}\chi(m_1^\rho+\eta(m_2^\rho,m_1^\rho))+{}^\rho I^\alpha_{(m_1^\rho+\eta(m_2^\rho,m_1^\rho))^{\frac{1}{\rho}-}}\chi(m_1^\rho)]\right|$$
$$\leq \frac{\eta(m_2^\rho,m_1^\rho)\rho}{2}\int_0^1\left|(1-\Lambda^\rho)^\alpha-\Lambda^{\rho\alpha}\right|\Lambda^{\rho-1}\left|\chi^{'}(m_1^\rho+(1-\Lambda^\rho)\eta(m_2^\rho,m_1^\rho))\right|d\Lambda$$
$$\leq \frac{\eta(m_2^\rho,m_1^\rho)\rho}{2}\left(\int_0^1\left|(1-\Lambda^\rho)^\alpha-\Lambda^{\rho\alpha}\right|\Lambda^{\rho-1}d\Lambda\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|(1-\Lambda^\rho)^\alpha-\Lambda^{\rho\alpha}\right|\Lambda^{\rho-1}\left|\chi^{'}(m_1^\rho+(1-\Lambda^\rho)\eta(m_2^\rho,m_1^\rho))\right|d\Lambda\right)^{\frac{1}{q}}$$
$$\leq \frac{\eta(m_2^\rho,m_1^\rho)\rho}{2}\left(\int_0^1\left|(1-\Lambda^\rho)^\alpha-\Lambda^{\rho\alpha}\right|\Lambda^{\rho-1}d\Lambda\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|(1-\Lambda^\rho)^\alpha-\Lambda^{\rho\alpha}\right|\Lambda^{\rho-1}\left(\mathcal{H}(|\chi^{'}|;\eta,\rho)-\mu^\rho(1-\Lambda^\rho)\eta^2(m_2^\rho,m_1^\rho))\right)d\Lambda\right)^{\frac{1}{q}}$$
$$=\frac{\eta(m_2^\rho,m_1^\rho)\rho}{2}\left(\frac{2}{\rho(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\right)^{1-\frac{1}{q}}\left(\frac{2}{\rho(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\mathcal{G}(|\chi^{'}|;\eta,\rho)\right.$$
$$\left.-\frac{\mu\eta^2(m_2^\rho,m_1^\rho)}{\rho}\left[\frac{2^{\alpha+2}-\alpha-4}{2^{\alpha+1}(\alpha+2)(\alpha+3)}\right]\right)^{\frac{1}{q}} \tag{4.11}$$

where

$$\int_0^1\left|(1-\Lambda^\rho)^\alpha-\Lambda^{\rho\alpha}\right|\Lambda^{\rho-1}d\Lambda=\frac{2}{\rho(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)$$

and

$$\int_0^1\left|(1-\Lambda^\rho)^\alpha-\Lambda^{\rho\alpha}\right|\Lambda^\rho\Lambda^{\rho-1}(1-\Lambda^\rho)d\Lambda=\frac{1}{\rho}\left[\frac{2^{\alpha+2}-\alpha-4}{2^{\alpha+1}(\alpha+2)(\alpha+3)}\right].$$

Hence the inequality is derived. □

Theorem 4.4. Suppose $\alpha,\rho>0$ and suppose $\mathcal{N}\subseteq R$ be an invex set with respect to $\eta:\mathcal{N}\times\mathcal{N}\to R$ with $m_1^\rho, m_2^\rho\in\mathcal{N}$ and $m_1^\rho < m_1^\rho+\eta(m_2^\rho,m_1^\rho)$. If $\chi:$

$mathcalN \to R$ is differentiable function on $(m_1^\rho, m_1^\rho + \eta(m_2^\rho, m_1^\rho)$ If $|\chi^{'}|^q$ is a strongly generalized-prequasi-invex function on$(m_1^\rho, m_1^\rho + \eta(m_2^\rho, m_1^\rho)$ for $q \geq 1$ then we have following inequality:(where $\frac{1}{r} + \frac{1}{q} = 1$)

$$\left|\frac{\chi(m_1^\rho) + \chi(m_1^\rho + \eta(m_2^\rho, m_1^\rho))}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2\eta(m_2^\rho, m_1^\rho)^\alpha}[{}^\rho I^\alpha_{m_1^+}\chi(m_1^\rho + \eta(m_2^\rho, m_1^\rho)) + {}^\rho I^\alpha_{(m_1^\rho + \eta(m_2^\rho, m_1^\rho))^{\frac{1}{\rho}-}}\chi(m_1^\rho)]\right| \leq \frac{\eta(m_2^\rho, m_1^\rho)}{2}\left(\frac{1}{\alpha r + 1}\right)^{\frac{1}{r}}\left(\mathcal{G}(|\chi^{'}|^q;\eta,\rho) - \frac{1}{6}\mu\eta^2(m_2^\rho, m_1^\rho)\right)^{\frac{1}{q}} \tag{4.12}$$

Proof .

$$\begin{aligned}
&\left|\frac{\chi(m_1^\rho) + \chi(m_1^\rho + \eta(m_2^\rho, m_1^\rho))}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2\eta(m_2^\rho, m_1^\rho)^\alpha}[{}^\rho I^\alpha_{m_1^+}\chi(m_1^\rho + \eta(m_2^\rho, m_1^\rho)) + {}^\rho I^\alpha_{(m_1^\rho + \eta(m_2^\rho, m_1^\rho))^{\frac{1}{\rho}-}}\chi(m_1^\rho)]\right| \\
&\leq \frac{\eta(m_2^\rho, m_1^\rho)\rho}{2}\int_0^1 \left|(1-\Lambda^\rho)^\alpha - \Lambda^{\rho\alpha}\right|\Lambda^{\rho-1}\left|\chi^{'}(m_1^\rho + (1-\Lambda^\rho)\eta(m_2^\rho, m_1^\rho))\right| d\Lambda \\
&\leq \frac{\eta(m_2^\rho, m_1^\rho)\rho}{2}\left(\int_0^1 \left|(1-\Lambda^\rho)^\alpha - \Lambda^{\rho\alpha}\right|^r \Lambda^{\rho-1} d\Lambda\right)^{\frac{1}{r}} \times \left(\int_0^1 \Lambda^{\rho-1}\left|\chi^{'}(m_1^\rho + (1-\Lambda^\rho)\eta(m_2^\rho, m_1^\rho))\right|^q d\Lambda\right)^{\frac{1}{q}} \\
&\leq \frac{\eta(m_2^\rho, m_1^\rho)\rho}{2}\left(\int_0^1 \left|(1-\Lambda^\rho)^\alpha - \Lambda^{\rho\alpha}\right|^r \Lambda^{\rho-1} d\Lambda\right)^{1-\frac{1}{q}} \\
&\times \left(\mathcal{G}(|\chi^{'}|;\eta,\rho)\int_0^1 \Lambda^{\rho-1}d\Lambda - \mu\eta^2(m_2^\rho, m_1^\rho)\int_0^1 \Lambda^{2\rho-1}(1-\Lambda^\rho)d\Lambda\right)^{\frac{1}{q}} \\
&= \frac{\eta(m_2^\rho, m_1^\rho)\rho}{2}\left(\frac{1}{(\alpha r+1)\rho}\right)^{\frac{1}{r}}\left(\frac{1}{\rho}\mathcal{G}(|\chi^{'}|^q;\eta,\rho) - \frac{1}{6\rho}\mu\eta^2(m_2^\rho, m_1^\rho)\right)^{\frac{1}{q}}
\end{aligned} \tag{4.13}$$

where

$$\begin{aligned}
\int_0^1 \left|(1-\Lambda^\rho)^\alpha - \Lambda^{\rho\alpha}\right|^r \Lambda^{\rho-1} d\Lambda &= \frac{1}{\rho}\int_0^1 \left|(1-v)^\alpha - v^\alpha\right|^r dv \\
&\leq \frac{1}{\rho}\int_0^1 (1-2v)^{\alpha r} dv \\
&= \frac{1}{\rho}\left[\int_0^{\frac{1}{2}} (1-2v)^{\alpha r} dv + \int_{\frac{1}{2}}^1 (2v-1)^{\alpha r} dv\right] \\
&= \frac{1}{\rho(\alpha r+1)}.
\end{aligned}$$

□

Lemma 4.5. Suppose that $\alpha, \rho > 0$ and suppose $\mathcal{N} \subseteq R$ is the invex set with respect to $\eta : \mathcal{N} \times \mathcal{N} \to R$ and $m_1^\rho, m_2^\rho \in \mathcal{N}$ with $m_1^\rho < m_1^\rho + t\eta(m_2^\rho, m_1^\rho)$. If $\chi : \mathcal{N} \to R$ is differentiable function such that $\chi^{'} \in L([m_1^\rho, m_1^\rho + \eta(m_2^\rho, m_1^\rho)])$ then we have following inequality:

$$\begin{aligned}
&\frac{\chi(m_1^\rho) + \chi(m_1^\rho + \eta(m_2^\rho, m_1^\rho))}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\eta(m_2^\rho, m_1^\rho)^\alpha}[{}^\rho I^\alpha_{m_1^+}\chi(m_1^\rho + \eta(m_2^\rho, m_1^\rho)) + {}^\rho I^\alpha_{(m_1^\rho + \eta(m_2^\rho, m_1^\rho))^{\frac{1}{\rho}-}}\chi(m_1^\rho)] \\
&= \frac{\eta(m_2^\rho, m_1^\rho)}{\alpha}\int_0^1 \Lambda^{\rho(\alpha+1)-1}[\chi^{'}(m_1^\rho + \Lambda^\rho\eta(m_2^\rho, m_1^\rho)) - \chi^{'}(m_1^\rho + (1-\Lambda^\rho)\eta(m_2^\rho, m_1^\rho))]d\Lambda.
\end{aligned} \tag{4.14}$$

Proof . This result can be obtained by direct use of integration by parts. □

Theorem 4.6. Suppose that $\alpha, \rho > 0$ and suppose $\mathcal{N} \subseteq R$ be an invex set with respect to $\eta : \mathcal{N} \times \mathcal{N} \to R$ with $m_1^\rho, m_2^\rho \in \mathcal{N}$ and $m_1^\rho < m_1^\rho + \eta(m_2^\rho, m_1^\rho)$. If $\chi : \mathcal{N} \to R$ is differentiable function on $(m_1^\rho, m_1^\rho + \eta(m_2^\rho, m_1^\rho)$ If $|\chi^{'}|^q$ is strongly generalized-prequasi-invex function with $\mu \geq 0$ for $q \geq 1$, so we have; (where $\frac{1}{r}+\frac{1}{q}=1$)

$$\left|\frac{\chi(m_1^\rho)+\chi(m_1^\rho+\eta(m_2^\rho,m_1^\rho))}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(m_2^\rho,m_1^\rho)^\alpha}[{}^\rho I^\alpha_{m_1^+}\chi(m_1^\rho+\eta(m_2^\rho,m_1^\rho))+{}^\rho I^\alpha_{(m_1^\rho+\eta(m_2^\rho,m_1^\rho))^{\frac{1}{\rho}}-}\chi(m_1^\rho)]\right|$$
$$\leq \frac{\eta(m_2^\rho,m_1^\rho)}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\Bigg[\left(\frac{1}{\alpha+1}\mathcal{H}(|\chi'|^q;\eta,\rho)-\frac{\mu\eta^2(m_2^\rho,m_1^\rho)}{(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\alpha+1}\mathcal{G}(|\chi'|^q;\eta,\rho)-\frac{\mu\eta^2(m_2^\rho,m_1^\rho)}{(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\Bigg].$$

Proof . Practicing Lemma 4.5, Holder's inequality and strongly generalized-prequasi-invex function, we obtain

$$\left|\frac{\chi(m_1^\rho)+\chi(m_1^\rho+\eta(m_2^\rho,m_1^\rho))}{\alpha\rho}-\frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{2\eta(m_2^\rho,m_1^\rho)^\alpha}\left[{}^\rho I^\alpha_{m_1^+}\chi(m_1^\rho+\eta(m_2^\rho,m_1^\rho))+{}^\rho I^\alpha_{(m_1^\rho+\eta(m_2^\rho,m_1^\rho))^{\frac{1}{\rho}}-}\chi(m_1^\rho)\right]\right|$$
$$\leq \frac{\eta(m_2^\rho,m_1^\rho)}{\alpha}\int_0^1\Lambda^{\rho(\alpha+1)-1}\left[\left|\chi'(m_1^\rho+\Lambda^\rho\eta(m_2^\rho,m_1^\rho))\right|+\left|\chi'(m_1^\rho+(1-\Lambda^\rho)\eta(m_2^\rho,m_1^\rho))\right|\right]d\Lambda$$
$$\leq \frac{\eta(m_2^\rho,m_1^\rho)}{\alpha}\left(\int_0^1\Lambda^{\rho(\alpha+1)-1}d\Lambda\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\Lambda^{\rho(\alpha+1)-1}\left|\chi'(m_1^\rho+\Lambda^\rho\eta(m_2^\rho,m_1^\rho))\right|^q d\Lambda\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\Lambda^{\rho(\alpha+1)-1}\left|\chi'(m_1^\rho+(1-\Lambda^\rho)\eta(m_2^\rho,m_1^\rho))\right|^q d\Lambda\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{\eta(m_2^\rho,m_1^\rho)}{\alpha}\left(\int_0^1\Lambda^{\rho(\alpha+1)-1}d\Lambda\right)^{1-\frac{1}{q}}\left[\mathcal{H}(|\chi'|^q;\eta,\rho)\left(\int_0^1\Lambda^{\rho(\alpha+1)-1}-\mu\eta^2(m_2^\rho,m_1^\rho)\int_0^1\Lambda^{\rho(\alpha+2)-1}(1-\Lambda^\rho)d\Lambda\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\mathcal{G}(|\chi'|^q;\eta,\rho)\int_0^1\Lambda^{\rho(\alpha+1)-1}-\mu\eta^2(m_2^\rho,m_1^\rho)\int_0^1\Lambda^{\rho(\alpha+2)-1}(1-\Lambda^\rho)d\Lambda\right)^{\frac{1}{q}}\right]$$
$$=\frac{\eta(m_2^\rho,m_1^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}}\Bigg[\left(\frac{1}{\rho(\alpha+1)}\mathcal{H}(|\chi'|^q;\eta,\rho)-\frac{\mu\eta^2(m_2^\rho,m_1^\rho)}{\rho(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\rho(\alpha+1)}\mathcal{G}(|\chi'|^q;\eta,\rho)-\frac{\mu\eta^2(m_2^\rho,m_1^\rho)}{\rho(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\Bigg]. \tag{4.15}$$

Hence we established the outcome. □

5 Conclusion

We defined strongly generalized-prequasi-invex function. Utilizing the definition we developed new counterparts of k-Riemann–Liouville Hermite-Hadamard inequalities in section 3. Hermite-Hadamard's integral inequalities involving Katugampola fractional integral by means of strongly generalized-prequasi-invex function are obtained in section 4. It has been noticed that these new results are generalizations of previously known results.

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