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Sequences of Cesàro type using lacunary notion

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Abstract

The scenario of this article is to introduce the space $\mathfrak{R}_s^t(p,\Delta)$ based on a general Riesz sequence space. Its completeness property is derived and its linear isomorphism property with $\ell(p)$ is proved. The Köthe-dual property of the space $\mathfrak{R}_s^t(p,\Delta)$ is also derived. Furthermore, its basis is constructed and some characterization of infinite matrices are given.

Keywords: Sequence spaces, Köthe-duals, Infinite matrix
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1 Introduction, background

By $\mathfrak{S}$ we mean the set of all real and any linear subspace of $\mathfrak{S}$ is called as sequence space. Let $\mathbf{N} = \{0, 1, 2, \cdots\}$; by $\mathbf{R}$ we represent the set of all real numbers. The Banach space ℓ_p is the set of all sequences $v = (v_i) \in \mathfrak{S}$ such that

$$\|v\|_p = \left(\sum_{j=0}^{\infty} |v_j|^p\right)^{\frac{1}{p}} < \infty,$$

for $1 \leq p < \infty$.

By ℓ_∞ we mean the set of all bounded sequence; c represents set of all convergent sequences and are Banach spaces with the norm $\|v\| = \sup_j |v_j|$.

It is important to note that the infinite matrix is consider as a linear operator one a sequence space to another one. We call a space $\mathcal{Y}$ to be FK space if it is a complete metric space with continuous coordinated $p_m : \mathcal{Y} \to \mathbb{C}$ where $p_m(v) = v_m$ for all $v \in \mathcal{Y}$ and $m \in \mathbb{N}$. A normed FK space is called a BK space as defined in [20], [10] and others.

Let $\theta = (t_j)$ be increasing integer sequence. Then, as in [11], it will be called lacunary sequence if $t_0 = 0$ and $t_j = t_j - t_{j-1} \to \infty$. By θ we will denote the intervals of the form $I_j = (t_{j-1}, t_j]$ and with q_j we will denote the ratio $\frac{t_j}{t_{j-1}}$.

It has been further studied by various authors as in [4], [8], [26], [27] and many others.

We now give a brief description of Cesàro convergence.

Consider the geometric series of the form $\sum_o^\infty v^j = \frac{1}{1-v}$ which is valid only for values of $-1 < v < 1$, here -1, 1 are excluded. It is important to note that at $v = 1$, there is a singularity. In geometric series, the sequence of partial

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sums converges to a real number. As an illustration, consider $v = -1$, Then, the series

$$\mathcal{Q} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{j=0}^{\infty} Q_j = \sum_{j=0}^{\infty} \Big(-1\Big)^j,$$

known as Grandi's series. It is obvious that the sequence of partial sums gives

$$\mathcal{P}_0 = Q_0 = 1, \mathcal{P}_1 = \mathcal{Q}_0 + \mathcal{Q}_1 = 0, \mathcal{P}_2 = \mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_2 = 1, \mathcal{P}_3 = \mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 = 0, \cdots .$$

Consequently, the sequence of partial sums does not converge to a real number. However, the sum of the geometric series $\sum_0^{\infty}(-1)^j = \frac{1}{1-(-1)} = \frac{1}{2}$, which is a real number. Thus, clearly, right hand side converges while the left hand side diverges, hence for consistency, we consider the averages of partial sums, that is,

$$\frac{\mathcal{P}_0}{1} = 1, \frac{\mathcal{P}_0 + \mathcal{P}_1}{2} = \frac{1}{2}, \frac{\mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2}{3} = \frac{2}{3}, \frac{\mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3}{4} = \frac{2}{4}, \cdots .$$

Thus, the sequence of average of partial sums gives

$$\mathcal{P}_j = \frac{1}{j+1} \sum_{i=0}^{j} Q_j = \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{2}{4}, \frac{3}{5}, \frac{3}{6}, \cdots .$$

It can be easily seen that

$$\mathcal{P}_j = \begin{cases} \frac{1}{2}, & \text{for } j = odd, \\ \frac{1}{2} + \frac{1}{2j+2}, & \text{for } j = even, \end{cases}$$

and hence as $j \to \infty$ it converges to $\frac{1}{2}$ and this kind of convergence is known as Cesàro convergence.

For $1 \le p \le \infty$, the author in [25] has defined the Cesàro sequence space ces_p is defined as

$$ces_p = \left\{ v = (v_k) : \sum_{i=1}^{\infty} \left(\frac{1}{i} \sum_{j=1}^{i} |v_j| \right)^p < \infty \right\},$$

and proved that it is a Banach space with the norm

$$||v||_{ces_p} = \left[\sum_{i=1}^{\infty} \left(\frac{1}{i} \sum_{j=1}^{i} |v_j| \right)^p \right]^{\frac{1}{p}},$$

where p is a fixed parameter greater or equal to 1; in fact, Jagers [16] and Leibowitz [18] showed that $ces_1 = \{0\}$, ces_p are separable reflexive Banach spaces for $1 < p < \infty$.

It was further studied by several authors like in [9], [24]. The author in [21] has introduced the Cesàro sequence spaces $\mathcal{X}_p$ and $\mathcal{X}_\infty$ of non-absolute type and has shown that $ces_p \subset \mathcal{X}_p$ is strict for $1 \le p \le \infty$.

The infinite Cesàro matrix $\mathfrak{C}$, which we mentioned above has the entries of the form:

$$\mathfrak{C} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and defined by Ernesto Cesàro (1859-1906) who was an Italian mathematician and worked in the field of differential geometry. He is known also for his 'averaging' method for the 'Cesàro summation' of divergent series, known as the Cesàro mean.

For fixed $j \in \mathbb{N}$, the sequence $e_j = \{e_j(i)\}$ defined by

$$e_j(i) = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{else where,} \end{cases}$$

is an element of ces_p and in this case, we see

$$\|e_j\|^p_{ces_p} = \sum_{i=j}^{\infty} \frac{1}{i^p}.$$

Consequently, we see $\|ces_p\|_{ces_p} \to 0$ as $j \to \infty$. Furthermore, the sequence of canonical vectors (e_j) is an unconditional, boundedly complete, and shrinking Schauder basis for ces_p as in [7]. Also, it is important to remark the following property which tell us that ces_p is a solid space. More precisely, if $v = \{v(i)\}$ and $w = \{w(i)\}$ are real sequences such that $v \in ces_p$ with $1 < p < \infty$ and $|v(i)| \leq |w(i)|$ for all $i \in \mathbb{N}$, then $\|v\|_{ces_p} \leq \|w\|_{ces_p}$.

In 1968 the Dutch Mathematical Society posted a problem to find the Köthe dual of Cesàro sequence spaces ces_p and Cesàro function spaces $Ces_{p[0,\infty)}$. Before, it was also known a result of Luxemburg and Zaanen [19] who have found the Köthe dual of $Ces[0,1]$ space. Also, in 1957 Alexiewicz, in his overlooked paper [2], found implicitly the Köthe dual of the weighted ces_∞-spaces. In 1974 problem was solved (isometrically) by Jagers [16] even for weighted Cesàro sequence spaces, but the proof is far from being easy and elementary. Later on some amount of papers appeared in the case of sequence spaces as well as for function spaces. Bennett [18] proved representation of the dual $(ces_p)^*$ for $1 < p < \infty$ as the corollary from factorization theorems for Cesàro and ℓ_p spaces.

Recall [23] that a modulus $\mathfrak{f}$ is a function from $R^+ \to R^+$ such that (i) $\mathfrak{M}(v) = 0$ if and only if $v = 0$, (ii) $\mathfrak{M}(v + w) \leq \mathfrak{M}(v) + \mathfrak{M}(w)$ for $v \geq 0$, $w \geq 0$, (iii) $\mathfrak{M}$ is increasing, (iv) $\mathfrak{M}$ is continuous from the right at origin.

A modulus function may be bounded or unbounded. For example the function $\mathfrak{M}(\varsigma) = \frac{\varsigma}{\varsigma+1}$ is bounded but the function $\mathfrak{M}(\varsigma) = \varsigma^p$ for $0 < p \leq 1$ is unbounded. Ruckle, used this notion to define the space

$$L(\mathfrak{M}) = \{u = (u_i) \in \mathfrak{S} : \sum_{i=1}^{\infty} \mathfrak{M}(|u_i|) < \infty\}.$$

For $\mathfrak{M}(u) = u^p$, then $L(\mathfrak{M})$ reduces to the well known space ℓ_p and is given by

$$\ell_p = \{u = (u_i) \in \mathfrak{S} : \sum_{i=1}^{\infty} (|u_i|^p < \infty\}.$$

Also, for $\mathfrak{M}(u) = u$, then $L(\mathfrak{M})$ reduces to the space ℓ_1 and is given by

$$\ell_1 = \{u = (u_i) \in \mathfrak{S} : \sum_{i=1}^{\infty} (|u_i| < \infty\}.$$

Several authors as can be found in [12], [15], [20], and some others have used a modulus function to construct some sequence spaces.

Let $\mathfrak{X}$ be a sequence space. Then the sequence space $\mathfrak{X}(\mathfrak{M})$ is defined by

$$\mathfrak{X}(\mathfrak{M}) = \{u = (u_j) \in \mathfrak{S} : \sum_{j=1}^{\infty} \mathfrak{M}(|u_j|) \in \mathfrak{X}\}.$$

The author in [17] gave an extension of $\mathfrak{X}(f)$ by considering a sequence of modulus functions $\mathfrak{M} = (\mathfrak{M}_j)$ and defined the space

$$\mathfrak{X}(\mathfrak{M}) = \{w = (w_m) \in \mathfrak{S} : \sum_{m=1}^{\infty} \mathfrak{M}_j(|w_m|) \in \mathfrak{X}\}.$$

As in [5], for a vector space $\mathcal{Y}$, a paranorm $g : \mathcal{Y} \to [0, \infty)$ is a function on $\mathcal{Y}$ such that (i) $g(\theta) = 0$, (ii) $g(v) = g(-v)$ and (iii) if $\{a_j\}$ is a sequence of scalars with $a_j \to a$ and $\{v_j\} \subset \mathcal{Y}$ with $g(v_j - v) \to 0$, then $g(a_j v_j - av) \to 0$ (continuity of multiplication), where θ is the zero vector in the linear space $\mathfrak{Y}$.

The pair $(\mathcal{Y}, g)$ is called a paranormed space if g is a paranorm on $\mathcal{Y}$ as can be seen in [3].

We call a space $\mathcal{Y}$ to be a Fréchet space if every point in the closure of a subset $\mathcal{A}$ of $\mathcal{Y}$ is a limit of a sequence of points of $\mathcal{A}$ or in simple language , we call a space to be a Frèchet space if it is a complete linear space.

Here as in [20], we have

$$\ell(p) = \left\{ v = (v_i) \in \mathfrak{S} : \sum_i |v_i|^{p_i} < \infty \right\},$$

where $0 < p_i \leq \mathcal{H} < \infty$ with $\mathcal{H} = \sup p_i < \infty$ and $\mathcal{G} = \max(1, \mathcal{H})$. It is complete with the paranorm

$$g(v) = \left(\sum_i |v_i|^{p_i}\right)^{\frac{1}{\mathcal{G}}}.$$

The author in [4] constructed the following space:

$$\mathfrak{C}_{(p)}\left[\mathfrak{M}, p\right] = \left\{v = (\varsigma_k) \in \mathfrak{S} : \sum_{i=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{f}\sum_{k=1}^{i}|\varsigma_i|\right]\right)^{p_i} < \infty\right\},$$

and proved some good results concerning about it.

This whole procedure gave the author the idea to work on and the author is able to introduce the space $\mathfrak{C}_{(p)}\left(\mathfrak{M}, \Omega, \theta\right)$ and construct its various properties.

2 Main results

In this section of text, we introduce the space $\mathfrak{C}^{\Omega}_{(p)}\left(\mathfrak{M}, \theta\right)$, and show it is a Fréchet space.

Following Başarir [5], Et. [9], Freedman [11], Ganie [13]-[14], Jagers [16], Ruckle [23], Savaş [24], we introduce the following spaces:

$$\mathfrak{C}_{(p)}\left[\mathfrak{M}, \Omega, \theta\right] = \left\{v = (\varsigma_k) : \sum_{i=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_i}\sum_{k\in I_i}|\Omega_k\varsigma_k|\right]\right)^{p_i} < \infty\right\},$$

where $p = (p_i)$ is a bounded sequence of positive real numbers with $\mathcal{H} = \sup p_i < \infty$ and $\mathcal{G} = \max(1, \mathcal{H})$ and $\Omega = (\Omega_k)$ is a sequence such that $\Omega_k \neq 0$ for all $k \in \mathbb{N}$. Also, for any complex η,
$eta|^{p_k} \leq \max(1, |\eta|^{\mathcal{H}}$.

We now begin with the following theorem.

Theorem 2.1. The space $\mathfrak{C}^{\Omega}_{(p)}\left(\mathfrak{M}, \theta\right)$ is linear spaces over $\mathbf{C}$.

Proof . Let $\varsigma, \tau \in \mathfrak{C}^{\Omega}_{(p)}\left(\mathfrak{M}, \theta\right)$, then

$$\sum_{i=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_i}\sum_{k\in I_i}|\Omega_k\varsigma_k|\right]\right)^{p_i} < \infty \quad \text{and} \quad \sum_{i=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_i}\sum_{k\in I_i}|\Omega_k\tau_k|\right]\right)^{p_i} < \infty.$$

Now, $a, b \in \mathbb{C}$, we have

$$\begin{aligned}&\sum_{i=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_i}\sum_{k\in I_i}|a(\Omega_k\varsigma_k) + b(\Omega_k\tau_k)|\right]\right)^{p_i}\\ &\qquad \leq \max(1, 2^{\mathcal{H}-1})\left[\sum_{i=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_i}\sum_{k\in I_i}|\Omega_k\varsigma_k|\right]\right)^{p_i} + \sum_{i=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_i}\sum_{k\in I_i}|\Omega_k\tau_k|\right]\right)^{p_i}\right] < \infty.\end{aligned}$$

Consequently, $a\varsigma_k + b\tau_k \in \mathfrak{C}^{\Omega}_{(p)}\left(\mathfrak{M}, \theta\right)$ and hence is linear spaces over $\mathbf{C}$. □

Theorem 2.2. For $1 \leq p_i < \infty$, the space $\mathfrak{C}^{\Omega}_{(p)}\left(\mathfrak{M}, \theta\right)$ is a Fréchet space paranormed by

$$g(\varsigma) = \left[\sum_{i=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_i}\sum_{k\in I_i}|\Omega_k\varsigma_k|\right]\right)^{p_i}\right]^{\frac{1}{\mathcal{G}}}, \tag{2.1}$$

where $\mathcal{H} = \sup p_i < \infty$ and $\mathcal{G} = \max(1, \mathcal{H})$.

Proof . To establish the result, we must show the completeness property of $\mathfrak{C}^{\Omega}_{(p)}(\mathfrak{M},\theta)$, as paranormed property is easy to prove. So, let $\varsigma^{(j)}=(\varsigma^{(j)}_i)_i$ be any Cauchy sequence in $\mathfrak{C}^{\Omega}_{(p)}(\mathfrak{M},\theta)$ for each $j\in\mathbb{N}$. Therefore, we have

$$g\left(\varsigma^{(i)}-\varsigma^{(j)}\right)\to 0\ \ as\, i,j\ \to\infty.$$

This shows by using (2.1) that

$$\sum_{n=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_n}\sum_{k\in I_n}|\Omega_k(\varsigma^{(i)}_k-\varsigma^{(j)}_k)|\right]\right)^{p_n}\to 0\, as\, i,j\to\infty \tag{2.2}$$

so that for each fixed k, $|\varsigma^{(i)}_k-\varsigma^{(j)}_k|\to 0\, as\, i,j\ \to\infty$.

Hence for each fixed k, $\varsigma^{(i)}_k$ is a Cauchy sequence in $\mathbf{C}$. But $\mathbf{C}$ being complete, it converges, say, $\varsigma^{(j)}_k\to\varsigma_k$ for $j\to\infty$.

Thus, for a given $\varepsilon>0$, we can find a positive integer $n_0>1$ such that

$$\sum_{n=1}^{n_0}\left(\mathfrak{M}\left[\frac{1}{h_n}\sum_{k\in I_n}|(\Omega_k(\varsigma^{(i)}_k-\varsigma^{(j)}_k)|\right]\right)^{p_n}<\varepsilon^{\mathcal{G}}\ for\, i,j>n_0. \tag{2.3}$$

We now let $j\to\infty$ in (2.3), we have

$$\sum_{n=1}^{n_0}\left(\mathfrak{M}\left[\frac{1}{h_n}\sum_{k\in I_n}|\Omega_k(\varsigma^{(i)}_k-\varsigma_k)|\right]\right)^{p_n}<\varepsilon^{\mathcal{G}}\ for\, i>n_0. \tag{2.4}$$

Now n_0 being arbitrary, so letting $n_0\to\infty$ in (2.4), we have

$$\sum_{n=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_n}\sum_{k\in I_n}|\Omega_k(\varsigma^{(i)}_k-\varsigma_k)|\right]\right)^{p_n}<\varepsilon^{\mathcal{G}}\ for\, i>n_0.$$

showing that $g\left(\varsigma^{(i)}-\varsigma\right)<\varepsilon$ for all $i\geq n_0$ and ε as small as we please but positive. This, shows that $(\varsigma^{(i)})$ converges to ς in the paranorm of $\mathfrak{C}_{(p)}(\mathfrak{M},\theta)$. But, $(\varsigma^{(i)})\in\mathfrak{C}^{\Omega}_{(p)}(\mathfrak{M},\theta)$ and $\mathfrak{M}$ is continuous, it follows that $\varsigma\in\mathfrak{C}^{\Omega}_{(p)}(\mathfrak{M},\theta)$. □

3 Inclusion relations on $\mathfrak{C}^{\Omega}_{(p)}(\mathfrak{M},\theta)$

In this section, we investigate some inclusion relations concerning $\mathfrak{C}^{\Omega}_{(p)}(\mathfrak{M},\theta)$.

We first consider the following definitions:

Definition 3.1. For any set $\mathcal{D}$ of sequences, the space of multipliers of $\mathcal{D}$, denoted by $S(\mathcal{D})$, is given by

$$S(\mathcal{D})=\{u\in\mathfrak{S}:u\varsigma\in\mathcal{D}\ for\ all\ \varsigma\in\mathcal{D}\}.$$

Definition 3.2. We say that g satisfies a Δ_2-condition, or that $g\in\Delta_2$, if there is a constant $K\geq 2$ such that $g(2t)\leq Kg(t)$ for all $t\geq 0$.

Theorem 3.3. If $p=(p_j)$ and $s=(s_j)$ are bounded sequences of positive real numbers with $0<p_j\leq s_j<\infty$ for each j, then $\mathfrak{C}^{\Omega}_{(p)}(\mathfrak{M},\theta)\subseteq C^{\Omega}_{(s)}(\mathfrak{M},\theta)$, for any modulus function $\mathfrak{M}$.

Proof . Let $\varsigma\in\mathfrak{C}^{\Omega}_{(p)}(\mathfrak{M},\theta)$, then

$$\sum_{j=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_j}\sum_{k\in I_i}|\Omega_k\varsigma_k|\right]\right)^{p_j}<\infty.$$

Thus, for sufficiently large j, say $j \geq j_0$, we have for ε small and positive that

$$\mathfrak{M}\left[\frac{1}{h_i}\sum_{k\in I_i}|\Omega_k\varsigma_k|\right] < \varepsilon,$$

for fixed $j_0 \in \mathbf{N}$. But $\mathfrak{M}$ being increasing with $p_j \leq s_j$, we conclude that

$$\sum_{j\geq j_0}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_j}\sum_{k\in I_i}|\Omega_k\varsigma_k|\right]\right)^{s_j} \leq \sum_{j\geq j_0}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_j}\sum_{k\in I_i}|\Omega_k\varsigma_k|\right]\right)^{p_j} < \infty.$$

Consequently, $\varsigma \in C^{\Omega}_{(s)}(\mathfrak{M},\theta)$, as desired. □

Theorem 3.4. If Δ_2-condition is satisfied by the modulus function $\mathfrak{M}$, then

$$\ell_\infty \subset S(\mathfrak{C}^{\Omega}_{(p)}(\mathfrak{M},\theta).$$

Proof . Let $\nu \in \ell_\infty$ with $\mathcal{T} = \sup_j |\nu_j|$ and $\varsigma \in \mathfrak{C}^{\Omega}_{(p)}(\mathfrak{M},\theta)$, implies

$$\sum_{j=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_i}\sum_{k\in I_i}|\Omega_k\varsigma_k|\right]\right)^{p_j} < \infty,$$

and as $\mathfrak{M}$ satisfies the Δ_2-condition, there exists a constant $\mathcal{B}$ such that

$$\begin{aligned}\sum_{j=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_i}\sum_{k\in I_i}|\Omega_k(\nu_k\varsigma_k)|\right]\right)^{p_j} &\leq \sum_{j=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_j}\sum_{k\in I_i}|\nu_k||\Omega_k\varsigma_k|\right]\right)^{p_j}\\ &\leq (\mathcal{B}(1+[\mathcal{T}]))^{\mathcal{H}}\sum_{j=1}^{\infty}\left(\mathfrak{M}\left[\frac{1}{h_i}\sum_{k\in I_i}|\Omega_k\varsigma_k|\right]\right)^{p_j}\\ &< \infty,\end{aligned}$$

where $[\mathcal{T}]$ represents the integer part of $\mathcal{T}$. Consequently, $\nu \in \mathfrak{C}^{\Omega}_{(p)}(\mathfrak{M},\theta)$. □

Conclusion: In this paper, Cesàro spaces have been described in details with examples, and we have introduced the space $\mathfrak{C}^{\Omega}_{(p)}(\mathcal{F},\theta)$ by employing lacunary sequences and sequences of strictly positive real numbers with Ω as defined in the text. Some basic properties and inclusions relations have been determined. The consequence of the results obtained in this paper are more general and extensive than the pre-existing known results.

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