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Fuzzy $\alpha-$ideals and $\alpha-$congruences of ADLs

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Abstract

In this paper, we introduce L-fuzzy $\alpha-$ideals and L-fuzzy $\alpha-$congruences on ADL A and we discuss the properties of these. Also, we prove that the homomorphic image and pre-image of an L-fuzzy $\alpha-$congruence is also an L-fuzzy $\alpha-$congruence on A under certain conditions. Finally, we establish a one-to-one correspondence between L-fuzzy $\alpha-$ideals and L-fuzzy $\alpha-$congruences on A.

Keywords: Almost Distributive Lattice (ADL), L-fuzzy $\alpha-$ideal, L-fuzzy $\alpha-$congruence
2020 MSC: 06D72, 06F15, 08A72

1 Introduction

The notion of a fuzzy subset of a set X as a function from X into $[0,1]$ was introduced by L.A. Zadeh [19]. J.A. Goguen [4] replaced the valuations set $[0,1]$, by means of a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L-fuzzy sets. Later Flip [3] further tried to make a more generalized study by replacing the valuation set by Partially ordered monoid. The partially ordered algebraic systems play an important role in algebra. Some important concepts in partially ordered systems are l-groups, l-rings, f-rings and lattices. Several algebraists took interest in the study of fuzzy subalgebras of several algebraic structures. Rosenfeld [12] and Kuroki [6] applied this concept in group theory and semi group theory, and developed the theory of a fuzzy subgroups and fuzzy subsemi groupoids respectively. In 1982, Liu [8] defined and studied fuzzy subrings as well as fuzzy ideals in rings. Subsequently, several researchers worked on fuzzy subrings and ideals of rings (Malik and Mordeson [9]), fuzzy ideals of lattices (Lehmke [7] and U.M. Swamy and D.V. Raju [14]), algebraic fuzzy systems and irreducibility (U.M. Swamy and D.V. Raju [15] and [16]), fuzzy ideals of a ring (Mukharjee and Sen [10]), fuzzy groups and level subgroups (Das [1]), fuzzy pseudo ideals in semi groups (Dutta [2]), fuzzy vector spaces and fuzzy topological vector spaces (Katsaras and Liu [5]) and on the truth values of fuzzy statements (U.M. Swamy, Rama Rao and Prabhakara [18]).

To make an abstract study, we consider a general complete lattice satisfying the infinite meet distributivity to have the truth values of fuzzy statements, is called a frame. In this context, several generalizations of Boolean algebras (Boolean rings) have come up into focus. U.M. Swamy and G.C. Rao [17] have introduced the notion an Almost Distributive Lattice (abbreviated as ADL) as a common abstraction of several lattice theoretic and ring theoretic generalizations of Boolean algebras and Boolean rings. Later, U.M. Swamy, Ch.S.S. Raj and Natnael Teshale A. [13] introduced the concept of L-fuzzy ideals of ADLs A and their lattice properties.

In this paper, we introduce L-fuzzy $\alpha-$ideals of ADL A, where $\alpha \in L-\{0\}$. It is well known that a lattice is algebraic if and only if it is isomorphic to an algebraic closed set system. We have proved that the class of L-fuzzy

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$\alpha-$ideals of ADL A is an algebraic fuzzy set system. Also, we introduce L-fuzzy $\alpha-$congruences on A. We show that the homomorphic image and pre-image of an L-fuzzy $\alpha-$congruence on A is an L-fuzzy $\alpha-$congruence under a certain conditions. Finally, we introduce a one-to-one correspondence between L-fuzzy $\alpha-$ideals and L-fuzzy $\alpha-$congruences on A.

2 Preliminaries

In this section, we recall some definitions and basic results mostly taken from [17] and [11].

Definition 2.1. An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following conditions for all a, b and $c \in A$.

1. $0 \wedge a = 0$
2. $a \vee 0 = a$
3. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
4. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
5. $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
6. $(a \vee b) \wedge b = b$

Any bounded below distributive lattice is an ADL. Any nonempty set X can be made into an ADL which is not a lattice by fixing an arbitrarily chosen element 0 in X and fix an arbitrary element $x_0 \in X$. For any $x, y \in X$, define $\wedge$ and $\vee$ on X by,

$$x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0 \end{cases} \qquad \text{and} \qquad x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases}$$

Then $(X, \wedge, \vee, x_0)$ is an ADL with x_0 as its zero element. This ADL is called the **discrete ADL**.

Definition 2.2. Let $A = (A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, define $a \leq b$ if $a = a \wedge b$ ($\Leftrightarrow a \vee b = b$). Then $\leq$ is a partial order on A with respect to which 0 is the smallest element in A.

Theorem 2.3. The following hold for any a, b and c in an ADL A.

(1) $a \wedge 0 = 0 = 0 \wedge a$ and $a \vee 0 = a = 0 \vee a$

(2) $a \wedge a = a = a \vee a$

(3) $a \wedge b \leq b \leq b \vee a$

(4) $a \wedge b = a \Leftrightarrow a \vee b = b$

(5) $a \wedge b = b \Leftrightarrow a \vee b = a$

(6) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (i.e., $\wedge$ is associative)

(7) $a \vee (b \vee a) = a \vee b$

(8) $a \leq b \Rightarrow a \wedge b = a = b \wedge a$ $\big(\Leftrightarrow a \vee b = b = b \vee a\big)$

(9) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$

(10) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(11) $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$

(12) $a \wedge b = \inf\{a, b\} \Leftrightarrow a \wedge b = b \wedge a \Leftrightarrow a \vee b = \sup\{a, b\}$.

Definition 2.4. Let I be a non empty subset of an ADL A with 0. Then I is called an $\alpha-$ideal of A if $(a]^{**} \subseteq I$ for all $a \in I$.

As a consequence, let A be an ADL with 0 and S a multiplicatively closed subset of A. Then $I = \{a \in A : a \wedge b = 0,\ for\ some\ b \in S\}$ is an $\alpha-$ideal of A. An element $m \in A$ is said to be maximal if, for any $x \in A$, $m \leq x$ implies $m = x$. It can be easily observed that m is maximal if and only if $m \wedge x = x$ for all $x \in A$.

Definition 2.5. Let θ be an equivalence relation on an ADL A. Then θ is called a congruence relation on A if, (a, b) and $(c, d) \in \theta \Rightarrow (a \wedge c, b \wedge d)$ and $(a \vee c, b \vee d) \in \theta$, for all $a, b, c, d \in A$.

Corollary 2.6. For any $x \in A, \theta_x = \{(a, b) \in A \times A : x \vee a = x \vee b\}$ is a congruence relation on A.

Definition 2.7. An L-fuzzy subset μ of X is a mapping from X into L, where L is a complete lattice satisfying the infinite meet distributive law. If L is the unit interval $[0, 1]$ of real numbers, then these are the usual fuzzy subsets of X.

3 Fuzzy $\alpha-$ideal

In this section, we introduce the notion of L-fuzzy $\alpha-$ideals of ADL A and their characterizations. In particular, we prove that the class of all L-fuzzy $\alpha-$ideals of A is an algebraic fuzzy set system.

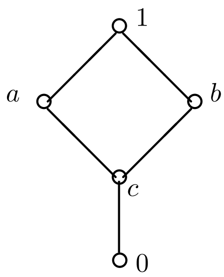
Definition 3.1. Let A be an ADL and L a frame. An L-fuzzy subset μ of A is said to be an L-fuzzy α-ideal of A if for all $a, b \in A$ and $\alpha \in L - \{0\}$,

(1) $\mu(0) = \alpha$

(2) $\mu(a \vee b) \geq \mu(a) \wedge \mu(b)$ and

(3) $\mu(a \wedge b) \geq \mu(a) \vee \mu(b)$.

Example 3.2. Let $D = \{0, x, y\}$ be a discrete ADL with 0 as its zero element defined above and $L = \{0, a, b, c, 1\}$ be the lattice represented by the Hasse diagram given below:



Consider $D \times L = \{(t, s) \mid t \in D \text{ and } s \in L\}$. Then $(D \times L, \wedge, \vee, 0)$ is an ADL under the pointwise operations $\wedge$ and $\vee$ on $D \times L$ and $0 = (0, 0)$, the zero element in $D \times L$. Now define $\mu : D \times L \to [0, 1]$ by

$$\mu(t, s) = \begin{cases} 1 & \text{if } (t, s) = (0, 0) \\ 0.5 & \text{if } t = 0 \ and \ s \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

for all $(t, s) \in D \times L$. Then μ is an L-fuzzy $\alpha-$ideal of the ADL $D \times L$ (note that $D \times L$ is not a lattice).

Corollary 3.3. Let μ be an L-fuzzy subset of A such that $\mu(0) = \alpha$, for all $\alpha \in L - \{0\}$. Then μ is an L-fuzzy $\alpha-$ideal of A if and only if $\mu(a \vee b) = \mu(a) \wedge \mu(b)$, for all a and $b \in A$.

Example 3.4. Let $A = \{0, a, b, c\}$ and $L = [0, 1]$ and let $\vee$ and $\wedge$ be binary operations on A defined as follows:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then, $(A, \wedge, \vee, 0)$ is an ADL which is not a lattice (since $b \wedge a = a \neq b = a \wedge b$). Now define an L-fuzzy subsets μ and ν of A by $\mu(0) = \alpha$, $\mu(a) = \mu(b) = 0.3$ and $\mu(c) = 0.6$; $\nu(0) = \alpha$, $\nu(a) = 0.3$, $\nu(b) = 0.5$ and $\nu(c) = 0.75$. Then for any $x, y \in A$, $\mu(x \vee y) = \mu(x) \wedge \mu(y)$. Therefore, μ is an L-fuzzy α-ideal of A, while ν is not an L-fuzzy α-ideal of A, since $\nu(b \vee a) = \nu(b) = 0.5 \neq \nu(b) \wedge \nu(a)$.

Corollary 3.5. Let μ be an L-fuzzy $\alpha-$ideal of A and S a non-empty subset of A. Then for any a and $b \in A$, we have the following:

(1) $a \leq b \Rightarrow \mu(b) \leq \mu(a)$ (μ is an antitone mapping)

(2) If $a \sim b$, then $\mu(a) = \mu(b)$

(3) $\mu(a \wedge b) = \mu(b \wedge a)$ and $\mu(a \vee b) = \mu(b \vee a)$

(4) If $a \in \langle S]$, then $\mu(a) \geq \bigwedge_{i=1}^{n} \mu(x_i)$ *for some* $x_1, x_2, ..., x_n \in S$

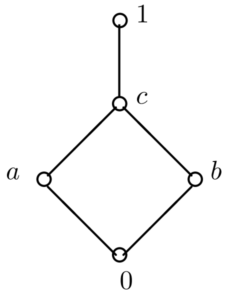
(5) If $a \in \langle b]$, then $\mu(b) \leq \mu(a)$

(6) If m is a maximal element in A, then $\mu(m) \leq \mu(a)$

(7) $\mu(m) = \mu(n)$, for any maximal elements m and n of A.

In the above result, every L-fuzzy $\alpha-$ideal of A is an antitone, but L-fuzzy subset of A may be an antitone without being an L-fuzzy $\alpha-$ideal; for consider the following.

Example 3.6. Let $A = \{0, a, b, c, 1\}$ be the lattice represented by the Hasse diagram given below:



Now define $\mu : A \to [0, 1]$ by $\mu(0) = \mu(a) = 1$, $\mu(b) = 0.4$ and $\mu(c) = \mu(1) = 0$. Then the mapping μ is an antitone but not an L-fuzzy $\alpha-$ideal of A, since $\mu(a \vee b) = \mu(c) = 0 \neq \mu(a) \wedge \mu(b)$.

In the following, we give a characterization for L-fuzzy $\alpha-$ideals of an ADL A.

Definition 3.7. Let A be an ADL with a maximal element m and for any L-fuzzy subsets λ and μ of A, we define the following L-fuzzy subsets on A,

$$\begin{aligned} \big(\mu + \lambda\big)(x) &= \vee\big\{\mu(y) \wedge \lambda(z) : y \vee z = x\big\} \\ \big(\mu.\lambda\big)(x) &= \vee\big\{\mu(y) \wedge \lambda(z) : y \wedge z = x\big\} \\ \text{and } \big(c\mu\big)(x) &= \vee\big\{\mu(y) : c \wedge y \wedge m = x \wedge m\big\}, \text{ for any } c \in A. \end{aligned}$$

Remark: If there are no $y, z \in A$ such that $y \wedge z = x$, then clearly $(\mu.\lambda)(x) = 0$ being the supremum of the empty set and the same is true in the case of $c\mu$, for any $c \in A$. Now, we prove the following.

Theorem 3.8. If λ and μ are L-fuzzy $\alpha-$ideals of A, then $\mu.\lambda = \mu \wedge \lambda$.

Proof . For any $x \in A$, it is clear that

$$\mu(x) \wedge \lambda(x) \leq (\mu.\lambda)(x) \text{ (since } x \wedge x = x).$$

To prove the other inequality, let x, y and $z \in A$ such that $y \wedge z = x$. Then $x = y \wedge z \leq z$, so that $\lambda(z) \leq \lambda(x)$, since λ is an antitone. Now,

$$\begin{aligned} \text{consider} \quad \mu(y) \wedge \mu(x) &= \mu(y \vee x) \\ &= \mu(y \vee (y \wedge z)) \\ &= \mu(y). \end{aligned}$$

Therefore $\mu(y) \leq \mu(x)$ and it follows that,

$$\mu(y) \wedge \lambda(z) \leq \mu(x) \wedge \lambda(x), \quad \text{for } y \wedge z = x.$$

Hence, it follows that $(\mu.\lambda)(x) \leq \mu(x) \wedge \lambda(x)$. Thus $\mu.\lambda = \mu \wedge \lambda$. $\square$

Theorem 3.9. If μ is an L-fuzzy subset of A with a maximal element m, then $c\mu \leq \mu$ for all $c \in A$ if and only if $\lambda.\mu \leq \mu$, for any L-fuzzy subset λ of A.

Proof . Suppose that $c\mu \leq \mu$ for all $c \in A$. Let λ be any L-fuzzy subset of A. Then, for any $x \in A$,

$$\begin{aligned} (\lambda.\mu)(x) &= \vee\{\lambda(y) \wedge \mu(z) : y \wedge z = x\} \\ &\leq \vee\{\mu(z) : y \wedge z = x\} \\ &\leq \vee\{\mu(z) : y \wedge z \wedge m = x \wedge m\} \\ &= (y\mu)(x) \leq \mu(x). \end{aligned}$$

Conversely, we suppose that $\lambda.\mu \leq \mu$ for any L-fuzzy subset λ of A. Let $c \in A$ and define an L-fuzzy subset λ of A by

$$\lambda(x) = \begin{cases} 1 & \text{if } x \leq c \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \text{Then,} \quad (c\mu)(x) &= \vee\{\mu(y) : c \wedge y \wedge m = x \wedge m\} \\ &\leq \vee\{\lambda(z) \wedge \mu(y) : z \wedge y = x,\ z \leq c\} \\ &= \vee\{\mu(y) : z \wedge y = x,\ z \leq c\} \\ &\leq \vee\{\lambda(z) \wedge \mu(y) : z \wedge y = x\} \\ &= (\lambda.\mu)(x) \leq \mu(x). \end{aligned}$$

Hence $c\mu \leq \mu$, for any $c \in A$. $\square$

Theorem 3.10. Let μ be an L-fuzzy subset of A with a maximal element m such that $\mu(0) = \alpha$, for any $\alpha \in L - \{0\}$. Then μ is an L-fuzzy $\alpha-$ideal of A if and only if $\mu + \mu = \mu$ and $c\mu \leq \mu$, for any $c \in A$.

Proof . Suppose that μ is an L-fuzzy $\alpha-$ideal of A. Then for any $x \in A$,

$$\begin{aligned} (\mu + \mu)(x) &= \vee\{\mu(y) \wedge \mu(z) : y \vee z = x\} \\ &= \vee\{\mu(y \vee z) : y \vee z = x\} = \mu(x) \\ \text{and} \quad (c\mu)(x) &= \vee\{\mu(y) : c \wedge y \wedge m = x \wedge m\} \leq \mu(x) \end{aligned}$$

for; if $c \wedge y \wedge m = x \wedge m$, then $\mu(c \wedge y \wedge m) = \mu(x \wedge m)$ (since m is maximal) and hence $\mu(c \wedge y) = \mu(x)$. Now, $\mu(x) = \mu(c \wedge y) \geq \mu(c) \vee \mu(y)$ (by theorem 2.8(2)) $\geq \mu(y)$.
Thus $c\mu \leq \mu$ for any $c \in A$. Conversely, we suppose that $\mu + \mu = \mu$ and $c\mu \leq \mu$, for any $c \in A$. Now we prove that μ is an L-fuzzy $\alpha-$ideal of A.

$$\begin{aligned} \text{Consider,} \quad \mu(x \vee y) &= (\mu + \mu)(x \vee y) \\ &= \vee\{\mu(s) \wedge \mu(t) : s \vee t = x \vee y\} \\ &\geq \mu(x) \wedge \mu(y). \end{aligned}$$

$$\begin{aligned} \text{Also, } \mu(x \wedge y) \geq (x\mu)(x \wedge y) &= \vee\{\mu(s) : x \wedge s \wedge m = x \wedge y \wedge m\} \\ &\geq \mu(y) \end{aligned}$$

$$\begin{aligned} \text{and } \mu(x \wedge y) \geq (y\mu)(x \wedge y) &= \vee\{\mu(s) : y \wedge s \wedge m = x \wedge y \wedge m\} \\ &= \vee\{\mu(s) : s \wedge y \wedge m = x \wedge y \wedge m\} \\ &\geq \mu(x). \end{aligned}$$

Thus, $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$. Therefore, μ is an L-fuzzy $\alpha-$ideal of A. $\square$

Let us recall that a complete lattice L is called an algebraic lattice if every element of L is the supremum of a set of compact elements of L. An element c in a lattice L is called compact if, for any $X \subseteq L$, $c \leq Sup\ X \Rightarrow c \leq Sup\ F$, for some $F \subset\subset X$.

Theorem 3.11. Let A be an ADL. Then the lattice $\mathcal{I}(A)$ of ideals of A is an algebraic lattice in which the finitely generated ideals are precisely compact elements.

Proof . Let $I \in \mathcal{I}(A)$. Then we observe that $I = \bigcup_{a\in I} \langle a] = Sup\ \{\langle a] : a \in I\}$. If I is compact in $\mathcal{I}(A)$, then there exists a finite subset of A say, $F = \{a_1, a_2, ..., a_n\}$ such that $I = Sup\ \{\langle a_1], \langle a_2], ..., \langle a_n]\} = \langle F]$. So, the compact elements in $\mathcal{I}(A)$ precisely are of the form $\langle F]$, where F is finite subset of A. On the other hand, suppose that $I = \langle F]$, where $F = \{a_1, a_2, ..., a_n\}$. Let $\{I_\alpha\}_{\alpha\in\Delta} \subseteq \mathcal{I}(A)$ such that $I \subseteq Sup\ \{I_\alpha\}_{\alpha\in\Delta} = \langle \bigcup_{\alpha\in\Delta} I_\alpha]$. For each $1 \le i \le n$, $a_i \in I$ and hence $a_i = \Big(\bigvee_{j=1}^{m_i} a_{ij}\Big) \wedge x_i$, for some $a_{ij} \in I_{\alpha_{ij}}, \alpha_{ij} \in \Delta$ and $x_i \in A$. Then we observe that $I \subseteq Sup\ \{I_{\alpha_{ij}} : 1 \le i \le n,\ 1 \le j \le m_i\}$. Thus I is compact in $\mathcal{I}(A)$. In particular, every principal ideal is compact in $\mathcal{I}(A)$. Since for any $I \in \mathcal{I}(A)$, $I = \bigcup_{a\in I} \langle a] = Sup\ \{\langle a] : a \in I\}$. It follows that,$\mathcal{I}(A)$ is an algebraic lattice. $\square$

Recalling that a subclass $\{\mu_i\}_{i\in\Delta}$ of L-fuzzy subsets $\mathcal{C}$ of a non-empty set X is called directed above if, for any i and $j \in \Delta$ there is $k \in \Delta$ such that $\mu_i \le \mu_k$ and $\mu_j \le \mu_k$ and a class $\mathcal{C}$ of X is said to be an algebraic fuzzy set system if $\mathcal{C}$ is closed under point-wise infimums and closed under the point-wise supremums of directed above subclasses of $\mathcal{C}$. We recall $\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{I}(A)$ is the set of all L-fuzzy $\alpha-$ideals of A.

Theorem 3.12. Let $(A, \wedge, \vee, 0)$ be an ADL and $L = [0, \alpha]$. Then the class $\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{I}(A)$ of all L-fuzzy $\alpha-$ideals of A is an algebraic fuzzy set system.

Proof . Let $\{\mu_i\}_{i\in\Delta}$ be a directed above class of L-fuzzy $\alpha-$ideals of A and μ be the point-wise supremum of $\{\mu_i\}_{i\in\Delta}$. That is, $\mu(x) = \bigvee_{i\in\Delta} \mu_i(x)$, for any $x \in A$. It follows that for any $x, y \in A$, $\mu(x \vee y) = \bigvee_{i\in\Delta} \mu_i(x \vee y) = \bigvee_{i\in\Delta} \mu_i(y \vee x) = \mu(y \vee x)$, since each μ_i is an L-fuzzy $\alpha-$ideal of A. Now we prove that μ is an L-fuzzy $\alpha-$ideal of A. Clearly, $\mu_i \le \mu$, for all $i \in \Delta$. In particular $\alpha = \mu_i(0) \le \mu(0)$ and hence $\mu(0) = \alpha$. Also for any x and $y \in A$,

$$\begin{aligned}
x \le y &\Rightarrow \mu_i(y) \le \mu_i(x), \text{ for all } i \in \Delta \quad \text{(since each } \mu_i \text{ is an antitone)}\\
&\Rightarrow \bigvee_{i\in\Delta} \mu_i(y) \le \bigvee_{i\in\Delta} \mu_i(x)\\
&\Rightarrow \mu(y) \le \mu(x). \text{ Therefore, } \mu \text{ is an antitone.}
\end{aligned}$$

In particular, $\mu(x \vee y) \le \mu(x)$ (since $x \le x \vee y$) and $\mu(x \vee y) = \mu(y \vee x) \le \mu(y)$ and therefore, $\mu(x \vee y) \le \mu(x) \wedge \mu(y)$. On the other hand for any $i, j \in \Delta$, there exists $k \in \Delta$ such that $\mu_i \le \mu_k$ and $\mu_j \le \mu_k$ and hence,

$$\begin{aligned}
\mu(x) \wedge \mu(y) &= \Big(\bigvee_{i\in\Delta} \mu_i(x)\Big) \wedge \Big(\bigvee_{j\in\Delta} \mu_j(y)\Big)\\
&= \bigvee_{i,j\in\Delta} \Big(\mu_i(x) \wedge \mu_j(y)\Big) \text{ (by infinite meet distributivity in } L)\\
&\le \bigvee_{k\in\Delta} \Big(\mu_k(x) \wedge \mu_k(y)\Big)\\
&= \bigvee_{k\in\Delta} \mu_k(x \vee y)\\
&= \mu(x \vee y).
\end{aligned}$$

Therefore $\mu(x) \wedge \mu(y) \le \mu(x \vee y)$. Hence $\mu(x \vee y) = \mu(x) \wedge \mu(y)$. Thus μ is an L-fuzzy $\alpha-$ideal of A and therefore the class $\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{I}(A)$ is closed under the point-wise supremum of $\{\mu_i\}_{i\in\Delta}$ and it is closed under point-wise infimum of $\{\mu_i\}_{i\in\Delta}$. Thus $\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{I}(A)$ is an algebraic fuzzy set system. $\square$

4 Fuzzy α-congruences

By an L-fuzzy subset θ of $A \times B$, where A and B are non-empty sets, we mean any mapping $\theta : A \times B \to L$ is called L-fuzzy relation on A and B. If $A = B$, any mapping $\theta : A \times A \to L$ is called L-fuzzy relation on A.

Definition 4.1. Let A be an ADL and L a frame. Any L-fuzzy subset of $A \times A$ is called an L-fuzzy relation on A.

Definition 4.2. An L-fuzzy subset θ of $A \times A$ is said to be an L-fuzzy α-congruence on A if and only if, for any $x, y, z \in A$, the following hold:

(1) $\theta(x, x) = \alpha$ and $\theta(x, y) \le \alpha$, for all $\alpha \in L - \{0\}$ ($\alpha-$reflexive)

(2) $\theta(x, y) = \theta(y, x)$ (Symmetric)

(3) $\theta(x, y) \wedge \theta(y, z) \le \theta(x, z)$ (Transitive)

(4) $\theta(x,y) \leq \theta(x \vee z, y \vee z) \wedge \theta(x \wedge z, y \wedge z)$.

Note: The above (4) equivalently that $\theta(a,b) \wedge \theta(c,d) \leq \theta(a \wedge c, b \wedge d) \wedge \theta(a \vee c, b \vee d)$, for all $a,b,c,d \in A$.

In the following, we prove that homomorphic image and pre-image of L-fuzzy α-congruence is an L-fuzzy α-congruence under certain conditions.

Theorem 4.3. Let $f : A \times A \to A^{'} \times A^{'}$ be a lattice homomorphism defined by $f(x,y) = (g(x), g(y))$, where g is a homomorphism from A to $A^{'}$, for all $x,y \in A$. If θ is an L-fuzzy α-congruence on $A^{'}$, then $f^{-1}(\theta)$ is an L-fuzzy α-congruence on A.

Proof . Define $f^{-1}(\theta) : A \times A \to L$ by $\big(f^{-1}(\theta)\big)(x,y) = \theta\big(f(x,y)\big)$, for all $x,y \in A$. Let θ be an L-fuzzy α-congruence on $A^{'}$. Then for any $x \in A$, $\Big(f^{-1}(\theta)\Big)(x,x) = \theta\big(f(x,x)\big) = \theta\big(g(x), g(x)\big) = \alpha$ (since θ is $\alpha-$ reflexive). For any $x,y \in A$, $\Big(f^{-1}(\theta)\Big)(x,y) = \theta\big(f(x,y)\big) = \theta\big(g(x), g(y)\big) \leq \alpha$ (since θ is $\alpha-$ reflexive). Therefore, $f^{-1}(\theta)$ is $\alpha-$ reflexive. Let $x,y \in A$. Then,

$$\begin{aligned}\Big(f^{-1}(\theta)\Big)(x,y) &= \theta(f(x,y))\\ &= \theta\big(g(x), g(y)\big)\\ &= \theta\big(g(y), g(x)\big) \text{ (since } \theta \text{ is symmetric)}\\ &= \theta\big(f(y,x)\big)\\ &= \Big(f^{-1}(\theta)\Big)(y,x).\end{aligned}$$

Thus $f^{-1}(\theta)$ is symmetric. Also, let $x,y,z \in A$. Then,

$$\begin{aligned}\Big(f^{-1}(\theta)\Big)(x,z) &= \theta(f(x,z))\\ &= \theta(g(x), g(z))\\ &\geq \theta(g(x), g(y)) \wedge \theta(g(y), g(z)) \text{ (since } \theta \text{ is symmetric)}\\ &= \theta(f(x,y)) \wedge \theta(f(y,z))\\ &= \Big(f^{-1}(\theta)\Big)(x,y) \wedge \Big(f^{-1}(\theta)\Big)(y,z).\end{aligned}$$

Thus, $f^{-1}(\theta)$ is transitive. Finally, for any $x,y,z \in A$. Then

$$\begin{aligned}(f^{-1}(\theta))(x,y) &= \theta(f(x,y))\\ &= \theta(g(x), g(y)\\ &\leq \theta\Big(g(x) \vee g(z), g(y) \vee g(z)\Big) \wedge \theta\Big(g(x) \wedge g(z), g(y) \wedge g(z)\Big)\\ &= \theta\Big(g(x \vee z), g(y \vee z)\Big) \wedge \theta\Big(g(x \wedge z), g(y \wedge z)\Big) \text{ (since } g \text{ is homomorphism)}\\ &= \theta\Big(f(x \vee z, y \vee z)\Big) \wedge \theta\Big(f(x \wedge z, y \wedge z)\Big) \text{ (since } f \text{ is homomorphism)}\\ &= \Big(f^{-1}(\theta)\Big)(x \vee z, y \vee z) \wedge \Big(f^{-1}(\theta)\Big)(x \wedge z, y \wedge z).\end{aligned}$$

Therefore, $f^{-1}(\theta)$ is an L-fuzzy $\alpha-$ congruence on A. $\square$

Theorem 4.4. Let $f : A \times A \to A^{'} \times A^{'}$ be an epimorphism defined by $f(x,y) = (g(x), g(y))$, where g is an epimorphism from A to $A^{'}$, for all $x,y \in A$. If θ is an L-fuzzy α-congruence on A, then $f(\theta)$ is an L-fuzzy α-congruence on $A^{'}$.

Proof . Define $f(\theta) : A \times A \to L$ by

$$\Big(f(\theta)\Big)(x,y) = \bigvee_{(a,b) \in f^{-1}(x,y)} \theta(a,b), \text{ for all } (a,b), (x,y) \in A \times A.$$

Let θ be an L-fuzzy α-congruence on A. For any $x,y,z \in A$ and $\alpha \in L - \{0\}$. Then
(1). $(f(\theta))(x,x) = \bigvee_{(a,a) \in f^{-1}(x,x)} \theta(a,a) = \alpha$ and $(f(\theta))(x,y) = \bigvee_{(a,b) \in f^{-1}(x,y)} \theta(a,b) \leq \alpha$ (since θ is $\alpha-$ reflexive). Thus, $f(\theta)$ is $\alpha-$ reflexive.
(2). $(f(\theta))(x,y) = \bigvee_{(a,b) \in f^{-1}(x,y)} \theta(a,b) = \bigvee_{(b,a) \in f^{-1}(y,x)} \theta(b,a)$ (since θ is symmetric)
$= (f(\theta))(y,x)$.

Thus $f(\theta)$ is symmetric.

(3). Consider
$$\Big(f(\theta)\Big)(x,y)\wedge\Big(f(\theta)\Big)(y,z)=\Big(\bigvee_{(a,b)\in f^{-1}(x,y)}\theta(a,b)\Big)\wedge\Big(\bigvee_{(b,c)\in f^{-1}(y,z)}\theta(b,c)\Big)$$
$$=\bigvee_{(a,b)\in f^{-1}(x,y),(b,c)\in f^{-1}(y,z)}\Big(\theta(a,b)\wedge\theta(b,c)\Big)$$
$$\leq\bigvee_{(a,c)\in f^{-1}(x,z)}\theta(a,c)\ \ (\text{since } \theta \text{ is transitive})$$
$$=\Big(f(\theta)\Big)(x,z).$$
Hence, $f(\theta)$ is transitive.
(4). Similarly, it can be verified that, $\Big(f(\theta)\Big)(x,y)\leq\Big(f(\theta)\Big)(x\vee z,y\vee z)\wedge\Big(f(\theta)\Big)(x\wedge z,y\wedge z)$.
Therefore, $f(\theta)$ is an L-fuzzy α-congruence on $A^{'}$. □

In the following theorems gives a one-to-one correspondence between L-fuzzy $\alpha-$ideals and L-fuzzy $\alpha-$congruence on A. We recall $\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{C}(A)$ is the set of all L-fuzzy $\alpha-$congruences on A.

5 One to one correspondence between $\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{I}(A)$ and $\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{C}(A)$

In this section, we will construct an L-fuzzy $\alpha-$ ideals corresponding to L-fuzzy $\alpha-$congruences on ADL A.

Definition 5.1. Let μ be an L-fuzzy $\alpha-$ideal of A. An L-fuzzy subset θ_μ can be defined by

$$\theta_\mu(x,y)=\begin{cases}\bigvee_{z\vee x=z\vee y}\mu(z) & \text{if } x\neq y\\ \alpha & \text{if } x=y,\end{cases}$$

for all $x,y,z\in A$.

Recalling from [17] that, An ADL A is said to be associative ADL, if the binary operation $\vee$ in A is associative; that is $(a\vee b)\vee c=a\vee(b\vee c)$, for all $a,b,c\in A$.

Theorem 5.2. Let μ be an L-fuzzy $\alpha-$ideal of an associative ADL A. Then θ_μ can be defined above, is an L-fuzzy $\alpha-$congruence on A.

Proof . Let μ be an L-fuzzy $\alpha-$ideal of A. For any $x\in A$, $\theta_\mu(x,x)=\alpha$ and clearly, $\theta_\mu(x,y)\leq\alpha$ for any $x,y\in A$. Hence, θ_μ is $\alpha-$reflexive. Clearly, $\theta_\mu(x,y)=\theta_\mu(y,x)$ if $x=y$. Suppose $x\neq y$. Then $_{\theta_\mu}(x,y)=\bigvee_{z\vee x=z\vee y}\mu(z)$
$$=\bigvee_{z\vee y=z\vee x}\mu(z)$$
$$=\theta_\mu(y,x).$$
Thus, θ_μ is symmetric. If $x=z$, then we have, $\theta_\mu(x,y)\wedge\theta_\mu(y,z)=\alpha\wedge\alpha=\alpha=\theta_\mu(x,z)$. Suppose that $x\neq z$. Then
$$\theta_\mu(x,y)\wedge\theta_\mu(y,z)=\Big(\bigvee_{t\vee x=t\vee y}\mu(t)\Big)\wedge\Big(\bigvee_{s\vee x=s\vee y}\mu(s)\Big)$$
$$=\bigvee_{t\vee x=t\vee y,s\vee x=s\vee y}\Big(\mu(t)\wedge\mu(s)\Big)$$
$$\leq\bigvee_{(t\vee s)\vee x=(t\vee s)\vee y}\mu(t\vee s)$$
$$\leq\bigvee_{c\vee x=c\vee z}\mu(c)$$
$$=\theta_\mu(x,z).$$
Thus, θ_μ is transitive. Finally, let $x,y,z,t\in A$ such that $a\vee x=a\vee y$ and $b\vee z=b\vee t$, for any $a,b\in A$. Since A is an associative ADL, then we get $(a\vee b)\vee(x\vee z)=(a\vee b)\vee(y\vee z)$ and $(a\vee b)\vee(y\vee z)=(a\vee b)\vee(y\vee t)$, which implies that $(a\vee b)\vee(x\vee z)=(a\vee b)\vee(y\vee t)$.
Now,
$$\theta_\mu(x,y)\wedge\theta_\mu(z,t)=\Big(\bigvee_{a\vee x=a\vee y}\mu(a)\Big)\wedge\Big(\bigvee_{b\vee z=b\vee t}\mu(b)\Big)$$
$$=\bigvee_{a\vee x=a\vee y,b\vee z=b\vee t}\Big(\mu(a)\wedge\mu(b)\Big)$$
$$\leq\bigvee_{(a\vee b)\vee(x\vee z)=(a\vee b)\vee(y\vee t)}\mu(a\vee b)$$
$$=\theta_\mu(x\vee z,y\vee t). \qquad (1)$$

Also, let $b\vee x = b\vee y$ and $c\vee z = c\vee t$. Then $(b\vee x)\wedge(c\vee z) = (b\vee y)\wedge(c\vee t)$ and thus $\Big((b\wedge c)\vee(x\wedge c)\vee(b\wedge z)\Big)\vee(x\wedge z) = \Big((b\wedge c)\vee(y\wedge c)\vee(b\wedge t)\Big)\vee(y\wedge t)$. Since $(b\vee x)\wedge c = (b\vee y)\wedge c$ and $(c\vee z)\wedge b = (c\vee t)\wedge b$, we have $(b\wedge c)\vee(x\wedge c)\vee(z\wedge b) = (b\wedge c)\vee(y\wedge c)\vee(t\wedge b)$. Since μ is an L-fuzzy $\alpha-$ideal of A, then

$\mu\Big((b\wedge c)\vee(x\wedge c)\vee(z\wedge b)\Big) = \mu(b\wedge c)\wedge\mu(x\wedge c)\wedge\mu(z\wedge b) \geq \mu(b)\wedge\mu(c)$. Now,

$$
\begin{aligned}
\theta_\mu(x,y)\wedge\theta_\mu(z,t) &= \Big(\bigvee_{b\vee x=b\vee y}\mu(a)\Big)\wedge\Big(\bigvee_{c\vee z=c\vee t}\mu(b)\Big)\\
&= \bigvee_{b\vee x=b\vee y, c\vee z=c\vee t}\Big(\mu(b)\wedge\mu(c)\Big)\\
&\leq \bigvee_{\Big((b\wedge c)\vee(x\wedge c)\vee(z\wedge b)\Big)\vee(x\wedge z)=\Big((b\wedge c)\vee(y\wedge c)\vee(t\wedge b)\Big)\vee(y\wedge t)}\mu\Big((b\wedge c)\vee(x\wedge c)\vee(z\wedge b)\Big)\\
&\leq \bigvee_{a\vee(x\wedge z)=a\vee(y\wedge t)}\mu(a)\\
&= \theta_\mu(x\wedge z, y\wedge t). \qquad (2)
\end{aligned}
$$

Thus from (1) and (2) we have, $\theta_\mu(x,y)\wedge\theta_\mu(z,t) \leq \theta_\mu(x\vee z, y\vee t)\wedge\theta_\mu(x\wedge z, y\wedge t)$. Therefore, θ_μ is an L-fuzzy $\alpha-$congruence on A. □

Theorem 5.3. Let θ be an L-fuzzy $\alpha-$ congruence on A and $L = [0,\alpha]$. Define the L-fuzzy subset μ_θ of A by $\mu_\theta(a) = \theta(a,0)$ for all $a\in A$. Then μ_θ is an L-fuzzy $\alpha-$ideal of A.

Proof . For any $x,y\in A$. Then $\mu_\theta(0,0) = \theta(0,0) = \alpha$ (since θ is L-fuzzy $\alpha-$congruence), for any $\alpha\in L-\{0\}$. Also, $\mu_\theta(x\vee y) = \theta(x\vee y,0) \geq \theta(x,0)\wedge\theta(y,0) = \mu_\theta(x)\wedge\mu_\theta(y)$. Finally, $\mu_\theta(x\wedge y) = \theta(x\wedge y,0) = \theta(x\wedge y, 0\wedge y) \geq \theta(x,0)\wedge\theta(y,y) = \mu_\theta(x)\wedge\alpha = \mu_\theta(x)$. Thus $\mu_\theta(x\wedge y)\geq\mu_\theta(x)$. Similarly, I can verified that $\mu_\theta(x\wedge y)\geq\mu_\theta(y)$. Form these, we have $\mu_\theta(x\wedge y)\geq\mu_\theta(x)\vee\mu_\theta(y)$. Hence the theorem. □

Theorem 5.4. Let $\lambda,\mu\in\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{I}$ and $\theta\in\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{C}$. Then $\mu_{(\theta_\lambda)} = \lambda$.

Proof . For each $a\in A$, $\mu_{(\theta_\lambda)}(a) = \theta_\lambda(a,0) = \bigvee_{b\vee a=b}\lambda(b) = \lambda(a)$ (since λ is an L-fuzzy $\alpha-$ideal, $b\vee a = b \Leftrightarrow a\leq b$, implies $\lambda(b)\leq\lambda(a)$). Thus, $\mu_{(\theta_\lambda)} = \lambda$. □

Theorem 5.5. Let $\lambda\in\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{I}$ and $\theta\in\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{C}$. Then $\theta_{(\lambda_\theta)} = \theta$.

Proof . Let θ be an L-fuzzy $\alpha-$congruence on A. For any $a,b\in A$,

$$
\begin{aligned}
\theta_{(\lambda_\theta)}(a,b) &= \bigvee_{c\vee a=c\vee b}\lambda_\theta(c), \text{ for all } c\in A\\
&= \bigvee_{c\vee a=c\vee b}\theta(c,0)\\
&= \theta(x,y).
\end{aligned}
$$

Therefore, $\theta_{(\lambda_\theta)} = \theta$. □

Finally, we conclude this paper with the following result.

Theorem 5.6. The mapping $\lambda\to\phi_\lambda : \mathcal{F}_{\mathcal{L}_\alpha}\mathcal{I}\to\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{C}$ and $\theta\to\mu_\theta : \mathcal{F}_{\mathcal{L}_\alpha}\mathcal{C}\to\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{I}$ are mutual inverses. More over, the maps are lattice isomorphism.

Proof . By the above theorems, the maps are mutual inverses and hence, there is a one-to-one correspondence between $\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{I}$ and $\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{C}$. Also, let $\lambda,\nu\in\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{I}$ such that $\lambda\leq\nu$. For any $a,b\in A$ such that $a\neq b$, $\phi_\lambda(a,b) = \bigvee_{c\vee a=c\vee b}\lambda(c) \leq \bigvee_{c\vee a=c\vee b}\nu(c) = \phi_\nu(a,b)$ (since $\lambda\leq\nu$). Thus, $\phi_\lambda\leq\phi_\nu$. Therefore, $\lambda\to\phi_\lambda$ is isotone. Finally, let $\theta,\phi\in\mathcal{F}_{\mathcal{L}_\alpha}\mathcal{C}$ such that $\phi\leq\theta$. For any $a\in A$, $\mu_\phi(a) = \phi(a,0)\leq\theta(a,0) = \mu_\theta(a)$. Thus, $\mu_\phi\leq\mu_\theta$. Therefore, $\theta\to\mu_\theta$ is isotone. Hence, the maps are lattice isomorphism. □

6 Conclusion

The notion of an Almost Distributive Lattice (ADL) is a common abstraction of several lattice theoretic and ring theoretic generalizations of Boolean algebra and Boolean rings. Later, G.C. Rao and M.S. Rao [11] introduced the concept of α-ideals in Almost Distributive Lattices. Here, we extend this result to the case of L-fuzzy α-ideals of Almost Distributive Lattices. In this paper, we make a thorough discussion on various lattice theoretic properties of the set of all L-fuzzy α-ideals and L-fuzzy α-congruences of an ADL with truth values in a complete lattice L satisfying the infinite meet distributive law. In particular, we proved that the class of L-fuzzy α-ideals of an ADL is an algebraic fuzzy set system. Also, it is noted that the homomorphic image and pre-image of L-fuzzy α-congruences of ADL is again an L-fuzzy α-congruence. Finally, we discussed a one-to-one correspondence between the class of L-fuzzy α-ideals and L-fuzzy α-congruences of an ADL.

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