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Identifying code number of some Cayley graphs

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Abstract

Let $\Gamma = (V, E)$ be a simple graph. A set C of vertices Γ is an identifying set of Γ if for every two vertices x and y the sets $N_\Gamma[x] \cap C$ and $N_\Gamma[y] \cap C$ are non-empty and different. Given a graph Γ, the smallest size of an identifying set of Γ is called the identifying code number of Γ and is denoted by $\gamma^{ID}(\Gamma)$. Two vertices x and y are twins when $N_\Gamma[x] = N_\Gamma[y]$. Graphs with at least two twin vertices are not identifiable graph. In this paper, we study identifying code number of some Cayley graphs.

Keywords: Domination, Identifying code, Cayley graph.
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1 Introduction

In this paper, all graphs are assumed to be finite, simple and undirected. We will often use the notation $\Gamma = (V, E)$ to denote the graph with non-empty vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma)$. An edge of Γ with endpoints u and v is denoted by $u - v$. For every vertex $x \in V(\Gamma)$, the *open neighborhood* of vertex x is denoted by $N_\Gamma(x)$ and defined as $N_\Gamma(x) = \{y \in V(\Gamma) \mid x - y\}$. Also the *close neighborhood* of vertex $x \in V(\Gamma)$, $N_\Gamma[x]$, is $N_\Gamma[x] = N_\Gamma(x) \cup \{x\}$. The *degree* of a vertex $x \in V(\Gamma)$ is $\deg_\Gamma(x) = \big|N_\Gamma(x)\big|$.

The *complement* of graph Γ is denoted by $\overline{\Gamma}$ is a graph with vertex set $V(\Gamma)$ which $e \in E(\overline{\Gamma})$ if and only if $e \notin E(\Gamma)$. For any $\Omega \subseteq V(\Gamma)$, the *induced subgraph* on Ω, denoted by $\Gamma[\Omega]$. The join of n graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, denoted by $\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$, is a graph obtained from $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ by joining each vertex of Γ_i to all vertices of Γ_j ($i \neq j$ and $1 \leq i, j \leq n$).

Let G be a non-trivial group, Ω be an inverse closed subset of G which does not contain the identity element of G, i.e. $\Omega = \Omega^{-1} = \{s^{-1} \mid s \in \Omega\}$ and $1 \notin \Omega$. The Cayley graph of Γ denoted by $Cay(G, \Omega)$, is a graph with vertex set G and two vertices a and b are adjacent if and only if $ab^{-1} \in \Omega$.

A set of vertices Γ such as C is a *dominating set* of graph Γ if for every vertex x of $V(\Gamma)$, is either in C or is adjacent to a vertex in C. It is clear that every isolated vertex is in every dominating set of Γ. Also, a set C is called a *separating set* of Γ if for each pair u, v of vertices of Γ, $N_\Gamma[u] \cap C \neq N_\Gamma[v] \cap C$ (equivalently, $(N_\Gamma[u] \triangle N_\Gamma[v]) \cap C \neq \emptyset$). If a dominating set C in graph Γ is a separating set of Γ, then we said that C is an identifying set of graph Γ and if Γ

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has an identifying set, then we said that Γ is an *identifiable graph*. Given a graph Γ, the smallest size of an identifying set of Γ is called the *identifying code number* of Γ and is denoted by $\gamma^{ID}(\Gamma)$. If for two distinct vertices x and y, $N_\Gamma[x] = N_\Gamma[y]$, then they are called *twins*. It is noteworthy that a graph Γ is identifiable if and only if Γ is twin free. In recent years much attention drawn to the domination theory which is very interesting branch in graph theory. The concept of domination expanded to other parameters of domination such as 2-rainbow domination, signed domination, Roman domination, total Roman domination number, and identifying code number. For more details, we refer reader to [2, 5, 13, 15, 17].

The identifying code concept was introduced by Karpovsky et al [12] in 1998. Later, several various families of graphs have been studied; cycles and paths [3, 9], trees [1], triangular and square grids [11] and triangle-free graphs [7]. Camarero et al [4], in 2015, provide a constructive method for finding a wide family of identifying codes over degree four Cayley graphs over finite Abelian groups. Also identifying codes have found applications to various fields. These applications include sensor network monitoring, identifying codes in random networks [8] and the structural analysis of RNA proteins [10].

This paper deals with the study of identifying code number of some Cayley graphs. We show that $Cay(G,\Omega)$ is not an identifiable graph if and only if $\Omega \subseteq N_{Cay(G,\Omega)}[s]$ for some $s \in \Omega$. Also for some finite Abelian group G, with $G = \langle\Omega\rangle, 1 \notin \Omega = \Omega^{-1}$, we determine $\gamma^{ID}\big(Cay(G,\Omega)\big)$.

2 Preliminaries

In this section, we give some facts that we are needed in section 3.

Theorem 2.1. [12] Let Γ be an identifiable graph with n vertices. Then $\gamma^{ID}(\Gamma) \geq \lceil log_2(n+1)\rceil$.

Lemma 2.2. [16] Let Γ be a graph and C be an identifying set of Γ. If $N_\Gamma[x] \bigtriangleup N_\Gamma[y] = \{y_1, y_2\}$, then $y_1 \in C$ or $y_2 \in C$.

Lemma 2.3. Let $n_i > 2$ and $\Gamma \cong K_{n_1,n_2,\ldots,n_k}$ be a complete multipartite graph. Then $\gamma^{ID}(\Gamma) = n - k$, where $n = n_1 + n_2 + \cdots + n_k$.

Proof . Let $X_i = \{x_{i1}, \ldots, x_{in_i}\}$, induced subgraph on X_i be empty graph and $V(\Gamma) = \cup_{i=1}^k X_i$. Let C be an identifying set of Γ with minimum cardinality such that $|C \cap X_i| \leq n_i - 2$ and $\{x_{i\ell}, x_{ih}\} \cap C = \emptyset$ for some $1 \leq i \leq k$. Then $N_\Gamma[x_{i\ell}] \cap C = N_\Gamma[x_{ih}] \cap C$, which is a contradiction. So $\gamma^{ID}(\Gamma) = |C| \geq n-k$. Now let $D = V(\Gamma)\backslash\{x_{i1} \in X_i \mid 1 \leq i \leq k\}$. Then $N_\Gamma[x_{ij}] \cap D = (D\backslash X_i) \cup \{x_{ij}\}$ and $N_\Gamma[x_{i1}] \cap D = D\backslash X_i$ for $2 \leq j \leq n_i, 1 \leq i \leq k$. This shows that $N_\Gamma[a] \cap D \neq \emptyset$ and $N_\Gamma[a] \cap D \neq N_\Gamma[b] \cap D$, for every a and b in Γ. So, D is an identifying set of Γ. Thus $\gamma^{ID}(\Gamma) \leq |D| = n - k$. Therefore, $\gamma^{ID}(\Gamma) = n - k$. $\square$

Theorem 2.4. ([9], [3]) Let $n \geq 4$ be a positive integer and $\Gamma \cong C_n$. Then

$$\gamma^{ID}(\Gamma) = \begin{cases} 3 & \text{if } n = 4, 5 \\ \frac{n}{2} & \text{if } n \geq 6 \text{ is even} \\ \frac{n+3}{2} & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

Theorem 2.5. [14] Let $n \geq 6$ be a positive integer. If n is even, then

$$\gamma^{ID}(\overline{C_n}) = \begin{cases} n - n/3 & \text{if } n \equiv 0 \pmod 3 \\ n - \lfloor n/3 \rfloor & \text{if } n \equiv 1 \pmod 3 \\ n - \lceil n/3 \rceil & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

and if n is odd, then

$$\gamma^{ID}(\overline{C_n}) = \begin{cases} n - n/3 & \text{if } n \equiv 0 \pmod 3 \\ n - \lceil n/3 \rceil & \text{if } n \equiv 1 \pmod 3 \\ n - \lfloor n/3 \rfloor & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Lemma 2.6. Let $G = \langle \Omega \rangle$ be a finite Abelian group, $1 \notin \Omega = \Omega^{-1}$ and $G \setminus \big(\Omega \cup \{1\}\big) = \Omega_1 \cup \Omega_2$ such that $\Omega_1 \cap \Omega_2 = \emptyset$. If $\Omega_1 \cup \{1\}$and $\Omega_2 \cup \{1\}$ are subgroups of G, then $\big|\Omega_2 \cup \{1\}\big| \;\big|\; \big[G : \Omega_1 \cup \{1\}\big]$.

Proof . Let $\Omega_1 \cup \{1\} = H$ and $\Omega_2 = \{x_1, \dots, x_t\}$. For $i \neq j$ and $1 \leq i, j \leq t$, since $x_i x_j^{-1} \in \Omega_2$, $Hx_i \neq Hx_j$ and so cosets $H = Hx_0, Hx_1, Hx_2, \dots, Hx_t$ are distinct. If $G = \cup_{i=0}^{t} Hx_i$, then $(t+1) \;\big|\; [G : H]$. Otherwise, we have $G \neq \cup_{i=0}^{t} Hx_i$. Let $y_1 \in G \setminus \cup_{i=0}^{t} Hx_i$. Then for $0 \leq i \leq t$ and $0 \leq j \leq\ 1$, the cosets $Hx_i y_j$ are distinct, where $y_0 = 1$. If for $0 \leq i \leq t$ and $0 \leq j \leq 1$, $G = \cup_{i=0}^{t}(\cup_{j=0}^{1} Hx_i y_j)$, then $|G| = 2(t+1)|H|$ and so $(t+1) \;\big|\; [G : H]$. Since G is a finite group, there is $\ell \in \mathbb{N}$ such that $Hx_i y_j$ for $0 \leq i \leq t$ and $0 \leq j \leq \ell$ are distinct and $G = \cup_{i=0}^{t}(\cup_{j=0}^{\ell} Hx_i y_j)$. Hence $|G| = (t+1)\ell|H|$. Therefore, $(t+1) \;\big|\; [G : H]$. $\square$

3 Main results

In this section, we prove our main results.

Theorem 3.1. Let G be a finite group and $\Omega \subseteq G$ such that $1 \notin \Omega = \Omega^{-1}$. Then $Cay(G, \Omega)$ is not an identifiable graph if and only if $\Omega \subseteq N_{Cay(G,\Omega)}[s]$ for some $s \in \Omega$.

Proof . If $Cay(G, \Omega)$ is not an identifiable graph, then there are two distinct vertices x and y in G such that $N_{Cay(G,\Omega)}[x] = N_{Cay(G,\Omega)}[y]$. Since x is adjacent to y, there is vertex $s \in \Omega$ such that $y = sx$ and s is unique. Also $\Omega x = \Omega y$, because $N_{Cay(G,\Omega)}[x] = N_{Cay(G,\Omega)}[y]$. So, for every $s_i \in \Omega \backslash \{s\}$ there is $s_j \in \Omega$ such that $s_i x = s_j y$ or $s_j^{-1} s_i x = y$. Thus $s_j^{-1} s_i = s$ and so $s_i = s_j s$. This shows that s is adjacent to s_i. So $\Omega \subseteq N_{Cay(G,\Omega)}[s]$.
Conversely, suppose that $\Omega \subseteq N_{Cay(G,\Omega)}[s]$, for some $s \in \Omega$. Since $Cay(G, \Omega)$ is $|\Omega|$-regular graph and $1 \in N_{Cay(G,\Omega)}[s]$, we have $N_{Cay(G,\Omega)}[s] = \Omega \cup \{1\} = N_{Cay(G,\Omega)}[1]$. Therefore, $Cay(G, \Omega)$ is not an identifiable graph. $\square$

Corollary 3.2. Let $G = \langle a \rangle$ be a cyclic group of order $2n$, where n is even. If $\Omega = \{a^{2k+1} \mid 0 \leq k \leq n-1\} \cup \{a^n\}$, then $Cay(G, \Omega)$ is not an identifiable graph.

Proof . We have $N_{Cay(G,\Omega)}[a^n] = \Omega \cup \{1\}$ By Theorem 3.1, $Cay(G, \Omega)$ is not an identifiable graph. $\square$

Theorem 3.3. Let G be a finite group of order n and H be a proper subgroup of G. If $G \setminus H = \Omega$, then $Cay(G, \Omega)$ is an identifiable graph and $\gamma^{ID}\big(Cay(G, \Omega)\big) = [G : H](|H| - 1)$.

Proof . Since H is a subgroup of G and $H \neq G$, $G = \langle \Omega \rangle$. Also we have $\Omega = \Omega^{-1}$ and $1 \notin \Omega$. Let $[G : H] = k$ and $Ha_1, Ha_2, \dots, Ha_k$ be the distinct cosets of H in G, where $a_1 = 1$. For h_1 and h_2 in H, we have $(h_1 a_j)(h_2 a_j)^{-1} = h_1 h_2^{-1} \in H$ $(1 \leq j \leq k)$. So induced subgrphs on $Ha_1, Ha_2, \dots, Ha_k$ in $Cay(G, \Omega)$ are empty graph. Also for $ha_j \in Ha_j$ and $h' a_\ell \in Ha_\ell$ we have $(ha_j)(h' a_\ell)^{-1} \notin H$. Hence ha_j is adjacent to $h' a_\ell$. Thus $Cay(G, \Omega)$ is isomorphic to $K_{n_1,\dots,n_k}$ and $n_1 = \cdots = n_k = |H|$. By Lemma 2.3, $\gamma^{ID}\big(Cay(G, \Omega)\big) = n - k = [G : H](|H| - 1)$. $\square$

Corollary 3.4. Let G be a finite group of order $n = 2k$. If $a \in G$, $o(a) = 2$, $\Omega = G \backslash \{1, a\}$, then $\gamma^{ID}\big(Cay(G, \Omega)\big) = k$.

Proof . It is clear that $G \setminus \Omega = \{1, a\}$ is a subgroup of G. By Theorem 3.3, $\gamma^{ID}\big(Cay(G, \Omega)\big) = k$. $\square$

Corollary 3.5. Let $G = \langle \Omega \rangle$ be a finite group of order $2n \geq 6$, where $\Omega = \Omega^{-1}$, $1 \notin \Omega$ and $|\Omega| = n$. If the induced subgraph on Ω in $Cay(G, \Omega)$ is empty graph, then $\gamma^{ID}\big(Cay(G, \Omega)\big) = 2n - 2$.

Proof . Since $Cay(G, \Omega)$ is n-regular and induced subgraph on Ω in $Cay(G, \Omega)$ is empty graph, so for every $x \in \Omega$, we have $N_{Cay(G,\Omega)}[x] = G \backslash \Omega \cup \{x\}$. By Theorem 3.1, $Cay(G, \Omega)$ is an identifiable graph. Also, for every $y \in G \setminus \Omega$ we have $N_{Cay(G,\Omega)}[y] = \Omega \cup \{y\}$. Hence, $Cay(G, \Omega)$ is isomorphic to $K_{n,n}$. By Lemma 2.3, $\gamma^{ID}\big(Cay(G, \Omega)\big) = 2n - 2$. $\square$

Theorem 3.6. Let G be a finite Abelian group of order n and $\Omega = G \setminus \{1, a, b\}$, where $G = \langle \Omega \rangle$, $\Omega = \Omega^{-1}$.

i) Let $o(a) \in \{2, 4\}$. Then $Cay(G, \Omega)$ is not an identifiable graph.

ii) Let $o(a) = 3$. Then $\gamma^{ID}\big((Cay(G, \Omega)\big) = \frac{2n}{3}$.

iii) Let $o(a) = k$ and $k \geq 5$. Then if $k = 5$, then $\gamma^{ID}\big(Cay(G,\Omega)\big) = \frac{3n}{5}$, if k is even, then

$$\gamma^{ID}\big(Cay(G,\Omega)\big) = \begin{cases} t(k - k/3) & \text{if } k \equiv 0 \pmod 3 \\ t(k - \lfloor k/3 \rfloor) & \text{if } k \equiv 1 \pmod 3 \\ t(k - \lceil k/3 \rceil) & \text{if } k \equiv 2 \pmod 3, \end{cases}$$

and if $k \neq 5$ is odd, then

$$\gamma^{ID}\big(Cay(G,\Omega)\big) = \begin{cases} t(k - k/3) & \text{if } k \equiv 0 \pmod 3 \\ t(k - \lceil k/3 \rceil) & \text{if } k \equiv 1 \pmod 3 \\ t(k - \lfloor k/3 \rfloor) & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Proof .

i) If $o(a) = 2$, then $o(b) = 2$. It is clear that $N_{Cay(G,\Omega)}[ab] = \Omega \cup \{1\}$. If $o(a) = 4$, then $b = a^3$. We have $N_{Cay(G,\Omega)}[a^2] = \Omega \cup \{1\}$. By Theorem 3.1, $Cay(G,\Omega)$ is not an identifiable graph.

ii) Let $o(a) = 3$. Then $b = a^{-1} = a^2$ and so $G \setminus \Omega$ is a subgroup of G. By Theorem 3.3, $\gamma^{ID}\big(Cay(G,\Omega)\big) = \frac{2n}{3}$.

iii) Let $o(a) = k$ and $H = \langle a \rangle$. For every $x \in G \setminus H$, induced subgraph on Hx in $Cay(G,\Omega)$ is isomorphic to $\overline{C}_k$. If $t = [G : H]$ and $H = Hx_1, Hx_2, \ldots, Hx_t$ are distinct cosets of H in G, then by definition of Cayley graph, we have

$$Cay(G,\Omega) = Cay(G,\Omega)[Hx_1] + Cay(G,\Omega)[Hx_2] + \cdots + Cay(G,\Omega)[Hx_t].$$

Let $Cay(G,\Omega) = \Gamma$ and $Cay(G,\Omega)[Hx_i] = \Gamma_i$ for $1 \leq i \leq t$. Also let C be an identifying set of $Cay(G,\Omega)$ and $C \cap Hx_i = C_i$ for $1 \leq i \leq t$. If $1 \leq j \leq k$ and $N_{\Gamma_i}[a^j x_i] \cap C_i = \emptyset$, then $C_i = \{a^{j-1}x_i, a^{j+1}x_i\}$ and so $N_\Gamma[a^{j-1}x_i] \cap C = N_\Gamma[a^{j+1}x_i] \cap C$. It is impossible.
Also if $N_{\Gamma_i}[a^j x_i] \cap C_i = N_{\Gamma_i}[a^\ell x_i] \cap C_i$, then $N_\Gamma[a^j x_i] \cap C = N_\Gamma[a^\ell x_i] \cap C$, which is a contradiction. So C_i is an identifying set of Γ_i. Hence $\gamma^{ID}(\Gamma_i) \leq |C_i|$. Thus $\gamma^{ID}(\Gamma) \geq \gamma^{ID}(\Gamma_1) + \cdots + \gamma^{ID}(\Gamma_t)$. Now let D_i be an identifying set of Γ_i with minimum cardinality, for $1 \leq i \leq t$. It is easy to see that $D = \cup_{i=1}^t D_i$ is an identifying set of Γ. So $\gamma^{ID}(\Gamma) \leq |D| = \sum_{i=1}^t |D_i| = \sum_{i=1}^t \gamma^{ID}(\Gamma_i)$. Therefore

$$\gamma^{ID}(\Gamma) = \sum_{i=1}^{t} \gamma^{ID}(\Gamma_i) = t\gamma^{ID}(\overline{C_k}).$$

If $k = 5$, then $\gamma^{ID}(\Gamma) = \frac{n}{5}\gamma^{ID}(\overline{C_5}) = \frac{n}{5}\gamma^{ID}(C_5) = \frac{3n}{5}$.
Let $k \geq 6$. Then by Theorem 2.5, if k is even, then

$$\gamma^{ID}(\Gamma) = \begin{cases} t(k - k/3) & \text{if } k \equiv 0 \pmod 3 \\ t(k - \lfloor k/3 \rfloor) & \text{if } k \equiv 1 \pmod 3 \\ t(k - \lceil k/3 \rceil) & \text{if } k \equiv 2 \pmod 3, \end{cases}$$

and if k is odd, then

$$\gamma^{ID}(\Gamma) = \begin{cases} t(k - k/3) & \text{if } k \equiv 0 \pmod 3 \\ t(k - \lceil k/3 \rceil) & \text{if } k \equiv 1 \pmod 3 \\ t(k - \lfloor k/3 \rfloor) & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

□

Theorem 3.7. Let $G = \langle \Omega \rangle$ be a group of order n, $x \in \Omega$ and $o(x) = 2$. If $H = \big(\Omega \setminus \{x\}\big) \cup \{1\}$ is a subgroup of G, then $\gamma^{ID}\big(Cay(G,\Omega)\big) = n - 2$.

Proof . Since $o(x) = 2$, n is even. Let $H = \{1 = h_1, h_2, \ldots, h_t\}$. Then for $1 \leq i, j \leq t$, $h_i h_j^{-1} \in H$ and $(h_i x)(h_j x)^{-1} \in H$. So induced subgraphs on H and Hx in $Cay(G,\Omega)$ are isomorphic to complete graph K_t. Also for $1 \leq i \leq t$, we have $N_{Cay(G,\Omega)}[h_i] = H \cup \{h_i x\}$ and $N_{Cay(G,\Omega)}[h_i x] = \{h_i\} \cup Hx$. By Theorem 3.1, $Cay(G,\Omega)$ is an identifiable graph. Since

$Cay(G,\Omega)$ is a t-regular connected graph, $G = H \cup Hx$. Hence $n = 2t$. Let $D = G \setminus \{1, x\}$. Then $N_{Cay(G,\Omega)}[h_i] \cap D = H \setminus \{1\} \cup \{h_i x\}$ and $N_{Cay(G,\Omega)}[h_i x] \cap D = Hx \setminus \{x\} \cup \{h_i\}$ for $2 \leq i \leq t$. Also $N_{Cay(G,\Omega)}[1] \cap D = H \setminus \{1\}$ and $N_{Cay(G,\Omega)}[x] \cap D = Hx \setminus \{x\}$. Hence D is an identifying set of $Cay(G,\Omega)$ and so $\gamma^{ID}\big(Cay(G,\Omega)\big) \leq |D| = n-2$. Now let C be an identifying set of $Cay(G,\Omega)$ with minimum cardinality. Since $Cay(G,\Omega)$ is a transitive graph, we can assume that $1 \notin C$. We have $N_{Cay(G,\Omega)}[x] \triangle N_{Cay(G,\Omega)}[h_i x] = \{1, h_i\}$ for $2 \leq i \leq t$, By Lemma 2.2, $h_i \in C$. Also we have $N_{Cay(G,\Omega)}[1] \triangle N_{Cay(G,\Omega)}[h_i] = \{x, h_i x\}$. By Lemma 2.2, $x \in C$ or $h_i x \in C$. Without loss of generality, we may assume that $x \notin C$. So $h_i x \in C$. Hence $H \setminus \{1\} \cup Hx \setminus \{x\} \subseteq C$ and so $|C| \geq n-2$.
Therefore, $\gamma^{ID}\big(Cay(G,\Omega)\big) = n-2$. □

Theorem 3.8. Let G be an Abelian group of order n and H be a proper subgroup of G such that $[G:H] = t$. Also let $x \in G \setminus H$, $o(x) = 2$, $G \setminus (H \cup \{x\}) = \Omega$ and $G = \langle \Omega \rangle$. Then

$$\gamma^{ID}\big(Cay(G,\Omega)\big) = \begin{cases} \frac{3t}{2} & |H| = 3,\ t \geq 2 \\ 4 & |H| = 4,\ t = 2 \\ 2t-1 & |H| = 4,\ t \geq 3 \\ \frac{t}{2}(|H|-1) & |H| \geq 5,\ t \geq 2. \end{cases}$$

Proof . Since H is a subgroup of G and $o(x) = 2$, $\Omega = \Omega^{-1}$ and $1 \notin \Omega$. So $Cay(G,\Omega)$ is connected graph. Let $g \in G \backslash H$. Then $Hg \subseteq \Omega \cup \{x\}$ and induced subgraph on Hg in $Cay(G,\Omega)$ is empty graph. By Theorem 3.1, $Cay(G,\Omega)$ is an identifiable graph. By Lemma 2.6, $t = 2k$, for some $k \in \mathbb{N}$. Let $H = \{1 = h_1, h_2, \ldots, h_\alpha\}$, $G = \cup_{j=1}^{k} Hxy_j \cup_{j=1}^{k} Hy_j$, and C be an identifying set of $Cay(G,\Omega)$ with minimum cardinality, where $y_1 = 1$. If $\{h_i y_j, h_\ell y_j, h_i y_j x, h_\ell y_j x\} \cap C = \emptyset$, for $1 \leq i, \ell \leq \alpha$ and $1 \leq j \leq k$, then $N_{Cay(G,\Omega)}[h_i y_j] \cap C = N_{Cay(G,\Omega)}[h_\ell y_j] \cap C$. This is a contradiction. So $\big|C \cap (Hy_j \cup Hy_j x)\big| \geq \alpha - 1$. Hence $|C| \geq (\alpha-1)k$ and so $\gamma^{ID}\big(Cay(G,\Omega)\big) \geq (\alpha-1)k$.
Case 1: Let $\alpha = 3$ and $\big|C \cap (Hy_j \cup Hy_j x)\big| = 2$ for some $1 \leq j \leq k$.
If $C \cap (Hy_j \cup Hy_j x) = \{h_i y_j, h_\ell y_j x\}$, then $N_{Cay(G,\Omega)}[h_i y_j] \cap C = N_{Cay(G,\Omega)}[h_\ell y_j x] \cap C$, which is false.
If $C \cap (Hy_j \cup Hy_j x) = \{h_i y_j, h_\ell y_j\}$ or $C \cap (Hy_j \cup Hy_j x) = \{h_i y_j x, h_\ell y_j x\}$, then we have $N_{Cay(G,\Omega)}[h_i y_j] \cap C = N_{Cay(G,\Omega)}[h_\ell y_j x] \cap C$, which is not true. Hence, for every $1 \leq j \leq k$, $\big|C \cap (Hy_j \cup Hy_j x)\big| = 3$ and so $|C| \geq 3k$. It is easy to see that $D = \cup_{j=1}^{k} Hy_j$ is an identifying set of $Cay(G,\Omega)$ and so $\gamma^{ID}\big(Cay(G,\Omega)\big) \leq |D| = 3k$. Therefore, $\gamma^{ID}\big(Cay(G,\Omega)\big) = 3k$.
Case 2: Let $\alpha = 4$. If $t = 2$, then by Theorem 2.1, $\gamma^{ID}(G) \geq \lceil log_2(n+1) \rceil$, we have $\gamma^{ID}\big(Cay(G,\Omega)\big) \geq 4$. It is easy to see that H is an identifying set of $Cay(G,\Omega)$ and so $\gamma^{ID}\big(Cay(G,\Omega)\big) \leq 4$. Therefore, $\gamma^{ID}\big(Cay(G,\Omega)\big) = 4$.
Let $\alpha = 4$ and $t \geq 3$. Then $\big|C \cap (Hy_j \cup Hy_j x)\big| \geq 3$ for $1 \leq j \leq k$.
Now let $\big|C \cap (Hy_j \cup Hy_j x)\big| = 3$ for $j \in \{i, \ell\}$. Then there are two elements g_1 and g_2 in G such that $N_{Cay(G,\Omega)}[g_1] \cap C = N_{Cay(G,\Omega)}[g_2] \cap C$. It is impossible. So $|C| \geq 4(k-1) + 3 = 4k-1$.
Now let $D = (\cup_{j=1}^{k} Hy_j) \backslash \{h_4 y_k\}$. Then $N_{Cay(G,\Omega)}[h_i y_j] \cap D = \{h_i y_j\} \cup (D \backslash Hy_j)$ and $N_{Cay(G,\Omega)}[h_i y_j x] \cap D = D \backslash \{h_i y_j\}$ for $1 \leq i \leq \alpha$ and $1 \leq j \leq k-1$. Also we have

$$N_{Cay(G,\Omega)}[h_i y_k] \cap D = \{h_i y_k\} \cup (D \setminus Hy_k)$$

$$N_{Cay(G,\Omega)}[h_i y_k x] \cap D = D \setminus \{h_i y_k\}$$

$$N_{Cay(G,\Omega)}[h_4 y_k] \cap D = D \setminus Hy_k$$

$$N_{Cay(G,\Omega)}[h_4 y_k x] \cap D = D,$$

where $1 \leq i \leq 3$. Hence D is an identifying set of $Cay(G,\Omega)$ and so $\gamma^{ID}\big(Cay(G,\Omega)\big) \leq |D| = 4k-1$. Therefore, $\gamma^{ID}\big(Cay(G,\Omega)\big) = 4k-1$.
Case 3: Let $\alpha \geq 5$. Then $\gamma^{ID}\big(Cay(G,\Omega)\big) \geq k(\alpha-1)$.
Now let $D = \{h_i y_j \mid 1 \leq i \leq \alpha-3,\ 1 \leq j \leq k\} \cup \{h_{\alpha-2} y_j, h_{\alpha-1} y_j \mid 1 \leq j \leq k\}$. Then $N_{Cay(G,\Omega)}[h_i y_j] \cap D = \{h_i y_j\} \cup (D \setminus Hy_j)$ and $N_{Cay(G,\Omega)}[h_i y_j x] \cap D = D \setminus \{h_i y_j, h_{\alpha-2} y_j, h_{\alpha-1} y_j\}$ for $1 \leq i \leq \alpha-3$, $1 \leq j \leq k$. Also for $1 \leq j \leq k$, we have

$$N_{Cay(G,\Omega)}[h_{\alpha-2} y_j] \cap D = D \setminus \big(\{h_{\alpha-2} y_j x\} \cup Hy_j\big)$$

$$N_{Cay(G,\Omega)}[h_\alpha y_j x] \cap D = D \setminus \{h_{\alpha-2} y_j x, h_{\alpha-1} y_j x\}$$

$$N_{Cay(G,\Omega)}[h_{\alpha-1} y_j] \cap D = D \setminus \big(\{h_{\alpha-1} y_j x\} \cup Hy_j\big)$$

$$N_{Cay(G,\Omega)}[h_{\alpha-1} y_j x] \cap D = D \setminus \{h_{\alpha-2} y_j x\}$$

$$N_{Cay(G,\Omega)}[h_{\alpha-2}y_jx] \cap D = D \setminus \{h_{\alpha-1}y_jx\}$$
$$N_{Cay(G,\Omega)}[h_\alpha y_j] \cap D = D \setminus Hy_j.$$

Hence, D is an identifying set of $Cay(G,\Omega)$ and so $\gamma^{ID}\big(Cay(G,\Omega)\big) \le |D| = k(\alpha-1)$. Therefore, $\gamma^{ID}\big(Cay(G,\Omega)\big) = k(\alpha-1) = \frac{t}{2}(|H|-1)$.
□

Corollary 3.9. Let $G = \langle a \rangle$ be a cyclic group of order $2n$, where n is odd.
If $\Omega = \{a^{2i+1} \mid 0 \le i \le n-1\} \setminus \{a^n\}$, then

$$\gamma^{ID}\big(Cay(G,\Omega)\big) = \begin{cases} 3 & n = 3 \\ n-1 & n \ne 3. \end{cases}$$

Proof . It is easy to see that if $H = \langle a^2 \rangle$, then $G \setminus \Omega = H \cup \{a^n\}$. By Theorem 3.8,

$$\gamma^{ID}\big(Cay(G,\Omega)\big) = \begin{cases} 3 & n = 3 \\ n-1 & n \ne 3. \end{cases}$$

□

Theorem 3.10. Let $G_1 = \langle \Omega_1 \rangle, G_2 = \langle \Omega_2 \rangle$, $Cay\big(G_1 \times G_2, \Omega_1 \times \Omega_2\big) = \Gamma$, $Cay(G_1,\Omega_1) = \Gamma_1$ and $Cay(G_2,\Omega_2) = \Gamma_2$, where $1 \notin \Omega_i = \Omega_i^{-1}$ and $1 \le i \le 2$. If Γ_1 and Γ_2 are identifiable graphs, then Γ is an identifiable graph and $\gamma^{ID}(\Gamma) \le \gamma^{ID}(\Gamma_1) \cdot \gamma^{ID}(\Gamma_2)$.

Proof . Let $\Omega_1 = \{s_{1i} \mid 1 \le i \le \alpha\}, \Omega_2 = \{s_{2j} \mid 1 \le j \le \beta\}$ and $\Omega_1 \times \Omega_2 \subseteq N_\Gamma\big[(s_{1\ell}, s_{2k})\big]$ for some $(s_{1\ell}, s_{2k}) \in \Omega_1 \times \Omega_2$. Then for every $1 \le i \le \alpha$ and $1 \le j \le \beta$ with $(i \ne \ell, j \ne k)$, $(s_{1i}, s_{2j})(s_{1\ell}, s_{2k}) \in \Omega_1 \times \Omega_2$.
So there are $s_{1h} \in \Omega_1$ and $s_{2f} \in \Omega_2$ such that $(s_{1i}, s_{2j})(s_{1\ell}, s_{2k}) = (s_{1h}, s_{2f})$. Hence $s_{1i}s_{1\ell} = s_{1h}$ and $s_{2j}s_{2k} = s_{2f}$. Thus $N_{\Gamma_1}[s_{1\ell}] = \Omega_1 \cup \{1\}$. By Theorem 3.1, $Cay(G_1,\Omega_1)$ is not identifiable graph, which is a contradiction. Hence Γ is an identifiable graph. Let C_i be an identifying set of Γ_i with minimum cardinality, for $i \in \{1,2\}$ and $C = C_1 \times C_2$. For every $y_1 \in G_1$ and $y_2 \in G_2$, we have $N_{\Gamma_1}[y_1] \cap C_1 \ne \emptyset$ and $N_{\Gamma_2}[y_2] \cap C_2 \ne \emptyset$. So $N_\Gamma\big[(y_1,y_2)\big] \cap C \ne \emptyset$. This shows that C is a dominating set of Γ. Let (y_1,y_2) and (y_1', y_2') be two distinct vertices in $G_1 \times G_2$. Since $N_{\Gamma_1}[y_1] \cap C_1 \ne N_{\Gamma_1}[y_1'] \cap C_1$ and $N_{\Gamma_2}[y_2] \cap C_2 \ne N_{\Gamma_2}[y_2'] \cap C_2$, there are two elements $x \in C_1$ and $y \in C_2$ such that $y_1 - x \not- y_1'$ and $y_2 - y \not- y_2'$. Hence $(x,y) \in C \cap N_\Gamma\big[(y_1,y_2)\big]$ and $(x,y) \notin C \cap N_\Gamma\big[(y_1',y_2')\big]$. So $N_\Gamma\big[(y_1,y_2)\big] \cap C \ne N_\Gamma\big[(y_1',y_2')\big] \cap C$. Hence C is an identifying set of Γ. Therefore, $\gamma^{ID}(\Gamma) \le |C| = |C_1| \cdot |C_2| = \gamma^{ID}(\Gamma_1) \cdot \gamma^{ID}(\Gamma_2)$. □

For an integer $k \ge 1$, let $A_k = (V_k, E_k)$ be the graph with vertex set $V_k = \{x_1, \ldots, x_{2k}\}$ and edge set $E_k = \{x_i - x_j \mid |i-j| \le k-1\}$.

Also, let $\mathscr{A}$ be the closure of $\{A_i \mid i = 1,2,\ldots\}$ with respect to operation $\bowtie$. In the next theorem, Foucaud et al. showed that for any twin free graph $\Gamma \notin \{K_{1,n-1}\} \cup (\mathscr{A}, \bowtie) \cup (\mathscr{A}, \bowtie) \bowtie K_1$, $\gamma^{ID}(\Gamma) \le |V(\Gamma)| - 2$.

Theorem 3.11. [6] Let Γ be an identifiable graph of order n. Then $\gamma^{ID}(\Gamma) = n-1$ if and only if $\Gamma \ncong \overline{K_2}$ and $\Gamma \in \{K_{1,n-1}\} \cup (\mathscr{A}, \bowtie) \cup (\mathscr{A}, \bowtie) \bowtie K_1$.

Note: Foucaud et al [6], classify all graphs of order n with identifying code number $n-1$. In Theorem 3.7 and Corollary 3.5 we obtain graphs of order n with identifying code number of $n-2$.

Question: Which graphs of order n have identifying code number of $n-2$?

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