Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 371–378
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.25391.3001

Integral inequality for the polar derivatives of polynomials

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(Communicated by Michael Th. Rassias)

Abstract

Let $P(z)$ be a polynomial of degree n and for any complex number α, let

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$$

denote the polar derivative of $P(z)$ with respect to a complex number α. In this paper, we prove some L_r inequalities for the polar derivative of a polynomial have all zeros in $|z| \le 1$. Our theorem generalizes a result of Dewan and Mir [K. K. Dewan, A. Mir, *Inequalities for the polar derivative of a polynomial*, J. Interd. Math. 10 (2007), no. 4, 525–531] and includes as special cases several interesting many known results.

Keywords: Polar derivative, Inequality of polynomials, Integral inequality, Zeros.
2020 MSC: 30A10; 30C10

1 Introduction

Let $\mathbb{P}_n$ be the class of complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most n and $P'(z)$ be its derivative. For $P(z) \in \mathbb{P}_n$, then

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The inequality (1.1) was proved by Bernstein in 1912 [6], and it is best possible with equality holding for polynomials $P(z) = cz^n$, where c is an arbitrary complex.

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequality (1.1) can be sharpened. In fact, Erdös conjectured and Lax [8] proved that, if $P(z) \ne 0$ in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |P'(z)| \le \frac{n}{2} \max_{|z|=1} |P(z)|, \tag{1.2}$$

whereas if $P(z)$ has no zeros in $|z| > 1$, then

$$\max_{|z|=1} |P'(z)| \ge \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Inequality (1.3) is due to Turán[14].

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Received: November 2021 *Accepted:* May 2022

As an extension of (1.2) and (1.3), Malik [9] proved that, if $P(z) \neq 0$ in $|z| < k, k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|, \tag{1.4}$$

whereas if $P(z)$ has all its zeros in $|z| \leq k, k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.5}$$

For a complex number α and for $P(z) \in \mathbb{P}_n$, let

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

Note that $D_\alpha P(z)$ is a polynomial of degree at most $n-1$. This is the so-called polar derivative of $P(z)$ with respect to point α. It generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha \to \infty} \left\{ \frac{D_\alpha P(z)}{\alpha} \right\} = P'(z).$$

Now corresponding to a given n^{th} degree polynomial $P(z)$, we construct a sequence of polar derivatives

$$\begin{aligned} D_{\alpha_1} P(z) &= nP(z) + (\alpha_1 - z)\, P'(z) \\ &\vdots \\ D_{\alpha_1} D_{\alpha_2} \ldots D_{\alpha_t} P(z) &= (n-t+1) D_{\alpha_1} D_{\alpha_2} \ldots D_{\alpha_{t-1}} P(z) \\ &\quad + (\alpha_t - z) \left(D_{\alpha_1} D_{\alpha_2} \ldots D_{\alpha_{t-1}} P(z) \right)', t = 2, 3, \ldots, n. \end{aligned}$$

The points $\alpha_1, \alpha_2, ..., \alpha_t, t = 1, 2, ..., n$ may or may not be distinct. Like the t^{th} ordinary derivative $P^{(t)}(z)$ of $P(z)$, the t^{th} polar derivative $D_{\alpha_1} ... D_{\alpha_2} D_{\alpha_t} P(z)$ of $P(z)$ is a polynomial of degree $n-t$.
Where here and throughout we write

$$P_t(z) = D_{\alpha_1} D_{\alpha_2} ... D_{\alpha_t} P(z), \tag{1.6}$$

so that

$$\begin{aligned} P_t(z) = (n-t+1) P_{t-1}(z) + (\alpha_t - z) P'_{t-1}(z), t = 1, 2, ..., n, \\ P_0(z) = P(z). \end{aligned}$$

Aziz [1] was among the first to extend some of the above inequalities by replacing the derivative with the polar derivatives of polynomials. In fact, he extended (1.2) to the polar derivative of a polynomial and proved that if $P(z) \in \mathbb{P}_n$, $P(z) \neq 0$ in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{2}(|\alpha| + 1) \max_{|z|=1} |P(z)|. \tag{1.7}$$

Fucther, Rather, Ahangar and Suhail [13] proved that if $P(z)$ having all its zeros in $|z| < k$ where $k \leq 1$, then for $\delta \in \mathbb{C}$ with $|\delta| \leq 1$,

$$\max_{|z|=1} |D_\delta P(z)| \leq n \left(\frac{|\delta| + k}{1+k} \right) \max_{|z|=1} |P(z)|. \tag{1.8}$$

As a generalization of (1.7), Aziz and Shah [5] proved that if $P(z)$ have no zeros in the disk $|z| < 1$, for all real or complex number α_i with $|\alpha_i| \geq 1, i = 1, 2, \cdots, ..., t$, then for $|z| \geq 1$, we have

$$|P_t(z)| \leq \frac{n_t}{2} \left\{ (|\alpha_1 \alpha_2 \cdots \alpha_t||z|^{n-t} + 1) \max_{|z|=1} |P(z)| - (|\alpha_1 \alpha_2 \cdots \alpha_t||z|^{n-t} - 1) \min_{|z|=1} |P(z)| \right\} \tag{1.9}$$

where $P_t(z)$ is defined as (1.6), and

$$n_t = n(n-1) \cdots (n-t+1). \tag{1.10}$$

As an L_r analoge of (1.9), Mir and Baba [11] proved that if $P(z) \in \mathbb{P}_n$, $P(z)$ having no zeros in $|z| < 1$, then for every real or complex number $\alpha_i, i = 1, 2, \cdots, t, t \leq n-1$, with $|\alpha_i| \geq 1$, and real or complex β with $|\beta| \leq 1$, and for $r > 0$,

$$\left\{\int_0^{2\pi} \left|P_t(e^{i\theta}) + \beta \frac{mn_t(|\alpha_1 \cdots \alpha_t| - 1)}{2}\right|^r d\theta\right\}^{\frac{1}{r}} \leq n_t B_{\alpha_t} C_r \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{1.11}$$

where $P_t(z)$ is defined as (1.6), n_t is defined by (1.10), $m = \min_{|z|=1} |P(z)|$, and $B_{\alpha_t} = (|\alpha_1| + 1)(|\alpha_2| + 1) \cdots (|\alpha_t| + 1)$,

$$C_r = \left\{\int_0^{2\pi} \left|1 + e^{i\gamma}\right|^r d\gamma\right\}^{-\frac{1}{r}}. \tag{1.12}$$

If we put $r \to \infty$ in (1.11) and choose the argument of β with $|\beta| = 1$ subitably, we get (1.9).

In 2007, Dewan and Mir [7] proved the following result.

Theorem 1.1. If $P(z) \in \mathbb{P}_n$, $P(z)$ having all its zeros in $|z| \leq 1$, then for every real or complex number $\alpha_i, i = 1, 2, \cdots, t, t \leq n-1$, with $|\alpha_i| < 1$, then for $|z| \leq 1$

$$|P_t(z)| \leq \frac{n_t}{2}\left\{(|\alpha_1\alpha_2 \cdots \alpha_t||z|^{n-t} + 1) \max_{|z|=1} |P(z)| - (1 - |\alpha_1\alpha_2 \cdots \alpha_t||z|^{n-t}) \min_{|z|=1} |P(z)|\right\}, \tag{1.13}$$

where P_t, n_t are defined as (1.6), (1.10) respectively. The result is best possible and equality holds for the polynomial $P(z) = \frac{z^n+1}{2}$.

2 Main results

In this paper, we shall prove the following more general result which is an L_r norm generalization of (1.13).

Theorem 2.1. If $P(z) \in \mathbb{P}_n$, $P(z)$ having all its zeros in $|z| \leq 1$, then for every real or complex number $\alpha_i, i = 1, 2, \cdots, t, t \leq n-1$, with $|\alpha_i| < 1$, and real or complex β with $|\beta| \leq 1$, and for $r > 0$,

$$\left\{\int_0^{2\pi} \left|P_t(e^{i\theta}) + \beta \frac{mn_t(1 - |\alpha_1 \cdots \alpha_t|)}{2}\right|^r d\theta\right\}^{\frac{1}{r}} \leq n_t B_{\alpha_t} C_r \left\{\int_0^{2\pi} \left|P(e^{i\theta}) d\theta\right|^r\right\}^{\frac{1}{r}}, \tag{2.1}$$

where here and thoughout

$$\begin{aligned} &P_t(z) = D_{\alpha_1} D_{\alpha_2} ... D_{\alpha_t} P(z), \\ &B_{\alpha_t} = (|\alpha_1| + 1)(|\alpha_2| + 1) \cdots (|\alpha_t| + 1), \\ &n_t = n(n-1) \ldots (n-t+1). \end{aligned} \tag{2.2}$$

$m = \min_{|z|=1} |P(z)|$ and C_r is defined in (1.12).
In the limiting case, when $r \to \infty$, the above inequality is sharp and the equality in (2.1) holds for $P(z) = \frac{z^n+1}{2}$, where $\alpha_i < 1, i = 1, 2, \cdots, t$ are real.

Many interesting results easily follow from Theorem 2.1. Here, we mention a few of these.

Remark 2.1. If we let $r \to \infty$ in (2.1), and choose the argument of $|\beta|$ with $|\beta| = 1$ suitably, we get (1.13).

If we put $\beta = 0$ in (2.1), we have the following result.

Corollary 2.2. If $P(z) \in \mathbb{P}_n$, $P(z)$ having all zeros in $|z| \leq 1$, then for every complex number α_i with $|\alpha_i| < 1, i = 1, 2, \cdots, t$, and for $r > 0$,

$$\left\{\int_0^{2\pi} \left|P_t(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \leq n_t B_{\alpha_t} C_r \left\{\int_0^{2\pi} \left|P(e^{i\theta}) d\theta\right|^r\right\}^{\frac{1}{r}}. \tag{2.3}$$

where P_t, n_t, B_t, C_r are defined as Theorem 2.1.

If we put $t=1$ in (2.1), we get the following result.

Corollary 2.3. If $P(z)\in\mathbb{P}_n$, $P(z)$ having all its zeros in $|z|\le 1$, then for every complex number α with $|\alpha|<1$, and for $r>0$,

$$\left\{\int_0^{2\pi}\left|D_\alpha P\left(e^{i\theta}\right)+\beta\frac{mn(1-|\alpha|)}{2}\right|^r d\theta\right\}^{\frac{1}{r}}\le n(|\alpha|+1)C_r\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}, \tag{2.4}$$

where m, C_r are defined as Theorem 2.1.

If we let $r\to\infty$ in (2.4) and choose the argument of β with $|\beta|=1$ suitably, we get the following result, which is a special case of Theorem 1.1.

Corollary 2.4. If $P(z)\in\mathbb{P}_n$, $P(z)$ having all its zeros in $|z|\le 1$, then for every real or complex number α, with $|\alpha|<1$, then for $|z|\le 1$

$$|D_\alpha P(z)|\le\frac{n}{2}\left\{(|\alpha||z|^{n-1}+1)\max_{|z|=1}|P(z)|-(1-|\alpha||z|^{n-1})\min_{|z|=1}|P(z)|\right\}. \tag{2.5}$$

The result is best possible and equality holds for the polynomial $P(z)=\frac{z^n+1}{2}$.

3 Lemmas

For the proof of our Theorem 2.1, we need the following lemmas. The following lemma is due to Aziz [2].

Lemma 3.1. If all the zeros of an n^{th} degree polynomial $P(z)$ lie in a circular region $\mathcal{C}$ and if none of the points $\alpha_1,\alpha_2,\cdots,\alpha_t$ lie in the region $\mathcal{C}$, then each of the polar derivatives

$$D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}P(z),\quad t=1,2,\cdots,n-1,$$

has all of its zeros in $\mathcal{C}$.

Lemma 3.2. Let $P(z)$ be a polynomial of degree n have all its zeros in $|z|\le 1$, then for all real or complex numbers α_i with $|\alpha_i|<1, i=1,2,\ldots,t, t\le n-1$, and for $|z|\le 1$, we have

$$|P_t(z)|\le|Q_t(z)|-mn_t(1-|\alpha_1\alpha_2...\alpha_t||z|^{n-t}),$$

where $Q(z)=z^n\overline{P(\frac{1}{\bar z})}$ and m, n_t, P_t are defined as Theorem 2.1.

Proof of Lemma 2: Let $m=\min\limits_{|z|=1}|P(z)|$, we have $|\lambda m|<|P(z)|$ on $|z|=1$ for any λ with $|\lambda|<1$. By Rouche's theorem the polynomial $F(z)=P(z)+\lambda m$ has all its zeros in $|z|\le 1$.

Therefore the polynomial $G(z)=z^n\overline{F(\frac{1}{\bar z})}=Q(z)+\overline{\lambda}mz^n$ will have all its zeros in $|z|\ge 1$. Also $|F(z)|=|G(z)|$ for $|z|=1$. Therefore, for any complex number δ with $|\delta|>1$, the polynomial $F(z)-\delta G(z)$ has all its zeros in $|z|>1$. Hence by the repeated application of Laguerre's Theorem [10], if $\alpha_1,\alpha_2,\cdots,\alpha_t$ are complex numbers with $|\alpha_i|<1, i=1,2,\cdots,t$, the polynomial $D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}(F(z)-\delta G(z))$ has all its zeros in $|z|\ge 1$. Equivalently all the zeros of $D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}F(z)-\delta D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}G(z)$ lie in $|z|\ge 1$. This imples that for $|z|\le 1$,

$$D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}F(z)\le D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}G(z).$$

On substituting $F(z)$ and $G(z)$ in the above inequality, we obtain the following for every real or complex number α_i with $|\alpha_i|<1, i=1,2,...,t$ and for any real or complex number δ with $|\delta|>1$ and $|z|=1$,

$$|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}(P(z)+\delta m)|\le|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}(Q(z)+\overline{\delta}mz^n)|.$$

Equivalently

$$\begin{aligned}&|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}P(z)+m\delta n(n-1)\cdots(n-t+1)|\\ \le&|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}Q(z)+m\overline{\delta}n(n-1)\cdots(n-t+1)\alpha_1\alpha_2\cdots\alpha_t z^{n-t}|.\end{aligned}$$

This gives

$$\begin{aligned}&|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}P(z)+m\delta n(n-1)\cdots(n-t+1)|\\ \leq&|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}Q(z)|+m|\overline{\delta}|n(n-1)\cdots(n-t+1)|\alpha_1\alpha_2\cdots\alpha_t||z|^{n-t}.\end{aligned}$$

Chooseing argument of δ suitably on left hand side above and letting $|\delta|\to 1$, we get, for $|z|\leq 1$

$$\begin{aligned}|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}P(z)|\leq&|D_{\alpha_1}D_{\alpha_2}\cdots D_{\alpha_t}Q(z)|\\ &-mn(n-1)\cdots(n-t+1)(1-|\alpha_1\alpha_2\cdots\alpha_t||z|^{n-t}).\end{aligned}$$

This completes the proof of the Lemma.

Lemma 3.3. If $P(z)\in\mathbb{P}_n$, then for every complex α and $r>0$,

$$\left\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}\leq n(|\alpha|+1)\left\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}.$$

The above lemma is due to Rather [12].

Lemma 3.4. If $P(z)\in\mathbb{P}_n$ and $Q(z)=z^n\overline{P(\frac{1}{\overline{z}})}$, then for every $r>0$ and γ real,

$$\int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta})+e^{i\gamma}Q'(e^{i\theta})\right|^r d\theta d\gamma\leq 2\pi n^r\int_0^{2\pi}|P(e^{i\theta})|^r d\theta.$$

The above lemma is due to Aziz and Rather [4]. The following lemma is due to Aziz and Rather [3].

Lemma 3.5. If A,B and C are non-negative real numbers such that $B+C\leq A$, then for every real number γ,

$$|(A-C)e^{i\gamma}+(B+C)|\leq|Ae^{i\gamma}+B|.$$

4 Proof of the Theorem

Proof of the Theorem 2.1: Since $P(z)\in\mathbb{P}_n$ and $P(z)\neq 0$ in $|z|<1$, then the polynomial $Q(z)=z^n\overline{P(\frac{1}{\overline{z}})}\in\mathbb{P}_n$, and it can be easily verified that, for $0\leq\theta<2\pi$,

$$nP\left(e^{i\theta}\right)-e^{i\theta}P'\left(e^{i\theta}\right)=e^{i(n-1)\theta}\overline{Q'\left(e^{i\theta}\right)},$$

and

$$nQ\left(e^{i\theta}\right)-e^{i\theta}Q'\left(e^{i\theta}\right)=e^{i(n-1)\theta}\overline{P'\left(e^{i\theta}\right)}.$$

Hence

$$\begin{aligned}&nP\left(e^{i\theta}\right)+e^{i\gamma}nQ\left(e^{i\theta}\right)\\ =&e^{i\theta}P'\left(e^{i\theta}\right)+e^{i(n-1)\theta}\overline{Q'\left(e^{i\theta}\right)}+e^{i\gamma}\left(e^{i\theta}Q'\left(e^{i\theta}\right)+e^{i(n-1)\theta}\overline{P'\left(e^{i\theta}\right)}\right)\\ =&e^{i\theta}\left(P'\left(e^{i\theta}\right)+e^{i\gamma}Q'\left(e^{i\theta}\right)\right)+e^{i(n-1)\theta}\left(\overline{Q'\left(e^{i\theta}\right)}+e^{i\gamma}\overline{P'\left(e^{i\theta}\right)}\right),\end{aligned}$$

which gives

$$\begin{aligned}&n\left|P\left(e^{i\theta}\right)+e^{i\gamma}Q\left(e^{i\theta}\right)\right|\\ \leq&\left|P'\left(e^{i\theta}\right)+e^{i\gamma}Q'\left(e^{i\theta}\right)\right|+\left|\overline{Q'\left(e^{i\theta}\right)}+e^{i\gamma}\overline{P'\left(e^{i\theta}\right)}\right|\\ =&2\left|P'\left(e^{i\theta}\right)+e^{i\gamma}Q'\left(e^{i\theta}\right)\right|.\end{aligned}\tag{4.1}$$

Also, we have

$$
\begin{aligned}
& \left|D_\alpha P\left(e^{i\theta}\right)+e^{i\gamma} D_\alpha Q\left(e^{i\theta}\right)\right| \\
= & \left| nP\left(e^{i\theta}\right)+\left(\alpha-e^{i\theta}\right) P'\left(e^{i\theta}\right)+e^{i\gamma}\left(nQ\left(e^{i\theta}\right)+\left(\alpha-e^{i\theta}\right) Q'\left(e^{i\theta}\right)\right) \right| \\
= & \left| \left(nP\left(e^{i\theta}\right)-e^{i\theta} P'\left(e^{i\theta}\right)\right)+e^{i\gamma}\left(nQ\left(e^{i\theta}\right)-e^{i\theta} Q'\left(e^{i\theta}\right)\right)+\alpha\left(P'\left(e^{i\theta}\right)+e^{i\gamma} Q'\left(e^{i\theta}\right)\right) \right| \\
= & \left| \left(\overline{Q'(e^{i\theta})}+e^{i\gamma}\overline{P'(e^{i\theta})}\right) e^{i(n-1)\theta}+\alpha\left(P'(e^{i\theta})+e^{i\gamma} Q'\left(e^{i\theta}\right)\right) \right| \\
\leq & \left|\overline{P'\left(e^{i\theta}\right)+e^{i\gamma} Q'\left(e^{i\theta}\right)}\right|+|\alpha|\left|P'\left(e^{i\theta}\right)+e^{i\gamma} Q'\left(e^{i\theta}\right)\right| \\
= & (|\alpha|+1)\left|P'\left(e^{i\theta}\right)+e^{i\gamma} Q'\left(e^{i\theta}\right)\right| .
\end{aligned}
\tag{4.2}
$$

The above inequality (4.2), with the help of Lemma 3.4, gives, for each $r>0$,

$$
\begin{aligned}
& \int_0^{2\pi} \int_0^{2\pi}\left|D_\alpha P\left(e^{i\theta}\right)+e^{i\gamma} D_\alpha Q\left(e^{i\theta}\right)\right|^r d\theta d\gamma \\
\leq & (|\alpha|+1)^r \int_0^{2\pi} \int_0^{2\pi}\left|P'\left(e^{i\theta}\right)+e^{i\gamma} Q'\left(e^{i\theta}\right)\right|^r d\theta d\gamma \\
\leq & 2\pi n^r(|\alpha|+1)^r \int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta .
\end{aligned}
\tag{4.3}
$$

Further, let $T(z)=P(z)+e^{i\gamma}Q(z)$ is a polynomial of degree n so that $T_t(z)=P_t(z)+e^{i\gamma}Q_t(z)$ is a polynomial of degree $n-t, t\leq n-1$, we have by the respected application of Lemma 3.3, for $r>0$,

$$
\begin{aligned}
& \int_0^{2\pi}\left|D_{\alpha_1} D_{\alpha_2} \ldots D_{\alpha_t} P\left(e^{i\theta}\right)+e^{i\gamma} D_{\alpha_1} D_{\alpha_2} \ldots D_{\alpha_t} Q\left(e^{i\theta}\right)\right|^r d\theta \\
& \leq(n-t+1)^r\left(\left|\alpha_t\right|+1\right)^r \int_0^{2\pi}\left|D_{\alpha_1} D_{\alpha_2} \ldots D_{\alpha_{t-1}} P\left(e^{i\theta}\right)+e^{i\gamma} D_{\alpha_1} D_{\alpha_2} \ldots D_{\alpha_{t-1}} Q\left(e^{i\theta}\right)\right|^r d\theta \\
& \vdots \\
& \leq(n-t+1)^r \ldots(n-1)^r\left(\left|\alpha_t\right|+1\right)^r \ldots\left(\left|\alpha_2\right|+1\right)^r \times \int_0^{2\pi}\left|P'\left(e^{i\theta}\right)+e^{i\gamma} Q'\left(e^{i\theta}\right)\right|^r d\theta .
\end{aligned}
\tag{4.4}
$$

Integraing both sides of (4.4) with respect to γ from 0 to 2π, we get with the help of (4.3) that for each $r>0$,

$$
\int_0^{2\pi} \int_0^{2\pi}\left|P_t\left(e^{i\theta}\right)+e^{i\gamma} Q_t\left(e^{i\theta}\right)\right|^r d\theta d\gamma \leq 2\pi n_t^r B_{\alpha_t}^r \int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta,
\tag{4.5}
$$

where n_t, B_{α_t} are defined as (2.2).

Now by Lemma 3.2, for each $\theta, 0\leq\theta<2\pi$ and $\alpha_i, 1\leq i\leq t, t\leq n-1$ with $|\alpha_i|<1$, we have

$$
\left|P_t(e^{i\theta})\right| \leq |Q_t(e^{i\theta})| - mn_t(1-|\alpha_1\alpha_2 ... \alpha_t|),
$$

This imples

$$
|P_t(e^{i\theta})|+\frac{mn_t}{2}(1-|\alpha_1\alpha_2...\alpha_t|) \leq |Q_t(e^{i\theta})|-\frac{mn_t}{2}(1-|\alpha_1\alpha_2...\alpha_t|),
\tag{4.6}
$$

Take

$$
A=|P_t(e^{i\theta})|, \quad B=|Q_t(e^{i\theta})|, \quad C=\frac{mn_t}{2}(1-|\alpha_1\alpha_2...\alpha_t|).
$$

in Lemma 3.5, we get

$$
B+C\leq A-C\leq A.
$$

Hence for every real γ, with the help of Lemme 3.5, that

$$
\left|\left\{|Q_t(e^{i\theta})|-\frac{mn_t}{2}(1-|\alpha_1\alpha_2...\alpha_t|)\right\} e^{i\gamma}+\left\{|P_t(e^{i\theta})|+\frac{mn_t}{2}(1-|\alpha_1\alpha_2...\alpha_t|)\right\}\right| \leq \left||Q_t(e^{i\theta})|e^{i\gamma}+|P_t(e^{i\theta})|\right| .
\tag{4.7}
$$

This implies for each $r>0$,

$$
\int_0^{2\pi}|F(\theta)+e^{i\gamma}G(\theta)|^r d\theta \leq \int_o^{2\pi}\left||Q_t(e^{i\theta})|e^{i\gamma}+|P_t(e^{i\theta})|\right|^r d\theta,
\tag{4.8}
$$

where

$$
F(\theta)=|P_t(e^{i\theta})|+\frac{mn_t}{2}(1-|\alpha_1\alpha_2...\alpha_t|); \quad G(\theta)=|Q_t(e^{i\theta})|-\frac{mn_t}{2}(1-|\alpha_1\alpha_2...\alpha_t|).
$$

Integrating both sides of (4.8) with respect to γ from 0 to 2π, we get with the help of (4.5), that for each $r>0$,

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}|F(\theta)+e^{i\gamma}G(\theta)|^r d\theta d\gamma\\ \leq&\int_0^{2\pi}\int_0^{2\pi}\left||Q_t(e^{i\theta})|e^{i\gamma}+|P_t(e^{i\theta})|\right|^r d\theta d\gamma\\ \leq&2\pi n_t^r B_{\alpha_t}^r\int_0^{2\pi}|P(e^{i\theta})|^r d\theta.\end{aligned}\tag{4.9}$$

Now for any real γ and $s\geq 1$, it can be easily verified that $|s+e^{i\gamma}|\geq|1+e^{i\gamma}|$. Which implies for each $r>0$,

$$\int_0^{2\pi}|s+e^{i\gamma}|^r d\gamma\geq\int_0^{2\pi}|1+e^{i\gamma}|^r d\gamma.$$

If $F(\theta)\neq 0$, we take $s=\left|\frac{F(\theta)}{G(\theta)}\right|$, then by (4.6), $s\geq 1$ and we get

$$\int_0^{2\pi}\left|F(\theta)+e^{i\gamma}G(\theta)\right|^r d\gamma=|F(\theta)|^r\int_0^{2\pi}\left|s+e^{i\gamma}\right|^r d\gamma\geq|F(\theta)|^r\int_0^{2\pi}\left|1+e^{i\gamma}\right|^r d\gamma.\tag{4.10}$$

This inequality is true if $F(\theta)=0$.
Integrating both sides of (4.10) with respect to θ from 0 to 2π, we get, for every $r>0$,

$$\int_0^{2\pi}\int_0^{2\pi}|F(\theta)+e^{i\gamma}G(\theta)|^r d\theta d\gamma\geq\int_0^{2\pi}\left||P_t(e^{i\theta})|+\frac{mn_t}{2}(1-|\alpha_1\alpha_2\cdots\alpha_t|)\right|\int_0^{2\pi}\left|1+e^{i\gamma}\right|^r d\gamma.\tag{4.11}$$

Now using the fact that for every complex number β with $|\beta|\leq 1$,

$$\left|P_t(e^{i\theta})+\beta\frac{mn_t}{2}(1-|\alpha_1\alpha_2\cdots\alpha_t|)\right|\leq\left||P_t(e^{i\theta})|+\frac{mn_t}{2}(1-|\alpha_1\alpha_2\cdots\alpha_t|)\right|,\tag{4.12}$$

which gives on using (4.9), (4.11), and (4.12), we get (2.1). This completes the proof of Theorem 2.1.

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