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The boundedness of bilinear Fourier integral operators on $L^2 \times L^2$

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Abstract

In this paper, the regularity of bilinear Fourier integral operators on $L^2 \times L^2$ are determined in the framework of Besov spaces. Our result improves the $L^2 \times L^2 \to L^1$ boundedness of those operators with symbols in the bilinear Hörmander classes.

Keywords: Bilinear Fourier integral operators, bilinear Hörmander symbol classes, Phase function and Besov spaces
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1 Introduction

The results of this paper extend previous findings on bilinear pseudodifferential operators, which were introduced and thoroughly researched by Coifman and Meyer [11, 12, 13]. They have the following form

$$\mathrm{Op}_a(u_1, u_2)(x) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} e^{ix(\xi_1+\xi_2)} a(x, \xi_1, \xi_2) \hat{u}_1(\xi_1) \hat{u}_2(\xi_2) d\xi_1 d\xi_2. \tag{1.1}$$

The Calderón-Zygmund hypothesis has had an impact on the research of multi-linear operators. Indeed, many of Coifman-Meyer's pioneering discoveries [11, 12, 13] are the construction of pseudodifferential operators in terms of Calderón-Zygmund type singular integrals. Their multi-linear technique has had a huge impact on operator theory and partial differential equations. For example in [13], the boundedness of a class of translation invariant bilinear operators on Lebesgue spaces has been proved.

Moreover, Grafakos and Torres [15] treated a bilinear Calderón-Zygmund theory that allowed those conclusions to be extended to non-translation invariant bilinear pseudodifferential operators whose symbols depend on the space variable.

A brief examination and discussion of applications to partial differential equations can be found in [5], and a thorough analysis of bilinear pseudo-differential operators with symbols in bilinear Hörmander classes can be found in [9].

For a more recent contribution on Triebel-Lizorkin and local Hardy spaces, see [19].

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Another set of results concerns bilinear (and multi-linear) operators with non-smooth symbols. [6] was the first to notice their continuous features on modulation spaces.

These operators, unlike conventional bilinear pseudodifferential operators studied in, for example, [13], are handled using time–frequency analysis techniques (see also [2, 3, 7, 8, 9, 14, 20, 21]).

It's merely a matter of adding an appropriate oscillation component to the bilinear pseudodifferential operator defined in the form (1.1) to get a bilinear Fourier integral operator.

In this paper, we often use the notation ξ for the pair $(\xi_1, \xi_2) \in \mathbb{R}^{2n}$.

We're interested in a particular class of bilinear Fourier integral operators having the following form

$$I_{\phi,a}(u_1, u_2)(x) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} e^{i\phi(x,\xi)} a(x, \xi) \hat{u}_1(\xi_1) \hat{u}_2(\xi_2) d\xi_1 d\xi_2.$$

where $a \in BS^{m_1,m_2}_{\delta,\rho}$ and $\phi \in \Phi$ (see the next section for definitions).

The aim of this work is to extend results obtained in [17] for a bilinear Fourier integral operators. We will treat the global boundedness of $I_{\phi,a}$ on $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

Let us now describe the paper's structure. The crucial notations and preliminaries that will be used throughout the work are introduced in the second section.

Following that, we'll go over some basic tools and necessary lemmas which will serve as the starting point for our research. And the last section is devoted to prove our main goal.

We will conclude this section by stating why bilinear Fourier integral operators are important to examine. We discuss the problem of confining solutions of certain hyperbolic partial differential equations along half-space subspaces, which is inspired by some restriction difficulties. We present a typical problem that may arise in the case of the wave equation on $\mathbb{R}^{2n} \times]0, \infty[$.

Consider the wave equation on $\mathbb{R}^{2n} \times]0, \infty[$ with coordinates (x, t), where $x = (x_1; x_2) \in \mathbb{R}^{2n}$ and $t > 0$

$$\begin{cases} \partial_t^2 f(x,t) &= \Delta_x f(x,t), \\ f(x,0) &= v_1(x_1) v_2(x_2), \\ \partial_t f(x,0) &= u_1(x_1) u_2(x_2). \end{cases}$$

For each fixed t, the solution $f(x, t)$ can be written as a sum of Fourier integral operators with phase functions $\phi_\pm(x, \xi) = x\xi \pm t\sqrt{|\xi_1|^2 + |\xi_2|^2}$, where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{2n}$ is the dual variable of (x_1, x_2).

When we consider the restriction of the solution $f(x, t)$ along the diagonal $x_1 = x_2$, we obtain two bilinear Fourier integral operators with phases ϕ_+ and ϕ_- acting on the pairs of functions (u_1, u_2) and (v_1, v_2). When the initial data u_1, u_2, v_1, v_2 lie in $L^2(\mathbb{R}^n)$, it is natural to study the boundedness of these bilinear Fourier integral operators.

2 Preliminaries

2.1 Notations and definitions

In this sequel we define the class of amplitudes and phase functions that appear in the definition of operators treated here. But at first we recall some notations that will be used.

We assume $n \in \mathbb{N}$ throughout the whole paper unless otherwise noted. In particular $n \neq 0$. For all $x, \zeta \in \mathbb{R}^n$ we denote

$$x\zeta := \sum_{j=0}^{n} x_j \zeta_j \quad \textit{and} \quad \langle \zeta \rangle := (1 + |\zeta|^2)^{1/2}.$$

We set for all $R > 0$ and $L > n$

$$S_R(u)(x) = R^n \int_{\mathbb{R}^n} \frac{|u(y)|}{(1 + R|x - y|)^L} dy,$$

and simply write $S(u)$ for $R = 1$.

In addition, for all $u \in C^\infty(\mathbb{R}^n)$ and $\tau \in \mathbb{R}^n$ we define

$$\chi_\tau(u)(x) = u(x - \tau)$$

Partial derivatives with respect to a variable $x \in \mathbb{R}^n$ scaled with the factor $-i$ are denoted by

$$D_x^\alpha := (-i)^{|\alpha|}\partial_x^\alpha := (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index and $|\alpha| = \sum_{j=1}^n \alpha_j$ is the length of α. let us denote by $B(x,r)$ the euclidean ball centered at x with radius $r > 0$.

The usual inner product of $u, v \in L^2(\mathbb{R}^n)$ is denoted by $\langle u, v\rangle$, and the notation $|u(x)| \lesssim |v(x)|$ means that $|u(x)| \le C|v(x)|$ for some unspecified constant $C > 0$.

Let $\mathcal{S}(\mathbb{R}^n)$ be the space of rapidly decreasing smooth functions (Schwarz space), we define the Fourier transform $\hat{u}$ and its inverse $\mathcal{F}^{-1}(u)$ of $u \in \mathcal{S}(\mathbb{R}^n)$ by

$$\hat{u}(\zeta) = \mathcal{F}(u)(\zeta) = \int_{\mathbb{R}^n} e^{-ix\zeta} u(x)dx \quad and \quad \mathcal{F}^{-1}(u)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\zeta x} u(x) d\zeta$$

Definition 2.1 (The amplitude). Let $m_1, m_2 \in \mathbb{R}$ and $0 \le \delta, \rho \le 1$. We denote $BS_{\delta,\rho}^{m_1,m_2}$ the class of all functions $a(x, \xi_1, \xi_2) \in C^\infty(\mathbb{R}^{3n})$ such that for all $\alpha, \beta_1, \beta_2 \in \mathbb{N}^n$ there exists $C_{\alpha,\beta_1,\beta_2} > 0$ we have

$$\left|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} a(x, \xi_1, \xi_2)\right| \le C_{\alpha,\beta_1,\beta_2} \langle \xi_1\rangle^{m_1+\delta|\alpha|-\rho|\beta_1|} \langle \xi_2\rangle^{m_2+\delta|\alpha|-\rho|\beta_2|}$$

Remark 2.2. $BS_{\delta,\rho}^{m_1,m_2}$ is called the bilinear Hörmander class.

Definition 2.3 (The strong non-degeneracy condition). A real value function $\phi(x, \xi_1, \xi_2) \in C^2(\mathbb{R}^n \times \mathbb{R}^n\backslash\{0\} \times \mathbb{R}^n\backslash\{0\})$ satisfies the strong non-degeneracy condition, if there exist $c_1, c_2 > 0$ such that

$$\left|det \frac{\partial^2 \phi(x, \xi_1, \xi_2)}{\partial x \partial \xi_1}\right| \ge c_1 \quad for \;\; all \;\; (x, \xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^n\backslash\{0\} \times \mathbb{R}^n\backslash\{0\},$$

and

$$\left|det \frac{\partial^2 \phi(x, \xi_1, \xi_2)}{\partial x \partial \xi_2}\right| \ge c_2 \quad for \;\; all \;\; (x, \xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^n\backslash\{0\} \times \mathbb{R}^n\backslash\{0\}.$$

Definition 2.4 (The phase function). We denote by Φ the space of all real valued function $\phi(x, \xi_1, \xi_2) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n\backslash\{0\} \times \mathbb{R}^n\backslash\{0\})$, such that $\phi(x, \xi_1, \xi_2)$ is positively homogenous of degree 1 jointly in the variables (ξ_1, ξ_2), and satisfies the strong non-degeneracy condition.

Example 2.5. For all $k \in \mathbb{R}$ the phase function $\phi(x, \xi_1, \xi_2) = kx(\xi_1 + \xi_2)$ is belonging to Φ

2.2 Basic tools

In this section we will use a dyadic partition of unity $(\psi_j)_{j\ge0} \subset C_0^\infty(\mathbb{R}^n)$ such that

$$\sum_{j\ge0} \psi_j(\xi) = 1 \quad \forall \xi \in \mathbb{R}^n,$$

and

$$\text{supp } \psi_j \subseteq \{\xi \in \mathbb{R}^n : 2^{j-1} \le |\xi| \le 2^{j+1}\} \quad \forall j \ge 1.$$

The dyadic partition of unity can be constructed such that supp $\psi_0 \subset \overline{B(0_{\mathbb{R}^n}, 2)}$, $\psi_j(\xi) := \psi_1(2^{1-j}\xi)$ for all $j \ge 1$ holds

$$\left|\partial_\xi^\alpha \psi_j(\xi)\right| \le C2^{-|\alpha|j} \left\|\partial_\xi^\alpha \psi_1\right\|_{L^\infty} \quad \forall \alpha \in \mathbb{N}_0^n, \; j \ge 0.$$

Moreover, we note that

$$f(x) = \sum_{j\ge0} \psi_j(D_x) f(x) \quad \forall f \in \mathcal{S}(\mathbb{R}^n),$$

where

$$\psi_j(D_x) f = \mathcal{F}^{-1}[\psi_j(\xi)\hat{f}(\xi)].$$

Definition 2.6. Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$. Then the Besov space $B^s_{p,q}(\mathbb{R}^n)$ is defined by

$$B^s_{p,q}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{B^s_{p,q}} < \infty\},$$

where

$$if\, q < \infty \quad \|u\|_{B^s_{p,q}} := \left(\sum_{j \geq 0} 2^{jsq} \|\psi_j(D_x)u\|^q_{L^p} \right)^{\frac{1}{q}},$$

and

$$if\, q = \infty \quad \|u\|_{B^s_{p,\infty}} := \sup_{j \geq 0} 2^{jsq} \|\psi_j(D_x)u\|_{L^p}.$$

Here the exponent s is the order of $B^s_{p,q}$, p is called integration exponent, and q is called summation exponent.

Remark 2.7. 1. It is well known that the definition of Besov spaces $B^s_{p,q}$ is independent of the choice of $(\psi_j)_{j\geq 0}$.
2. For all $s > 0$. Then $B^s_{\infty,\infty}$ are called Hölder-Zygmund spaces.
3. $B^s_{2,2}$ is identical with Sobolev spaces H^s

Lemma 2.8. Let $s \in \mathbb{R}, 1 \geq p, q_1, q_2 \geq \infty$, and $\tau > 0$. Then

$$if\, q_1 \leq q_2 \; then \quad B^s_{p,q_1} \hookrightarrow B^s_{p,q_2},$$

and

$$B^{s+\tau}_{p,\infty} \hookrightarrow B^s_{p,1}.$$

Proof . See [1] □

We will investigate the decomposition of a in the rest of this section, assuming $m_1, m_2 \in \mathbb{R}$ and $a \in BS^{m_1,m_2}_{0,0}$. Let $(\psi_j)_{j\geq 0}$ be a dyadic partition of unity and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$supp\varphi \subset [-1, 1]^n,$$

and

$$\sum_{\alpha \in \mathbb{Z}^n} \varphi(\xi - \alpha) = 1 \quad \forall \xi \in \mathbb{R}^n.$$

We decompose as follows using these functions

$$a(x, \xi_1, \xi_2) = \sum_{k \in \mathbb{N}^2} \sum_{j \in \mathbb{N}} a_{j,k}(x, \xi_1, \xi_2) = \sum_{k \in \mathbb{N}^2} \sum_{\alpha \in \mathbb{Z}^{2n}} \sum_{j \in \mathbb{N}} a_{j,\alpha,k}(x, \xi_1, \xi_2) \tag{2.1}$$

where $k = (k_1, k_2), \alpha = (\alpha_1, \alpha_2)$ and

$$a_{j,k}(x, \xi_1, \xi_2) := [\psi_j(D_x)a](x, \xi_1, \xi_2)\psi_{k_1}(\xi_1)\psi_{k_2}(\xi_2),$$

and

$$a_{j,\alpha,k}(x, \xi_1, \xi_2) := a_{j,k}(x, \xi_1, \xi_2)\varphi(\xi_1 - \alpha_1)\varphi(\xi_2 - \alpha_2).$$

In order to prove our main result, we must use the following lemmas.

Lemma 2.9. Let $2 \geq r \geq \infty$, and let Λ be a finite subset of $\mathbb{Z}^n$. For each $N \geq 0$, we have

$$\left(\sum_{\alpha_1 \in \Lambda} \sum_{\alpha_2 \in \mathbb{Z}^n} + \sum_{\alpha_2 \in \Lambda} \sum_{\alpha_1 \in \mathbb{Z}^n} + \sum_{\mu \in \Lambda} \sum_{\alpha_1 + \alpha_2 = \mu} \right) \left| \langle I_{\phi, a_{j,\alpha,k}}(u_1, u_2), v \rangle \right|$$
$$\lesssim 2^{k_1 m_1 + k_2 m_2 - jN} \left(card(\Lambda) \right)^{1/2} \|u_1\|_{L^2} \|u_2\|_{L^2} \|v\|_{L^r}$$

for all $j \in \mathbb{N}$ and $k = (k_1, k_2) \in \mathbb{N}^2$.

Proof . Let $\widetilde{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that

$$supp\,\widetilde{\varphi} \subset [-2,2]^n \quad and \quad \widetilde{\varphi}\,|_{[-1,1]^n} \equiv 1.$$

For all $\alpha_1, \alpha_2, \mu \in \mathbb{Z}^n$ and $j \geq 0$, we set

$$u_{i,\alpha_i} = \widetilde{\varphi}(D-\alpha_i)u_i, \quad v_{j,\mu} = \widetilde{\varphi}\left(\frac{D-\mu}{2^{j+2}}\right)v.$$

Then, we have $\widetilde{\varphi}\varphi = \varphi$ that

$$I_{\phi,a_{j,\alpha,k}}(u_1,u_2) = I_{\phi,a_{j,\alpha,k}}(u_{1,\alpha_1},u_{2,\alpha_2}). \tag{2.2}$$

Hence, it follows from (2.2) that

$$\begin{aligned}\langle I_{\phi,a_{j,\alpha,k}}(u_1,u_2),v\rangle &= \langle I_{\phi,a_{j,\alpha,k}}(u_1,u_2),v_{j,\alpha_1+\alpha_2}\rangle \\ &= \langle I_{\phi,a_{j,\alpha,k}}(u_{1,\alpha_1},u_{2,\alpha_2}),v_{j,\alpha_1+\alpha_2}\rangle.\end{aligned} \tag{2.3}$$

By (2.3), [17, Lemma 3.2] with N replaced by $\varrho = N + n/r + n/2$ and Schwarz's inequality, we have

$$\begin{aligned}\sum_{\alpha_1\in\Lambda}\sum_{\alpha_2\in\mathbb{Z}^n}|\langle I_{\phi,a_{j,\alpha,k}}(u_1,u_2),v\rangle| &= \sum_{\alpha_1\in\Lambda}\sum_{\alpha_2\in\mathbb{Z}^n}|\langle I_{\phi,a_{j,\alpha,k}}(u_{1,\alpha_1},u_{2,\alpha_2}),v_{j,\alpha_1+\alpha_2}\rangle| \\ &\lesssim 2^{k_1m_1+k_2m_2-j\varrho}\sum_{\alpha_1\in\Lambda}\sum_{\alpha_2\in\mathbb{Z}^n}\int_{\mathbb{R}^n}S(u_{1,\alpha_1})(x)S(u_{2,\alpha_2})(x)|v_{j,\alpha_1+\alpha_2}(x)|dx \\ &\lesssim 2^{k_1m_1+k_2m_2-j\varrho} \\ &\quad\times\sum_{\alpha_1\in\Lambda}\int_{\mathbb{R}^n}S(u_{1,\alpha_1})(x)\left(\sum_{\alpha_2\in\mathbb{Z}^n}(S(u_{2,\alpha_2})(x))^2\right)^{1/2}\left(\sum_{\alpha_2\in\mathbb{Z}^n}|v_{j,\alpha_1+\alpha_2}(x)|^2\right)^{1/2}dx \\ &\lesssim 2^{k_1m_1+k_2m_2-j\varrho}(card(\Lambda))^{1/2} \\ &\quad\times\int_{\mathbb{R}^n}\left(\sum_{\alpha_1\in\mathbb{Z}^n}(S(u_{1,\alpha_1})(x))^2\right)^{1/2}\left(\sum_{\alpha_2\in\mathbb{Z}^n}(S(u_{2,\alpha_2})(x))^2\right)^{1/2}\left(\sum_{\mu\in\mathbb{Z}^n}|v_{j,\mu}(x)|^2\right)^{1/2}dx.\end{aligned}$$

It follows from [17, Lemma 3.1] that

$$\begin{aligned}\left(\sum_{\alpha_1\in\mathbb{Z}^n}(S(u_{1,\alpha_1})(x))^2\right)^{1/2} &\lesssim \left(\sum_{\alpha_1\in\mathbb{Z}^n}S(|u_{1,\alpha_1}|^2)(x)\right)^{1/2} \\ &\lesssim \left(S(S(|u_1|^2))(x)\right)^{1/2} \\ &\lesssim \left(S(|u_1|^2)(x)\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\left(\sum_{\mu\in\mathbb{Z}^n}|v_{j,\mu}(x)|^2\right)^{1/2} &\lesssim 2^{n(1+j/2)}\left(S_{2^{j+2}}(|v|^2)(x)\right)^{1/2} \\ &\lesssim 2^{jn}\left(\int_{\mathbb{R}^n}\frac{dy}{(1+2^j|x-y|)^q}\right)^{1/q}\left(\int_{\mathbb{R}^n}|v(y)|^r dy\right)^{1/r} \\ &\lesssim 2^{jn(1/2+1/r)}\|v\|_{L^r},\end{aligned}$$

where we use Holder's inequality with $1/r + 1/q = 1/2$ in the second inequality.
As a result of schwarz's and Young's inequalities, we get

$$\begin{aligned}\sum_{\alpha_1\in\Lambda}\sum_{\alpha_2\in\mathbb{Z}^n}|\langle I_{\phi,a_{j,\alpha,k}}(u_1,u_2),v\rangle| &\lesssim 2^{k_1m_1+k_2m_2-jN}card(\Lambda)\left(\int_{\mathbb{R}^n}\left(S(|u_1|^2)(x)\right)^{1/2}\left(S(|u_2|^2)(x)\right)^{1/2}dx\right)\|v\|_{L^r} \\ &\lesssim 2^{k_1m_1+k_2m_2-jN}card(\Lambda)\|\left(S(|u_1|^2)(x)\right)^{1/2}\|_{L^2}\|\left(S(|u_2|^2)(x)\right)^{1/2}\|_{L^2}\|v\|_{L^r} \\ &\lesssim 2^{k_1m_1+k_2m_2-jN}card(\Lambda)\|u_1\|_{L^2}\|u_2\|_{L^2}\|v\|_{L^r}.\end{aligned}$$

We may also estimate the total $\sum_{\alpha_2\in\Lambda}\sum_{\alpha_1\in\mathbb{Z}^n}$ in the same way.

Next, we consider the sum $\sum_{\mu\in\Lambda}\sum_{\alpha_1+\alpha_2=\mu}$. By (2.3), [17, Lemma 3.2] and Schwarz's inequality,

$$\begin{aligned}\sum_{\mu\in\Lambda}\sum_{\alpha_1+\alpha_2=\mu}|\langle I_{\phi,a_{j,\alpha,k}}(u_1,u_2),v\rangle| &= \sum_{\mu\in\Lambda}\sum_{\alpha_1+\alpha_2=\mu}|\langle I_{\phi,a_{j,\alpha,k}}(u_{1,\alpha_1},u_{2,\alpha_2}),v_{j,\alpha_1,\alpha_2}\rangle|\\ &\lesssim 2^{k_1m_1+k_2m_2-j\varrho}\sum_{\mu\in\Lambda}\sum_{\alpha_1\in\mathbb{Z}^n}\int_{\mathbb{R}^n}S(u_{1,\alpha_1})(x)S(u_{2,\mu-\alpha_1})(x)|v_{j,\mu}(x)|dx\\ &\lesssim 2^{k_1m_1+k_2m_2-j\varrho}\sum_{\mu\in\Lambda}\int_{\mathbb{R}^n}\left(\sum_{\alpha_1\in\mathbb{Z}^n}S^2(u_{1,\alpha_1})(x)\right)^{1/2}\left(\sum_{\alpha_1\in\mathbb{Z}^n}S^2(u_{2,\mu-\alpha_1})(x)\right)^{1/2}|v_{j,\mu}(x)|dx\\ &\lesssim 2^{k_1m_1+k_2m_2-j\varrho}card(\Lambda)\int_{\mathbb{R}^n}\left(\sum_{\alpha_1\in\mathbb{Z}^n}S^2(u_{1,\alpha_1})(x)\right)^{1/2}\left(\sum_{\alpha_2\in\mathbb{Z}^n}S^2(u_{2,\alpha_2})(x)\right)^{1/2}\left(\sum_{\mu\in\mathbb{Z}^n}|v_{j,\mu}(x)|^2\right)^{1/2}dx.\end{aligned}$$

The rest of the proof is the same as before. The proof has been established. □

We'll end this section by quoting Schur's lemma.

Lemma 2.10 (Schur's lemma). Let $(U_{k,l})_{k,l\geq 0}$ be a sequence of positive numbers satisfying

$$\sup_{k\geq 0}\sum_{l\geq 0}U_{k,l}<\infty \quad and \quad \sup_{l\geq 0}\sum_{k\geq 0}U_{k,l}<\infty,$$

then

$$\left(\sum_{k\geq 0}\sum_{l\geq 0}U_{k,l}V_kW_l\right)^2\lesssim\sum_{k\geq 0}V_k^2\sum_{l\geq 0}W_l^2,$$

for all positive sequences $(V_k)_{k\geq 0}$ and $(W_l)_{l\geq 0}$.

3 The boundedness of $I_{\phi,a}$ on $L^2\times L^2$

We have the following result concerning the boundedness on $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ of the bilinear Fourier integral operator.

Theorem 3.1. Let $I_{\phi,a}$ be the bilinear Fourier integral operator defined by

$$I_{\phi,a}(u_1,u_2)(x)=\frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^{2n}}e^{i\phi(x,\xi)}a(x,\xi)\hat{u_1}(\xi_1)\hat{u_2}(\xi_2)d\xi_1d\xi_2, \tag{3.1}$$

where $\xi=(\xi_1,\xi_2)\in\mathbb{R}^{2n},\phi\in\Phi(\mathbb{R}^{3n})$ and $a\in BS_{0,0}^{m_1,m_2}$.

For all $m_1,m_2<0$, $m_1+m_2=-\frac{n}{2}$ and $q\in[1,2]$, then $I_{\phi,a}$ can be extended as a bounded bilinear operator from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $B^0_{q,1}(\mathbb{R}^n)$.

Proof . From (2.1), we can write

$$\begin{aligned}\langle I_{\phi,a}(u_1,u_2),v\rangle &= \sum_{k\in\mathbb{N}^2}\sum_{j\geq 0}\sum_{\alpha\in\mathbb{Z}^{2n}}\left\langle I_{\phi,a_{j,\alpha,k}}(u_1,u_2),v\right\rangle\\ &=\left(\sum_{k_1<k_2}+\sum_{k_1\geq k_2}\right)\sum_{j\geq 0}\sum_{\alpha\in\mathbb{Z}^{2n}}\left\langle I_{\phi,a_{j,\alpha,k}}(u_1,u_2),v\right\rangle\end{aligned}$$

We only consider the first sum in the last line due to symmetry, because the argument below works for the second.

Let $(\psi_k)_{k\geq 0}$ be a dyadic partition of unity and $(\tilde{\psi}_k)_{k\geq 0}\subset\mathcal{S}(\mathbb{R}^n)$, such that

supp $\tilde{\psi}_0 \subset \overline{B(0_{\mathbb{R}^n}, 4)}$,
supp $\tilde{\psi}_k \subset \left\{\zeta \in \mathbb{R}^n : 2^{k-2} \le |\zeta| \le 2^{k+2}\right\} \quad \forall k \ge 1$,
$\tilde{\psi}_k|_{\text{supp }\psi_k} \equiv 1 \quad \forall k \ge 0$. since for all $i \in \{1,2\}$

$$\psi_{k_i}\tilde{\psi}_{k_i} \equiv \psi_{k_i},$$

and it holds that

$$\left\langle I_{\phi,a_{j,\alpha,k}}(u_1,u_2), v\right\rangle = \left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}), v\right\rangle,$$

with for all $i \in \{1,2\}$

$$u_{i,k_i}(x) = \tilde{\psi}_{k_i}(D_x)u_i(x) \quad \forall x \in \mathbb{R}^n.$$

We also use the decomposition

$$v(x) = \sum_{l\ge 0} v_l(x) \quad \forall x \in \mathbb{R}^n,$$

where $v_l(x) := \psi_l(D_x)v(x) = \mathcal{F}^{-1}[\psi_l(\zeta)\hat{v}(\zeta)]$. Then, we can write

$$\langle I_{\phi,a}(u_1,u_2), v\rangle = \sum_{k_1<k_2}\sum_{j\ge 0}\sum_{\alpha\in\mathbb{Z}^{2n}}\sum_{l\ge 0}\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}), v_l\right\rangle.$$

We also split the sum in the following manner.

$$\sum_{k_1<k_2}\sum_{j\ge 0}\sum_{\alpha\in\mathbb{Z}^{2n}}\sum_{l\ge 0}\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}), v_l\right\rangle = A_1 + A_2,$$

where

$$A_1 := \sum_{k_2-3\le j}\sum_{k_1<k_2}\sum_{j\ge 0}\sum_{\alpha\in\mathbb{Z}^{2n}}\sum_{l\ge 0}\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}), v_l\right\rangle,$$

and

$$A_2 := \sum_{k_2-3> j}\sum_{k_1<k_2}\sum_{j\ge 0}\sum_{\alpha\in\mathbb{Z}^{2n}}\sum_{l\ge 0}\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}), v_l\right\rangle.$$

Now we will estimate A_1. We have for all $l \ge 1$

$$\text{supp } \hat{v}_l \subset \left\{\zeta \in \mathbb{R}^n : 2^{l-1} \le |\zeta| \le 2^{l+1}\right\},$$

and for all $j \ge k_2 - 3, k_2 > k_1$ we have

$$\text{supp } \mathcal{F}\left[I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2})\right] \subset \overline{B(0_{\mathbb{R}^n}, 2^{j+6})},$$

it follows that $\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}), v_l\right\rangle = 0$ if $l \ge j+7$. Furthermore, we see that if supp $\chi_{\alpha_2}(\varphi) \cap$ supp $\psi_{k_2} = \varnothing$, then $a_{j,\alpha,k} = 0$, and consequently

$$\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}), v_l\right\rangle = 0.$$

So, from these observation, A_1 can be written as

$$A_1 = \sum_{l<j+7}\sum_{j\ge k_2-3}\sum_{k_1<k_2}\sum_{\alpha_1\in\mathbb{Z}^n}\sum_{\alpha_2\in\Lambda_2}\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}), v_l\right\rangle.$$

with

$$\Lambda_2 := \{\alpha \in \mathbb{Z}^n : \text{supp } \chi_{\alpha_2}(\varphi) \cap \text{supp } \psi_{k_2} \ne \varnothing\}.$$

Note that the number of elements of Λ_2 satisfies $card\,(\Lambda_2) \lesssim 2^{k_2 n}$. Using lemma 2.9 with $r = p$ and $N > 0$, we have

$$\begin{aligned}
|A_1| &\leq \sum_{l<j+7}\sum_{j\geq k_2-3}\sum_{k_1<k_2}\sum_{\alpha_1\in\mathbb{Z}^n}\sum_{\alpha_2\in\Lambda_2} |\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}),v_l\right\rangle| \\
&\lesssim \sum_{l<j+7}\sum_{j\geq k_2-3}\sum_{k_1<k_2} 2^{k_1m_1+k_2m_2-jN}[card\,(\Lambda_2)]^{\frac{1}{2}}\|u_{1,k_1}\|_{L^2(\mathbb{R}^n)}\|u_{2,k_2}\|_{L^2(\mathbb{R}^n)}\|v_l\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \sum_{l<j+7}\sum_{j\geq k_2-3}\sum_{k_1<k_2} 2^{-jN}\|u_{1,k_1}\|_{L^2(\mathbb{R}^n)}\|u_{2,k_2}\|_{L^2(\mathbb{R}^n)}\|v_l\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \sum_{j\geq k_2-3}\sum_{k_1<k_2} 2^{-jN}\|u_{1,k_1}\|_{L^2(\mathbb{R}^n)}\|u_{2,k_2}\|_{L^2(\mathbb{R}^n)}\left(\sum_{l<j+7}\|v_l\|_{L^p(\mathbb{R}^n)}\right) \\
&\lesssim \sum_{j\geq k_2-3}\sum_{k_1<k_2} 2^{-jN}(j+7)\|u_{1,k_1}\|_{L^2(\mathbb{R}^n)}\|u_{2,k_2}\|_{L^2(\mathbb{R}^n)}\left(\sup_{l\geq 0}\|v_l\|_{L^p(\mathbb{R}^n)}\right) \\
&\lesssim \sum_{j\geq 0} 2^{-jN}(j+7)(j+4)\left(\sum_{k_1\geq 0}\|u_{1,k_1}\|^2_{L^2(\mathbb{R}^n)}\right)^{\frac{1}{2}}\left(\sum_{k_2\geq 0}\|u_{2,k_2}\|^2_{L^2(\mathbb{R}^n)}\right)^{\frac{1}{2}}\|v\|_{B^0_{p,\infty}(\mathbb{R}^n)} \\
&\lesssim \|u_1\|_{L^2(\mathbb{R}^n)}\|u_2\|_{L^2(\mathbb{R}^n)}\|v\|_{B^0_{p,\infty}(\mathbb{R}^n)},
\end{aligned}$$

which implies the intended estimate.

For estimating A_2, we divide it as follows

$$A_2 = Q_1 + Q_2$$

where

$$Q_1 := \sum_{k_2-3\leq k_1<k_2}\sum_{j\leq k_2-2}\sum_{\alpha\in\mathbb{Z}^{2n}}\sum_{l\geq 0}\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}),v_l\right\rangle,$$

and

$$Q_2 := \sum_{k_1<k_2-3}\sum_{j\leq k_2-2}\sum_{\alpha\in\mathbb{Z}^{2n}}\sum_{l\geq 0}\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}),v_l\right\rangle.$$

Firs, we consider the estimate for Q_1. For all $k_1 < k_2$, $j \leq k_2 - 2$ we have

$$supp\,\left(\mathcal{F}[I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2})]\right) \subset \overline{B(0_{\mathbb{R}^n},2^{k_2+3})},$$

consequently, for all $l > k_2 + 3$ we have

$$\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}),v_l\right\rangle = 0.$$

In addition, by the fact $supp\,(\hat{v}_l) \subset supp\,(\psi_l)$, we see that if $\left(\alpha_1+\alpha_2+[-2^{j+2},2^{j+2}]^n\right)\cap supp\,(\psi_l) = \varnothing$, then

$$\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}),v_l\right\rangle = 0.$$

We can write Q_1 as a result of combining these observations by

$$Q_1 = \sum_{k_2-3\leq k_1\leq k_2}\sum_{j<k_2+3}\sum_{l\leq k_2+3}\sum_{\nu\in\Lambda_{j,l}}\sum_{\alpha_1+\alpha_2=\nu}\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}),v_l\right\rangle,$$

where

$$\Lambda_{j,l} = \left\{\nu\in\mathbb{Z}^n : \left(\nu+[-2^{j+2},2^{j+2}]^n\right)\cap supp\,(\psi_l) \neq \varnothing\right\},$$

with $card\,(\Lambda_{j,l}) \lesssim 2^{n(j+l)}$. Hence, it follows from lemma 2.9 with $r=p$ and $N>\frac{n}{2}$ such that

$$\begin{aligned}
|Q_1| &\leq \sum_{l<k_2+3}\sum_{j\geq k_2-3}\sum_{k_2-3\leq k_1<k_2}\sum_{\alpha_1+\alpha_2=\nu}\sum_{\nu\in\Lambda_{j,l}} |\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}),v_l\right\rangle| \\
&\lesssim \sum_{l<k_2+3}\sum_{j\geq k_2-3}\sum_{k_2-3\leq k_1<k_2} 2^{k_1m_1+k_2m_2-jN}[card\,(\Lambda_{j,l})]^{\frac{1}{2}}\|u_{1,k_1}\|_{L^2(\mathbb{R}^n)}\|u_{2,k_2}\|_{L^2(\mathbb{R}^n)}\|v_l\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \sum_{j<k_2-3}\sum_{l\leq k_2+3}\sum_{k_2-3\leq k_1<k_2} 2^{k_1m_1+k_2m_2-jN}2^{n(j+l)/2}\|u_{1,k_1}\|_{L^2(\mathbb{R}^n)}\|u_{2,k_2}\|_{L^2(\mathbb{R}^n)}\|v\|_{B^0_{p,\infty}(\mathbb{R}^n)} \\
&\lesssim \sum_{k_2-3\leq k_1<k_2} 2^{k_1m_1+k_2m_2}2^{k_2n/2}\|u_{1,k_1}\|_{L^2(\mathbb{R}^n)}\|u_{2,k_2}\|_{L^2(\mathbb{R}^n)}\|v\|_{B^0_{p,\infty}(\mathbb{R}^n)} \\
&\lesssim \sum_{k_2-3\leq k_1<k_2} 2^{k_1m_1-k_2m_2}\|u_{1,k_1}\|_{L^2(\mathbb{R}^n)}\|u_{2,k_2}\|_{L^2(\mathbb{R}^n)}\|v\|_{B^0_{p,\infty}(\mathbb{R}^n)}.
\end{aligned}$$

We put $k_1=k_2-k'$ with $k'\in\{0,1,2,3\}$, so for all $k_2-3\leq k_1<k_2$ we have

$$|Q_1| \lesssim \sum_{k_2\geq 0}\sum_{k'=0}^{3} 2^{-m_1k'}\|u_{1,k_2-k'}\|_{L^2(\mathbb{R}^n)}\|u_{2,k_2}\|_{L^2(\mathbb{R}^n)}.$$

By Schwarz's inequality, the right hand side of the last sum is estimated as

$$\sum_{k'=0}^{3} 2^{-m_1k'}\left(\sum_{k_2\geq 0}\|u_{1,k_2-k'}\|^2_{L^2}\right)^{1/2}\left(\sum_{k_2\geq 0}\|u_{2,k_2}\|^2_{L^2}\right)^{1/2} \lesssim \|u_1\|_{L^2}\|u_2\|_{L^2}$$

which gives the desired result.

Next, we gave a estimate for Q_2. For all $j<k_2-3, k_1<k_2-3$ we have

$$supp\,\left(\mathcal{F}[I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2})]\right)\subset\{\zeta\in\mathbb{R}^n : 2^{k_2-2}\leq|\zeta|2^{k_2+2}\},$$

it follows that for all $l\leq k_2-3$ or $k_2+3\leq l$ we have

$$\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}),v_l\right\rangle = 0.$$

As before, if $\operatorname{supp}\chi_{\alpha_2}(\varphi)\cap\operatorname{supp}\psi_{k_1}=\varnothing$, then

$$\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}),v_l\right\rangle = 0.$$

Moreover, we obtain

$$Q_2=\sum_{j<k_2-3}\sum_{k_1<k_2-3}\sum_{l=k_2-2}^{k_2+2}\sum_{\alpha_1\in\mathbb{Z}^n}\sum_{\alpha_2\in\Lambda_2}\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}),v_l\right\rangle.$$

By lemma 2.9, we have

$$\begin{aligned}
|Q_2| &\leq \sum_{l=k_2-2}^{k_2+2}\sum_{j<k_2-3}\sum_{k_1<k_2-3}\sum_{\alpha\in\mathbb{Z}^n}\sum_{\alpha_2\in\Lambda_2}\left|\left\langle I_{\phi,a_{j,\alpha,k}}(u_{1,k_1},u_{2,k_2}),v_l\right\rangle\right| \\
&\lesssim \sum_{l=k_2-2}^{k_2+2}\sum_{j<k_2-3}\sum_{k_1<k_2-3} 2^{k_1m_1+k_2m_2-jN}2^{k_1n/2}\|u_{1,k_1}\|_{L^2(\mathbb{R}^n)}\|u_{2,k_2}\|_{L^2(\mathbb{R}^n)}\|v_l\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \sum_{k_1<k_2-3} 2^{(k_2-k_1)m_1}\|u_{1,k_1}\|_{L^2(\mathbb{R}^n)}\|u_{2,k_2}\|_{L^2(\mathbb{R}^n)}\|v\|_{B^0_{p,\infty}(\mathbb{R}^n)} \\
&\lesssim \sum_{k_1\geq 0}\sum_{k_2\geq 0} 2^{|k_1-k_2|m_1}\|u_{1,k_1}\|_{L^2(\mathbb{R}^n)}\|u_{2,k_2}\|_{L^2(\mathbb{R}^n)}\|v\|_{B^0_{p,\infty}(\mathbb{R}^n)}.
\end{aligned}$$

Since $m_1 < 0$ we have

$$\sup_{k_1\geq 0}\sum_{k_2\geq 0} 2^{|k_1-k_2|m_1} < \infty\,,$$

and

$$\sup_{k_2\geq 0}\sum_{k_1\geq 0} 2^{|k_1-k_2|m_1} < \infty\,.$$

So, by Schur's lemma we have

$$\left(\sum_{k_1\geq 0}\|u_{1,k_1}\|^2_{L^2}\right)^{1/2}\left(\sum_{k_2\geq 0}\|u_{2,k_2}\|^2_{L^2}\right)^{1/2}\|v\|_{B^0_{p,\infty}} \lesssim \|u_1\|_{L^2}\|u_2\|_{L^2}\|v\|_{B^0_{p,\infty}}.$$

So, by duality we proved that $I_{\phi,a}$ is bounded from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ to $B^0_{q,1}$, where $\frac{1}{q}+\frac{1}{p}=1$. □

Corollary 3.2. Let $I_{\phi,a}$ be a class of bilinear Fourier integral operators defined as (3.1), then for all $m_1, m_2 < 0$ and $m_1+m_2=-\frac{n}{2}$ we have

$$I_{\phi,a}: L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^1(\mathbb{R}^n) \quad \textit{are bounded.}$$

Proof . By $B^0_{1,1}\hookrightarrow L^1$ and theorem 3.1 we get our desired result. □

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References

[1] H. Abels, *Pseudodifferential and singular integral operators. An introduction with applications*, de Gruyter, Berlin, 2012.

[2] OF. Aid and A. Senoussaoui, *The boundedness of h-admissible Fourier integral operators on Bessel potential spaces*, Turk. J. Math. **43** (2019), no. 5, 2125 – 2141.

[3] O.F. Aid and A. Senoussaoui, *H^s-Boundeness of a class of a Fourier integral operators*, Math. Slovaca **71** (2021), no. 4, 889–902.

[4] K. Asada and D. Fujiwara, *On some oscillatory transformations in $L^2(\mathbb{R}^n)$*, Japanese J. Math. **4** (1978), no. 2, 299–361.

[5] Á. Bényi and T. Oh, *On a class of bilinear pseudodifferential operators*, J. Funct. Spaces Appl. **2013** (2013), 1–5.

[6] Á. Bényi and K.A. Okoudjou, *Bilinear pseudodifferential operators on modulation spaces*, J. Fourier Anal. Appl. **10** (2004), 301–313.

[7] Á. Bényi and K.A. Okoudjou, *Modulation spaces estimates for multilinear pseudodifferential operators*, Stud. Math. **172** (2006), 169–180.

[8] Á. Bényi, K. Groechenig, C. Heil and K. Okoudjou, *Modulation spaces and a class of bounded multilinear pseudodifferential operators*, J. Oper. Theory **54** (2005), 389–401.

[9] Á. Bényi, D. Maldonado, V. Naibo and R.H. Torres, *On the Hörmander classes of bilinear pseudodifferential operators*, Integr. Equ. Oper. Theory **67** (2010), 341–364.

[10] M. Cappiello and J. Toft, *Pseudo-differential operators in a Gelfand–Shilov setting*, Math. Nachr. **290** (2017), 738–755.

[11] R.R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[12] R.R. Coifman and Y. Meyer, *Commutateurs d'intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) **28** (1978), 177–202.

[13] R.R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels, Astérisque*, No. 57, Société Mathématique de France, 1979.

[14] E. Cordero and KA. Okoudjou, *Multilinear localization operators*, J. Math. Anal. Appl. **325** (2007), 1103–1116.

[15] L. Grafakos and R.H. Torres, *Multilinear Calderón–Zygmund theory*, Adv. Math. **165** (2002), 124–164.

[16] L. Grafakos, *Modern fourier analysis*, Third edition, Springer, New York, 2014.

[17] N. Hamada, N. Shida and N. Tomita, *On the ranges of bilinear pseudo-differential operators of $S_{0,0}$-type on $L^2 \times L^2$*, J. Funct. Anal. **280** (2021), no. 3, 108826.

[18] T. Kato, *Bilinear pseudo-differential operators with exotic class symbols of limited smoothness*, J. Fourier Anal. Appl. **27** (2021), 1–56.

[19] K. Koezuka and N. Tomita, *Bilinear pseudo-differential operators with symbols in $BS^m_{1,1}$ on Triebel–Lizorkin spaces*, J. Fourier Anal. Appl. **24** (2018) 309–319.

[20] S. Molahajloo, K.A. Okoudjou and G.E. Pfander, *Boundedness of multilinear pseudodifferential operators on modulation spaces*, J. Fourier Anal. Appl. **22** (2016), 1381–1415.

[21] S. Rodríguez-López, D. Rule and W. Staubach, *Global boundedness of a class of multilinear Fourier integral operators*, Forum Math. Sigma **9** (2021), 1–45.