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L^∞-regularity result for an obstacle problem with degenerate coercivity in Musielak-Sobolev spaces

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Abstract

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. In this paper we give an existence result of bounded solution, in Musielak spaces, for unilateral problems associated to the nonlinear elliptic equation

$$-\operatorname{div} a(x,u,\nabla u) + g(x,u,\nabla u) = f \quad \text{in } \Omega,$$

where the nonlinearity g does not satisfy the well known sign condition and f is an integrable source.

Keywords: Musielak-Orlicz-Sobolev spaces, Elliptic equations, L^∞-estimates, Bounded solutions, Unilateral problems, Degenerate coercivity
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1 Introduction

Modular spaces have received significant attention in recent years, Musielak spaces are the adequate setting to give mathematical models to various physical phenomenons. Also, there has been an interesting development in functional analysis of the setting, minimal assumptions ensuring the density of smooth functions in the modular topology in Musielak-Orlicz-Sobolev spaces were given (see [5]), namely

(Φ_1) The function φ (resp. $\overline{\varphi}$) is locally integrable, that is for any constant number $c > 0$ and for any compact set $K \subset \Omega$ we have $\displaystyle\int_K \varphi(x,c)\,dx < \infty$.

(Φ_2) There exists a function $\varphi_0 : [0,1/2] \times [0,\infty) \longrightarrow [0,\infty)$ such that $\varphi_0(.,s)$ and $\varphi_0(x,.)$ are nondecreasing functions and for all $x,y \in \overline{\Omega}$ with $|x-y| < \frac{1}{2}$ and for any constant $c > 0$

$$\varphi(x,s) \leq \varphi_0(|x-y|,s)\varphi(y,s) \text{ with } \limsup_{\varepsilon\to 0^+} \varphi_0(\varepsilon, c\varepsilon^{-N}) < \infty.$$

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With only the Log-Hölder continuity of the type: there exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$,

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\frac{A}{\log(\frac{1}{|x-y|})}} \text{ for all } \quad t \geq 1,$$

the same density results have been given by Benkirane et al. in [8].

These new results among others are necessary to start the analysis of partial differential equations in the setting. We list some models of equations to which the present result can be applied,

$$-\text{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) + \frac{\log(1+|u|)}{(1+|u|)^p}|\nabla u|^p = f,$$

here $\varphi(x,t) = t^p$, $p > 1$, $h(t) = \dfrac{1}{(1+|t|)^\theta}$, $0 \leq \theta < 1$ and

$$-\text{div}\left(h(u)\exp\Big(|\nabla u| + h(u)\Big)\nabla u\right) + \frac{\exp\Big(|\nabla u| + h(u)\Big)}{(e+|u|)^3\log(e+|u|)}|\nabla u|^2 = f,$$

here $\varphi(x,t) = t^2\exp(t)$ and $h(t) = \dfrac{1}{(e+|t|)\log(e+|t|)}$.

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, and let φ be a Musielak function satisfying the $\triangle_2$-condition. Let $\psi : \Omega \to \overline{\mathbb{R}}$ be a measurable function such that $\mathbf{K}_\psi = \{v \in W_0^1 L_\varphi(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}$ is a nonempty set. Let us consider the strongly nonlinear elliptic equation

$$\begin{cases} -\text{div } a(x,u,\nabla u) + g(x,u,\nabla u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

Consider the unilateral problem associated to equation (1.1) as follows

$$\begin{cases} u \in \mathbf{K}_\psi, \ a(\cdot,u,\nabla u) \in (L_{\overline{\varphi}}(\Omega))^N, \ g(\cdot,u,\nabla u) \in L^1(\Omega), \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla(u-v)dx + \int_\Omega g(x,u,\nabla u)(u-v)dx \\ \leq \displaystyle\int_\Omega f(u-v)dx, \quad \forall v \in \mathbf{K}_\psi \cap L^\infty(\Omega), \end{cases} \tag{1.2}$$

where a satisfies the following condition

$$a(x,s,\xi).\xi \geq \overline{\varphi}_x^{-1}(\varphi(x,h(|s|)))\varphi(x,|\xi|) \tag{1.3}$$

where $h : \mathbb{R} \to \mathbb{R}^+$ is a continuous decreasing function with unbounded primitive. The Hamiltonian $g(x,u,\nabla u)$ is not assumed to satisfy the well known sign condition but grows naturally at most like $\varphi(x,|\nabla u|)$, namely

$$|g(x,s,\xi)| \ \leq \ \beta(s)\varphi(x,|\xi|), \tag{1.4}$$

where $\beta : \mathbb{R} \to \mathbb{R}^+$ is a continuous function, while the source term $f \in L^1(\Omega)$.

We list some previews contributions concerning problem (1.1), in the framework of Sobolev spaces $W_0^{1,p}(\Omega)$, $(p > 1)$, existence of bounded solution of (1.1) has been proved by Boccardo et al. [20], in the case where h is constant, f can be changed by $f - \text{div } F$ with $f \in L^m(\Omega)$, $m > \frac{N}{p}$ and $F \in L^r(\Omega)$ where $r > \frac{N}{p-1}$, see also [31].

The same result has been obtained by Boccardo et al. [21] but this time h is not necessarily constant, $p = 2$ and $f \in L^m(\Omega)$, $m > \frac{N}{2}$ and in [43] the authors have find the result when $p > 1$ and $f \in L^m(\Omega)$, $m > \max(\frac{N}{p}, 1)$.

In Orlicz spaces, if M is the N-function defining the Orlicz spaces, for $g \equiv 0$ and $g \neq 0$, Benkirane et al. in [44, 16] have established existence of bounded solutions for problem (1.1) under conditions $\mathcal{A}(x,s,\xi).\xi \geq \overline{M}^{-1}(M(h(|s|)))M(|\xi|)$, $|g(x,s,\xi)| \leq \beta(s)M(|\xi|)$ and either

$$f \in L^N(\Omega),$$

or

$$\begin{cases} f \in L^m(\Omega) \text{ with } m = \frac{rN}{r+1} \text{ for some } r > 0, \\ \quad\quad and \int_{.}^{+\infty} \left(\frac{t}{M(t)}\right)^r dt < +\infty. \end{cases}$$

Let us recall some contributions concerning problem (1.2) in some particular cases. In the classical Lebesgue spaces ($\varphi(x,t) = t^p$, $1 < p < +\infty$), existence of bounded solution for problem (1.2) has been given in [22] with $f \equiv 0$ and in [23] for quasilinear operators without lower order terms (i.e. $\beta = 0$) and data satisfying

$$f \in L^m(\Omega),\ m > \frac{N}{2}$$

and then under smallness a condition on the data f in [**?**] with

$$f \in L^m(\Omega),\ m > \max\left(1; \frac{N}{p}\right) \tag{1.5}$$

using symmetrization methods.

In the non standard growth setting, the studies of variational inequalities (i.e. where $f \in W^{-1}E_{\overline{M}}(\Omega)$) were initiated by Gossez and Mustonen in [35] to investigate the obstacle problem (1.2) in the case $g(x,u,\nabla u) = g(x,u)$ by assuming some regularity conditions on the obstacle function ψ. Vast works were interested on existence of solutions for problem like (1.2) either in the variational case, see for example [2] or with L^1-data (see [3, 4, 30]).

In Orlicz spaces where $\varphi(x,t) = M(t)$ (without x-dependence), existence of bounded solution for problem (1.2) has been investigated by Benkirane et al. in [17] where the vector field a satisfies the following condition

$$a(x,s,\xi)\cdot\xi \geq \overline{M}^{-1}(M(h(|s|))M(|\xi|), \tag{1.6}$$

Our aim here is to handle a more general case, precisely, we prove existence of bounded solutions for problem (1.2) in Musielak structure, the result extends all works mentioned above.

The paper is organized as follows: in Section 2 we recall some preliminaries and auxiliary results about Musielak spaces. Section 3 concerns the basic assumptions and the main result, while in section 4 we give the proof of the main result.

2 Preliminaries

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. For further definitions and properties we refer the reader to [5, 37, 13, 39].

2.1 Musielak-Orlicz function

Let Ω be an open subset of $\mathbb{R}^N$ and let φ be real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the following conditions

(a) $\varphi(x,.)$ is an N-function, i.e., convex, nondecreasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,t) > 0$ for all $t > 0$ and

$$\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t} = 0 \quad \text{for almost all } x \in \Omega,$$

$$\lim_{t\to \infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t} = \infty \quad \text{for almost all } x \in \Omega.$$

(b) $\varphi(.,t)$ is a measurable function.

A function $\varphi(x,t)$, which satisfies the condition (a) and (b), is called a Musielak-Orlicz function. For a Musielak-Orlicz function $\varphi(x,t)$ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its nonnegative reciprocal function with respect to t and φ_x^{-1} that is,

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t.$$

For any two Musielak-Orlicz functions φ and γ we introduce the following ordering:
(c) If there exists two positive constants c and T such that for almost all $x \in \Omega$

$$\varphi(x,t) \leq \gamma(x,ct) \text{ for } t \geq T,$$

then we write $\varphi \prec \gamma$ and we say that γ dominates φ globally if $T = 0$ and near infinity if $T > 0$.
(d) If for every positive constant c and almost everywhere $x \in \Omega$ we have

$$\lim_{t\to 0}\Big(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)}\Big) = 0 \quad \textit{or} \quad \lim_{t\to\infty}\Big(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)}\Big) = 0,$$

then we write $\varphi \prec\prec \gamma$ at 0 or near ∞ respectively, and we say that φ increases essentially more slowly than γ at 0 or near ∞ respectively.
We recall that the Musielak function φ is said to satisfy the Δ_2-condition (or φ is doubling) if for some $k > 0$, and a non-negative function c, integrable on Ω, we have

$$\varphi(x,2t) \leq k\varphi(x,t) + c(x) \text{ for all } x \in \Omega \text{ and all } t \geq 0.$$

2.2 Musielak-Orlicz-Sobolev spaces

For a Musielak function φ and a measurable function $u : \Omega \to \mathbb{R}$ we define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x,|u(x)|)\,dx.$$

The set $K_\varphi(\Omega) = \Big\{u : \Omega \to \mathbb{R} \quad \textit{measurable} : \varrho_{\varphi,\Omega}(u) < \infty\Big\}$ is called the Musielak class (or the Musielak-Orlicz class or generalized Orlicz class). The Musielak space (or Musielak-Orlicz space or generalized Orlicz space) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \Big\{u : \Omega \to \mathbb{R} \quad \textit{measurable} : \varrho_{\varphi,\Omega}(\frac{u}{\lambda}) < \infty \quad \text{for some} \quad \lambda > 0\Big\}.$$

For a Musielak function φ we put

$$\overline{\varphi}(x,s) = \sup_{t\geq 0}\Big\{st - \varphi(x,t)\Big\}.$$

$\overline{\varphi}$ is called the Musielak function complementary to φ (or conjugate of φ) in the sense of Young with respect to s.
we say that a sequence of function $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \varrho_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0.$$

In the space $L_\varphi(\Omega)$ we can define two norms, the first is called the Luxemburg norm, that is

$$\|u\|_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})\,dx \leq 1\Big\}$$

and the second so-called the Orlicz norm, that is

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_{\overline{\varphi}}\leq 1} \int_\Omega |u(x)\,v(x)|\,dx,$$

where $\overline{\varphi}$ is the Musielak function complementary to φ. These two norms are equivalent and we have a Musielak class $K_\varphi(\Omega)$ is a convex subset of the Musielak space $L_\varphi(\Omega)$.
The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $(E_{\overline{\varphi}}(\Omega))^* = L_\varphi(\Omega)$. We have $E_\varphi(\Omega) = K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if φ satisfies the Δ_2-condition for large values of t or for all values of t, according to whether Ω has finite measure or not.
We define

$$W^1L_\varphi(\Omega) = \Big\{u \in L_\varphi(\Omega) : D^\alpha \in L_\varphi(\Omega), \forall|\alpha| \leq 1\Big\}$$

$$W^1E_\varphi(\Omega) = \Big\{u \in E_\varphi(\Omega) : D^\alpha \in E_\varphi(\Omega), \forall |\alpha| \leq 1\Big\}$$

where $\alpha = (\alpha_1, ..., \alpha_N)$, $|\alpha| = |\alpha_1| + ... + |\alpha_N|$ and $D^\alpha u$ denotes the distributional derivatives. The space $W^1L_\varphi(\Omega)$ is called the Musielak-Sobolev space. For $u \in W^1L_\varphi(\Omega)$, let

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq 1} \varrho_{\varphi,\Omega}(D^\alpha u) \text{ and } \|u\|^1_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1\Big\}$$

these functionals are convex modular and a norm on $W^1L_\varphi(\Omega)$ respectively.

The pair $\langle W^1L_\varphi(\Omega), \|u\|^1_{\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition

$$\text{there exists a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) > c.$$

The space $W^1L_\varphi(\Omega)$ is identified to a subspace of the product $\Pi_{|\alpha|\leq 1}L_\varphi(\Omega) = \Pi L_\varphi$; this subspace is $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$ closed.

We denote by $\mathfrak{D}(\Omega)$ the Schwartz space of infinitely smooth function with compact support in Ω and by $\mathfrak{D}(\overline{\Omega})$ the restriction of $\mathfrak{D}(\mathbb{R}^N)$ on Ω. The space $W^1_0L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$ closure of $\mathfrak{D}(\Omega)$ in $W^1L_\varphi(\Omega)$ and the space $W^1_0E_\varphi(\Omega)$ as the (norm) closure of the Schwarz space $\mathfrak{D}(\Omega)$ in $W^1L_\varphi(\Omega)$.

For two complementary Musielak functions φ and $\overline{\varphi}$ we have

i)The Young inequality:

$$ts \leq \varphi(x, t) + \overline{\varphi}(x, s) \text{ for all } t, s \geq 0, x \in \Omega.$$

ii) The Hölder inequality:

$$\Big|\int_\Omega u(x)\, v(x)\, dx\Big| \leq 2\|u\|_{\varphi,\Omega}\, \|v\|_{\overline{\varphi},\Omega}, \text{ for all } u \in L_\varphi(\Omega), v \in L_{\overline{\varphi}}(\Omega).$$

We say that a sequence of function u_n converges to u for the modular convergence in $W^1L_\varphi(\Omega)$ (respectively in $W^1_0L_\varphi(\Omega)$) if we have

$$\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0, \text{ for some } \lambda > 0.$$

Define the following space of distributions

$$W^{-1}L_{\overline{\varphi}}(\Omega) = \Big\{f \in \mathfrak{D}'(\Omega) : f = \sum_{|\alpha|\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ where } f_\alpha \in L_{\overline{\varphi}}(\Omega)\Big\}$$

and

$$W^{-1}E_{\overline{\varphi}}(\Omega) = \Big\{f \in \mathfrak{D}'(\Omega) : f = \sum_{|\alpha|\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ where } f_\alpha \in E_{\overline{\varphi}}(\Omega)\Big\}.$$

2.3 Some technical lemmas

Definition 2.1. Recall that an open domain $\Omega \subset \mathbb{R}^N$ has the segment property (see [34]) if there exist a locally finite open covering O_i of the boundary $\partial\Omega$ of Ω and a corresponding vectors y_i such that if $x \in \overline{\Omega} \cap O_i$ for some i, then $x + ty_i \in \Omega$ for $0 < t < 1$.

Lemma 2.2. [8] Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, $(N \geq 2)$ and let φ be a Musielak function satisfying the log-Holder continuity such that

$$\overline{\varphi}(x, 1) \leq c \text{ a.e in } \Omega \text{ for some } c > 0,$$

then $\mathfrak{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ and in $W^1_0L_\varphi(\Omega)$ for the modular convergence.

Consequently, the action of a distribution in $W^{-1}L_{\overline{\varphi}}(\Omega)$ on an element u of $W^1_0L_\varphi(\Omega)$ is well defined.

Lemma 2.3. [7]**(The Nemytskii operator)** Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak functions. Let $f : \Omega \times \mathbb{R}^{p_1} \to \mathbb{R}^{p_2}$ be a Caratheodory function such that

$$|f(x,s)| \leq c(x) + k_1 \psi_x^{-1}(\varphi(x, k_2|s|)),$$

for almost every $x \in \Omega$ and all $s \in \mathbb{R}^{p_1}$, where k_1, k_2 are real positive constant and $c \in E_\psi(\Omega)$. Then the Nemytskii operator N_f, defined by $N_f(u)(x) = f(x, u(x))$ is continuous from $\Big(\mathbf{P}(E_\varphi(\Omega), \frac{1}{k_2})\Big)^{p_1} = \prod \Big\{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega) < \frac{1}{k_2}\Big\}$ into $\Big(L_\psi\Big)^{p_2}$ for the modular convergence.
Furthermore, if $c \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\Big(\mathbf{P}(E_\varphi(\Omega), \frac{1}{k_2})\Big)^{p_1}$ into $\Big(E_\gamma(\Omega)\Big)^{p_2}$.

Lemma 2.4. If $f_n \subset L^1(\Omega)$ with $f_n \to f \in L^1(\Omega)$ a.e. in Ω, $f_n, f \geq 0$ a.e. in Ω and $\displaystyle\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx$, then $f_n \to f$ in $L^1(\Omega)$.

We will use the following real functions of a real variable

$$T_k(s) = \max(-k, \min(k, s)), \quad G_k(s) = s - T_k(s) \quad \forall k > 0,$$

and

$$\phi_\lambda(s) = s\exp(\lambda s^2), \text{ where } \lambda \text{ is a positive real number.}$$

Lemma 2.5. If c and d are positive real numbers such that $\lambda = \Big(\dfrac{c}{2d}\Big)^2$ then

$$d\phi_\lambda'(s) - c|\phi_\lambda(s)| \geq \frac{d}{2}, \quad \forall s \in \mathbb{R}.$$

2.4 Decreasing rearrangement

We recall the definition of decreasing rearrangement of a real-valued measurable function u in a measurable subset Ω of $\mathbb{R}^N$ having finite measure. Let $|E|$ stands the Lebesgue measure of a subset E of Ω. The distribution function of u, denoted by μ_u, is a map which informs about the content of level sets of u, that is

$$\mu_u(t) = \Big|\{x \in \Omega : |u(x)| > t\}\Big|, \quad t \geq 0.$$

The decreasing rearrangement of u is defined as the generalized inverse function of μ_u, that is the function $u^* : [0, |\Omega|] \to [0, +\infty]$, defined as

$$u^*(s) = \inf\Big\{t \geq 0 : \mu_u(t) \leq s\Big\}, \quad s \in [0, |\Omega|].$$

In other words, u^* is the (unique) non-increasing, right-continuous function in $[0, +\infty)$ equi-distributed with u. Furthermore, for every $t \geq 0$

$$u^*(\mu_u(t)) \leq t. \tag{2.1}$$

We also recall that

$$u^*(0) = ess\sup|u|. \tag{2.2}$$

3 Statement of the problem and main result

Through this paper Ω will be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, satisfying the segment property and φ is a doubling Musielak function. There exists an Orlicz functions q (see remark 3.1 below) such that

$$q(t) \leq \varphi(x,t).$$

Let us consider the following convex set

$$\mathbf{K}_\psi = \Big\{v \in W_0^1 L_\varphi(\Omega) : v \geq \psi \text{ a. e. in } \Omega\Big\} \tag{3.1}$$

where $\psi : \Omega \to \overline{\mathbb{R}}$ is a measurable function. On the convex $\mathbf{K}_\psi$ we assume that

(A_1) $\psi^+ \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$,

(A_2) For each $v \in \mathbf{K}_\psi \cap L^\infty(\Omega)$, there exists a sequence $\{v_j\} \subset \mathbf{K}_\psi \cap W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ such that $v_j \to v$ for the modular convergence.

Let $A : D(A) \subset W_0^1 L_\varphi(\Omega) \to W^{-1} L_{\overline{\varphi}}(\Omega)$ be the mapping (non-everywhere defined) given by

$$\mathrm{A}\, u = -\operatorname{div}\, a(x, u, \nabla u),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function satisfying, for almost every $x \in \Omega$ and for all $s \in \mathbb{R}, \xi, \eta \in \mathbb{R}^N (\xi \neq \eta)$ the following conditions

(A_3) The vector field $a(.,.,.)$ verifies the degenerate coercivity,

$$a(x, s, \xi).\xi \geq \overline{\varphi}_x^{-1}(\varphi(x, h(|s|)))\varphi(x, |\xi|) \tag{3.2}$$

where $h : \mathbb{R} \to \mathbb{R}_+^*$ is a continuous decreasing function such that : $h(0) \leq 1$ and its primitive $H(s) = \int_0^s h(t)\, dt$ is unbounded.

(A_4) There exist a function $c(x) \in E_{\overline{\varphi}}(\Omega)$ and some positive constants k_1, k_2, k_3 and k_4 and a Musielak function $P \prec\prec \varphi$ such that

$$|a(x, s, \xi)| \leq c(x) + k_1 \overline{P}_x^{-1}(\varphi(x, k_2|s|)) + k_3 \overline{\varphi}_x^{-1}(\varphi(x, k_4|\xi|)). \tag{3.3}$$

(A_5) a is strictly monotone

$$\Big(a(x, s, \xi) - a(x, s, \eta)\Big) \cdot \Big(\xi - \eta\Big) > 0 \tag{3.4}$$

Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Caratheodory function satisfying

(A_6) For all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ and for almost every $x \in \Omega$, g satisfies the natural growth,

$$|g(x, s, \xi)| \leq \beta(s)\varphi(x, |\xi|), \tag{3.5}$$

where $\beta : \mathbb{R} \to \mathbb{R}_+$ is a continuous function. We assume that the function $t \to \dfrac{\beta(t)}{\overline{q}^{-1}(q(h(|t|)))}$ belongs to $L^1(\mathbb{R})$ with $q(t) \leq \varphi(x, t)$ for all $(x, t) \in \Omega \times \mathbb{R}^+$. So that defining

$$\gamma(s) = \int_0^s \frac{\beta(t)}{\overline{q}^{-1}(q(h(|t|)))}\, dt,$$

for all $s \in \mathbb{R}$, we have that the function γ is bounded.

For that concerns the right hand, we assume one of the following two assumptions: Either

$$f \in L^N(\Omega), \tag{3.6}$$

or

$$\begin{cases} f \in L^m(\Omega) \text{ with } m = \frac{rN}{r+1} \text{ for some } r > 0, \\ \quad \textit{and} \displaystyle\int_{.}^{+\infty} \Big(\frac{t}{q(t)}\Big)^r dt < +\infty. \end{cases} \tag{3.7}$$

Remark 3.1. Notice that, in particular case, in variable exponent spaces when $\varphi(x, t) = t^{p(x)}$, the Orlicz function $q(t)$ plays the role of t^{p^-}, where $p^- = ess \inf_{x \in \Omega} p(x)$. Moreover, $q(t)$ is an N-function satisfying the Δ_2-condition, for its construction, see [27, Lemma A.4].

Lemma 3.2. Let φ be a Musielak function and q is an Orlicz function such that $q(t) \leq \varphi(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}^+$ then

$$\overline{q}^{-1}(q(t)) \leq \overline{\varphi}_x^{-1}(\varphi(x,t)) \text{ for all } (x,t) \in \Omega \times \mathbb{R}^+,$$

where $\overline{q}$ and $\overline{\varphi}$ are the complementary functions of q and φ respectively.

Proof . Let $s,t \in \mathbb{R}^+$ and $x \in \Omega$. We have $q(t) \leq \varphi(x,t)$, then

$$st - q(t) \geq st - \varphi(x,t).$$

Passing to the sup over $t \geq 0$

$$\sup_{t\geq 0}\{st - q(t)\} \geq \sup_{t\geq 0}\{st - \varphi(x,t)\}.$$

That means

$$\overline{q}(s) \geq \overline{\varphi}(x,s) := \overline{\varphi}_x(s), \text{ for all } s \in \mathbb{R}^+.$$

It follows that for all $s \in \mathbb{R}^+$,

$$\overline{q}^{-1}(s) \leq \overline{\varphi}_x^{-1}(s)$$

Taking $s = q(t)$, since $\overline{\varphi}_x^{-1}$ is increasing, we have $\forall t \in \mathbb{R}^+$, $\overline{q}^{-1}(q(t)) \leq \overline{\varphi}_x^{-1}(q(t)) \leq \overline{\varphi}_x^{-1}(\varphi(x,t))$. □

Our main result reads as the following.

Theorem 3.3. Suppose that the assumptions $(A_1)-(A_6)$, either (3.6) or (3.7) are satisfied and the modular function φ verifies the assumptions (Φ_1) and (Φ_2), mentioned in the introduction. Then the following obstacle problem

$$\begin{cases} u \in \mathbf{K}_\psi \cap L^\infty(\Omega), \quad a(\cdot,u,\nabla u) \in (L_{\overline{\varphi}}(\Omega))^N, \quad g(\cdot,u,\nabla u) \in L^1(\Omega), \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla(u-v)\,dx + \int_\Omega g(x,u,\nabla u)(u-v)\,dx \\ \displaystyle\leq \int_\Omega f(u-v)\,dx, \quad \forall v \in \mathbf{K}_\psi \cap L^\infty(\Omega). \end{cases} \tag{3.8}$$

has at least one weak bounded solution.

4 Proof of the main result

The proof of theorem 3.3 is divided into eight steps.

Step 1: Approximate problems. For $n \in \mathbb{N}^*$ let us denote by m^* either N or m according as we assume (3.6) or (3.7). Define $f_n := T_n(f)$, $A_n(u) := -\operatorname{div} a(x,T_n(u),\nabla u)$ and $g_n(x,s,\xi) := T_n(g(x,s,\xi))$. We can easily see that we have $|g_n(x,s,\xi)| \leq |g(x,s,\xi)|$ and $|g_n(x,s,\xi)| \leq n$. Let us consider the sequence of approximate problem,

$$\begin{cases} u_n \in \mathbf{K}_\psi \cap D(A_n), \\ \displaystyle\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(u_n-v)\,dx + \int_\Omega g(x,u_n,\nabla u_n)(u_n-v)\,dx \\ \displaystyle\leq \int_\Omega f_n(u_n-v)\,dx, \quad \forall v \in \mathbf{K}_\psi. \end{cases} \tag{4.1}$$

Let ν be large enough. By (3.4) and Cauchy-Swhartz's inequality one has

$$\begin{aligned} -a(x,T_n(s),\xi)\cdot\nabla\psi^+ &\geq -\frac{1}{\nu}a(x,T_n(s),\xi)\cdot\xi - a(x,T_n(s),\nu\nabla\psi^+)\cdot\nabla\psi^+ \\ &\quad - \overline{\varphi}_x^{-1}(\varphi(x,h(|T_n(s)|)))\frac{\nu-1}{2\nu}\frac{|a(x,T_n(s),\nu\nabla\psi^+)|}{\overline{\varphi}_x^{-1}(\varphi(x,h(|T_n(s)|)))\frac{\nu-1}{2}}|\xi|. \end{aligned}$$

Then, Young's inequality enables us to get

$$\begin{aligned} -a(x,T_n(s),\xi)\cdot\nabla\psi^+ &\geq -\frac{1}{\nu}a(x,T_n(s),\xi)\cdot\xi - a(x,T_n(s),\nu\nabla\psi^+)\cdot\nabla\psi^+ \\ &\quad - \overline{\varphi}_x^{-1}(\varphi(x,h(|T_n(s)|)))\frac{\nu-1}{2\nu}\overline{\varphi}\Big(x,\frac{|a(x,T_n(s),\nu\nabla\psi^+)|}{\overline{\varphi}_x^{-1}(\varphi(x,h(|T_n(s)|)))\frac{\nu-1}{2}}\Big) \\ &\quad - \overline{\varphi}_x^{-1}(\varphi(x,h(|T_n(s)|)))\frac{\nu-1}{2\nu}\varphi(x,|\xi|). \end{aligned}$$

Let us define the positive real number depending on x, $\rho_n(x) := \overline{\varphi}_x^{-1}(\varphi(x,h(n)))\dfrac{\nu-1}{2\nu}$ and the function γ_n by

$$\begin{aligned}\gamma_n(x) &:= a(x,T_n(s),\nu\nabla\psi^+)\cdot\nabla\psi^+\\ &\quad+\overline{\varphi}_x^{-1}(\varphi(x,h(0)))\frac{\nu-1}{2\nu}\overline{\varphi}\Big(x,\frac{|a(x,T_n(s),\nu\nabla\psi^+)|}{\overline{\varphi}_x^{-1}(\varphi(x,h(|T_n(s)|)))\frac{\nu-1}{2}}\Big).\end{aligned}$$

For each $n\in\mathbb{N}$ the function γ_n belongs to $L^1(\Omega)$. Thus we have

$$a(x,T_n(s),\xi)\cdot(\xi-\nabla\psi^+)\geq\rho_n(x)\varphi(x,|\xi|)-\gamma_n(x).$$

Then, by lemma 3.2 $\rho_n(x)\geq\rho_n^*:=\overline{q}^{-1}(q(h(n)))\dfrac{\nu-1}{2\nu}$, and we have

$$a(x,T_n(s),\xi)\cdot(\xi-\nabla\psi^+)\geq\rho_n(x)\varphi(x,|\xi|)-\gamma_n(x)\geq\rho_n^*\varphi(x,|\xi|)-\gamma_n(x).$$

By [13], the operator A_n satisfies the conditions with respect to ψ^+, then the variational inequality (4.1) has at least a solution u_n.

Step 2: Preliminary results.

Lemma 4.1. Let u_n be a solution of (4.1). For all $t,\epsilon\in\mathbb{R}_+^*$ with $t>\|\psi^+\|_\infty$, one has the following inequality:

$$\begin{aligned}&\int_{\{t\leq u_n\leq t+\epsilon\}}a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{e}^{\gamma(u_n^+)}\,dx\\ &\leq\int_{\{u_n>t\}}f_n^+\,\mathrm{e}^{\gamma(u_n^+)}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}((u_n^+-\|\psi^+\|_\infty)^+))dx.\end{aligned}\tag{4.2}$$

Proof Let ϵ,t,k in $\mathbb{R}_+^*$ with $t>\|\psi^+\|_\infty$. Define

$$v=u_n-\eta\,\mathrm{e}^{\gamma(T_k(u_n^+))}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(T_k(w_n)))$$

where $w_n=(u_n^+-\|\psi^+\|_\infty)^+$ and $\eta=\mathrm{e}^{-\gamma(k)}$. Thus, using v as test function in (4.1) and then using (3.2) we get

$$\begin{aligned}&\int_\Omega\overline{\varphi}_x^{-1}(\varphi(x,h(|T_k(u_n^+|)))\varphi(x,|\nabla T_k(u_n^+|)\\ &\times\frac{\beta(T_k(u_n^+))}{\overline{q}^{-1}(q(h(|T_k(u_n^+|)))}\,\mathrm{e}^{\gamma(T_k(u_n^+))}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(T_k(w_n)))\,dx\\ &+\int_{\{t-\|\psi^+\|_\infty<T_k(w_n)\leq t-\|\psi^+\|_\infty+\epsilon\}}a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(w_n)\,\mathrm{e}^{\gamma(T_k(u_n^+))}\,dx\\ &+\int_\Omega g_n(x,u_n,\nabla u_n)\,\mathrm{e}^{\gamma(T_k(u_n^+))}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(T_k(w_n)))\,dx\\ &\leq\int_\Omega f_n\,\mathrm{e}^{\gamma(T_k(u_n^+))}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(T_k(w_n)))\,dx.\end{aligned}\tag{4.3}$$

By lemma 3.2 $\dfrac{\overline{\varphi}_x^{-1}(\varphi(x,h(|T_k(u_n^+|)))}{\overline{q}^{-1}(q(h(|T_k(u_n^+|)))}\geq1$ and then (4.3) becomes

$$\begin{aligned}&\int_\Omega\beta(T_k(u_n^+))\varphi(x,|\nabla T_k(u_n^+|)\,\mathrm{e}^{\gamma(T_k(u_n^+))}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(T_k(w_n)))\,dx\\ &+\int_{\{t-\|\psi^+\|_\infty<T_k(w_n)\leq t-\|\psi^+\|_\infty+\epsilon\}}a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(w_n)\,\mathrm{e}^{\gamma(T_k(u_n^+))}\,dx\\ &+\int_\Omega g_n(x,u_n,\nabla u_n)\,\mathrm{e}^{\gamma(T_k(u_n^+))}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(T_k(w_n)))\,dx\\ &\leq\int_\Omega f_n\,\mathrm{e}^{\gamma(T_k(u_n^+))}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(T_k(w_n)))\,dx.\end{aligned}\tag{4.4}$$

Now, we will pass to the limit as k tends to $+\infty$ in (4.4). In the first integral in the left-hand side of (4.4) the integrand function is nonnegative, so that Fatou's lemma allows us to get

$$\begin{aligned}&\int_\Omega\beta(u_n^+)\varphi(x,|\nabla u_n^+|)\,\mathrm{e}^{\gamma(u_n^+)}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(w_n))\,dx\\ &\leq\liminf_{k\to\infty}\int_\Omega\beta(T_k(u_n^+))\varphi(x,|\nabla T_k(u_n^+|)\,\mathrm{e}^{\gamma(T_k(u_n^+))}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(T_k(w_n)))\,dx\end{aligned}$$

Observe that the second integral in the left-hand side of (4.4) reads as

$$\begin{aligned}&\int_{\{t-\|\psi^+\|_\infty<T_k(w_n)\leq t-\|\psi^+\|_\infty+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(w_n)\,\mathrm{e}^{\gamma(T_k(u_n^+))}\ dx\\ &=\int_{\{t<u_n^+\leq t+\epsilon\}\cap\{0<u_n^+-\|\psi^+\|_\infty<k\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n^+\,\mathrm{e}^{\gamma(T_k(u_n^+))}\ dx.\end{aligned}$$

It follows by applying the monotone convergence theorem, that

$$\begin{aligned}&\lim_{k\to+\infty}\int_{\{t-\|\psi^+\|_\infty<T_k(w_n)\leq t-\|\psi^+\|_\infty+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(w_n)\,\mathrm{e}^{\gamma(T_k(u_n^+))}\ dx\\ &=\int_{\{t<u_n^+\leq t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n^+\,\mathrm{e}^{\gamma(u_n^+)}\ dx,\end{aligned}$$

while for the remaining terms in (4.4), being g_n and f_n bounded, we apply the Lebesgue's dominated convergence theorem. Consequently, letting k tends to $+\infty$ in (4.4) we obtain

$$\begin{aligned}&\int_\Omega \beta(u_n^+)\varphi(x,|\nabla u_n^+|)\,\mathrm{e}^{\gamma(u_n^+)}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(w_n))\,dx\\ &+\int_{\{t<u_n^+\leq t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{e}^{\gamma(u_n^+)}\ dx\\ &+\int_\Omega g_n(x,u_n,\nabla u_n)\,\mathrm{e}^{\gamma(u_n^+)}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(w_n))\,dx\\ &\leq\int_\Omega f_n\,\mathrm{e}^{\gamma(u_n^+)}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(w_n))\,dx.\end{aligned}\tag{4.5}$$

Due to the fact that $u_n^+\geq\psi^+$, the function w_n is equal to zero if $u_n\leq 0$. By virtue of (3.5) we get

$$\begin{aligned}&\int_\Omega \beta(u_n^+)\varphi(x,|\nabla u_n^+|)\,\mathrm{e}^{\gamma(u_n^+)}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(w_n))\,dx\\ &+\int_\Omega g_n(x,u_n,\nabla u_n)\,\mathrm{e}^{\gamma(u_n^+)}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(w_n))\,dx\\ &=\int_\Omega \beta(u_n^+)\varphi(x,|\nabla u_n^+|)\,\mathrm{e}^{\gamma(u_n^+)}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(w_n))\,dx\\ &+\int_{\{u_n>0\}} g_n(x,u_n,\nabla u_n)\,\mathrm{e}^{\gamma(u_n^+)}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(w_n))\,dx\geq 0.\end{aligned}$$

Hence, (4.5) is reduced to

$$\begin{aligned}&\int_{\{t<u_n^+\leq t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{e}^{\gamma(u_n^+)}\ dx\\ &\leq\int_\Omega f_n\,\mathrm{e}^{\gamma(u_n^+)}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}(w_n))\,dx.\end{aligned}$$

Since $T_\epsilon(G_{t-\|\psi^+\|_\infty}(w_n))$ is different from zero only on the subset

$$\{w_n>t-\|\psi^+\|_\infty\}=\{u_n^+>t\},$$

and $f_n\leq f_n^+$ we finally have

$$\begin{aligned}&\int_{\{t<u_n^+\leq t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{e}^{\gamma(u_n^+)}\ dx\\ &\leq\int_{\{u_n>t\}} f_n^+\,\mathrm{e}^{\gamma(u_n^+)}\,T_\epsilon(G_{t-\|\psi^+\|_\infty}((u_n^+-\|\psi^+\|_\infty)^+))\,dx.\end{aligned}$$

Lemma 4.2. Let u_n be a solution of (4.1). For all $t,\epsilon\in\mathbb{R}_+^*$, one has the following inequality:

$$\begin{aligned}&\int_{\{-t-\epsilon\leq u_n<-t\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{e}^{\gamma(u_n^-)}\ dx\\ &\leq\int_{\{u_n<-t\}} f_n^-\,\mathrm{e}^{\gamma(u_n^-)}\,T_\epsilon(G_t(u_n^-))dx.\end{aligned}\tag{4.6}$$

Proof. For all $k>0$, define the function $v=u_n+\mathrm{e}^{\gamma(T_k(u_n^-))}\,T_\epsilon(G_t(T_k(u_n^-)))$, the choice of v as test function in (4.1), yields

$$\begin{aligned}
&-\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)\,\mathrm{e}^{\gamma(T_k(u_n^-))}\,T_\epsilon(G_t(T_k(u_n^-)))\\
&\times\frac{\beta(T_k(u_n^-))}{\overline{q}^{-1}(q(h(|T_k(u_n^-)|)))}\,dx\\
&-\int_{\{t<T_k(u_n^-)\le t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)\,\mathrm{e}^{\gamma(T_k(u_n^-))}\,dx\\
&-\int_\Omega g_n(x,u_n,\nabla u_n)\,\mathrm{e}^{\gamma(T_k(u_n^-))}\,T_\epsilon(G_t(T_k(u_n^-)))\,dx\\
&\le-\int_\Omega f_n\,\mathrm{e}^{\gamma(T_k(u_n^-))}\,T_\epsilon(G_t(T_k(u_n^-)))\,dx.
\end{aligned}\tag{4.7}$$

The first integral in the left-hand side of (4.7) is written as

$$\begin{aligned}
&-\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)\,\mathrm{e}^{\gamma(T_k(u_n^-))}\,T_\epsilon(G_t(T_k(u_n^-)))\\
&\times\frac{\beta(T_k(u_n^-))}{\overline{q}^{-1}(q(h(|T_k(u_n^-)|)))}\,dx\\
&=\int_{\{-k<u_n\le 0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{e}^{\gamma(u_n^-)}\,T_\epsilon(G_t(u_n^-))\\
&\times\frac{\beta(T_k(u_n^-))}{\overline{q}^{-1}(q(h(|T_k(u_n^-)|)))}\,dx.
\end{aligned}$$

By the monotone convergence theorem, we have

$$\begin{aligned}
&-\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)\,\mathrm{e}^{\gamma(T_k(u_n^-))}\,T_\epsilon(G_t(T_k(u_n^-)))\\
&\times\frac{\beta(T_k(u_n^-))}{\overline{q}^{-1}(q(h(|T_k(u_n^-)|)))}\,dx\\
&\to\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{e}^{\gamma(u_n^-)}\,T_\epsilon(G_t(u_n^-))\frac{\beta(T_k(u_n^-))}{\overline{q}^{-1}(q(h(|T_k(u_n^-)|)))}\,dx.
\end{aligned}$$

as $k\to+\infty$. For the second integral in the left-hand side of (4.7), we write

$$\begin{aligned}
&-\int_{\{t<T_k(u_n^-)\le t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)\,\mathrm{e}^{\gamma(T_k(u_n^-))}\,dx\\
&=\int_{\{t<T_k(u_n^-)\le t+\epsilon\}\cap\{-k<u_n\le 0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{e}^{\gamma(u_n^-)}\,dx\\
&=\int_{\{-t-\epsilon<u_n\le -t\}\cap\{-k<u_n\le 0\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{e}^{\gamma(u_n^-)}\,dx.
\end{aligned}$$

Applying again the monotone convergence theorem, we obtain

$$\begin{aligned}
&-\int_{\{t<T_k(u_n^-)\le t+\epsilon\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n^-)\,\mathrm{e}^{\gamma(T_k(u_n^-))}\,dx\\
&\to\int_{\{-t-\epsilon<u_n\le -t\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{e}^{\gamma(u_n^-)}\,dx.
\end{aligned}$$

as $k\to+\infty$. For the remaining terms in (4.7), being g_n and f_n bounded, we apply the Lebesgue's dominated convergence theorem. Consequently, letting k tends to $+\infty$ in (4.7) we obtain

$$\begin{aligned}
&\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{e}^{\gamma(u_n^-)}\,T_\epsilon(G_t(u_n^-))\frac{\beta(u_n^-)}{\overline{q}^{-1}(q(h(|u_n^-|)))}\,dx\\
&+\int_{\{-t-\epsilon<u_n)\le -t\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla u_n\,\mathrm{e}^{\gamma(u_n^-)}\,dx\\
&-\int_\Omega g_n(x,u_n,\nabla u_n)\,\mathrm{e}^{\gamma(u_n^-)}\,T_\epsilon(G_t(u_n^-))\,dx\\
&\le-\int_\Omega f_n\,\mathrm{e}^{\gamma(u_n^-)}\,T_\epsilon(G_t(u_n^-))\,dx.
\end{aligned}\tag{4.8}$$

Since $u_n^- = |u_n|$ on the set $\{x \in \Omega : u_n(x) \le 0\}$, using (3.2) we obtain

$$\begin{aligned}
&\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, \mathrm{e}^{\gamma(u_n^-)} \, T_\epsilon(G_t(u_n^-)) \frac{\beta(u_n^-)}{\overline{q}^{-1}(q(h(|u_n^-|)))} \, dx \\
&- \int_\Omega g_n(x, u_n, \nabla u_n) \, \mathrm{e}^{\gamma(u_n^-)} \, T_\epsilon(G_t(u_n^-)) \, dx \\
&\ge \int_\Omega \overline{\varphi}_x^{-1}(\varphi(x, h(|u_n|)))\varphi(x, |\nabla u_n|) \, \mathrm{e}^{\gamma(u_n^-)} \, T_\epsilon(G_t(u_n^-)) \\
&\times \frac{\beta(u_n^-)}{\overline{q}^{-1}(q(h(|u_n^-|)))} \, dx \\
&- \int_\Omega g_n(x, u_n, \nabla u_n) \, \mathrm{e}^{\gamma(u_n^-)} \, T_\epsilon(G_t(u_n^-)) \, dx
\end{aligned}$$

By lemma 3.2 $\dfrac{\overline{\varphi}_x^{-1}(\varphi(x, h(|T_k(u_n^-)|)))}{\overline{q}^{-1}(q(h(|T_k(u_n^-)|)))} \ge 1$ and then

$$\begin{aligned}
&\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, \mathrm{e}^{\gamma(u_n^-)} \, T_\epsilon(G_t(u_n^-)) \frac{\beta(u_n^-)}{\overline{q}^{-1}(q(h(|u_n^-|)))} \, dx \\
&- \int_\Omega g_n(x, u_n, \nabla u_n) \, \mathrm{e}^{\gamma(u_n^-)} \, T_\epsilon(G_t(u_n^-)) \, dx \\
&\ge \int_\Omega \beta(u_n^-)\varphi(x, |\nabla u_n|) \, \mathrm{e}^{\gamma(u_n^-)} \, T_\epsilon(G_t(u_n^-)) \, dx \\
&- \int_\Omega g_n(x, u_n, \nabla u_n) \, \mathrm{e}^{\gamma(u_n^-)} \, T_\epsilon(G_t(u_n^-)) \, dx \ge 0, \text{ by (3.5)}
\end{aligned}$$

Observing that $-f_n \le f_n^-$ and $\{u_n^- > t\} \cap \{u_n \le 0\} = \{u_n < -t\}$, we have finally

$$\begin{aligned}
&\int_{\{-t-\epsilon < u_n \le -t\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, \mathrm{e}^{\gamma(u_n^-)} \, dx \\
&\le \int_{\{u_n < -t\}} f_n^- \, \mathrm{e}^{\gamma(u_n^-)} \, T_\epsilon(G_t(u_n^-)) \, dx.
\end{aligned}$$

Lemma 4.3. Let u_n be a solution of (4.1). There exists a constant c_0, not depending on n, such that for almost every $t > \|\psi^+\|_\infty$ and all $\epsilon > 0$, one has the following inequality:

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_{\{|u_n| > t\}} \overline{\varphi}_x^{-1}(\varphi(x, h(|u_n|)))\varphi(x, |\nabla u_n|) \, dx \le c_0 \int_{\{|u_n| > t\}} |f_n| \, dx. \tag{4.9}$$

Proof. Being γ bounded, summing up both inequalities (4.2) and (4.6) there exists a constant c_0 not depending on n such that for almost every $t > \|\psi^+\|_\infty$ and all $\epsilon > 0$

$$\int_{\{t < |u_n| \le t+\epsilon\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx \le \epsilon c_0 \int_{\{|u_n| > t\}} |f_n| \, dx.$$

Using (3.2), dividing by ϵ and then letting ϵ tends to o^+ we obtain (4.9).

Lemma 4.4. Let $K(t) = \dfrac{q(t)}{t}$ and $\mu_n(t) = |\{x \in \Omega : |u_n(x)| > t\}|$, for all $t > 0$. We have for almost every $t > \|\psi^+\|_\infty$:

$$h(t) \le \frac{2q(1)(-\mu_n'(t))}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}} K^{-1}\left(\frac{c_0 \displaystyle\int_{\{|u_n| > t\}} |f_n| \, dx}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}} \right). \tag{4.10}$$

where C_N stands for the measure of the unit ball in $\mathbb{R}^N$ and c_0 is the constant which appears in (4.9).

Proof . The hypotheses made on the N-function q, which are not a restriction, allow to affirm that the function $Q(t) = \dfrac{1}{K^{-1}(t)}$ is decreasing and convex (see [40]). By lemma 3.2, $\varphi(x, |\nabla u_n|) \ge q(|\nabla u_n|) \ge 0$ and $\overline{\varphi}_x^{-1}(\varphi(x, h(|u_n|))) \ge \overline{q}^{-1}(q(h(|u_n|))) \ge 0$, then

$$\overline{\varphi}_x^{-1}(\varphi(x, h(|u_n|)))\varphi(x, |\nabla u_n|) \ge \overline{q}^{-1}(q(h(|u_n|)))q(|\nabla u_n|)$$

Using the fact that Q is decreasing and then Jensen's inequality yield

$$\begin{aligned}
&Q\left(\frac{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{\varphi}_x^{-1}(\varphi(x,h(|u_n|)))\varphi(x,|\nabla u_n|)dx}{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{q}^{-1}(q(h(|u_n|)))|\nabla u_n|dx}\right)\\
&\le Q\left(\frac{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{q}^{-1}(q(h(|u_n|)))q(|\nabla u_n|)dx}{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{q}^{-1}(q(h(|u_n|)))|\nabla u_n|dx}\right)\\
&= Q\left(\frac{\displaystyle\int_{\{t<|u_n|\le t+k\}} K(|\nabla u_n|)\overline{q}^{-1}(q(h(|u_n|)))|\nabla u_n|dx}{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{q}^{-1}(q(h(|u_n|)))|\nabla u_n|dx}\right)\\
&\le \frac{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{q}^{-1}(q(h(|u_n|)))dx}{\displaystyle\int_{\{t<|u_n|\le t+k\}} \overline{q}^{-1}(q(h(|u_n|)))|\nabla u_n|dx}\\
&\le \frac{\overline{q}^{-1}(q(h(t)))(-\mu_n(t+k)+\mu_n(t))}{\overline{q}^{-1}(q(h(t+k)))\displaystyle\int_{\{t<|u_n|\le t+k\}} |\nabla u_n|dx}.
\end{aligned}$$

Taking into account that $\overline{q}^{-1}(q(h(t))) \le \overline{q}^{-1}(q(1))$, using the convexity of Q and then letting $k \to 0^+$, we obtain for almost every $t > 0$,

$$\begin{aligned}
&\frac{\overline{q}^{-1}(q(1))}{\overline{q}^{-1}(q(h(t)))}\, Q\left(\frac{-\frac{d}{dt}\displaystyle\int_{\{|u_n|>t\}} \overline{\varphi}_x^{-1}(\varphi(x,h(|u_n|)))\varphi(x,|\nabla u_n|)dx}{\overline{q}^{-1}(q(1))(-\frac{d}{dt}\displaystyle\int_{\{|u_n|>t\}} |\nabla u_n|dx)}\right)\\
&\le \frac{-\mu_n'(t)}{-\frac{d}{dt}\displaystyle\int_{\{|u_n|>t\}} |\nabla u_n|dx}.
\end{aligned}$$

Now we recall the following inequality from [40]:

$$-\frac{d}{dt}\int_{\{|u_n|>t\}} |\nabla u_n|dx \ge NC_N^{1/N}\mu_n(t)^{1-\frac{1}{N}} \quad \text{for almost every } t>0. \tag{4.11}$$

Combining (4.9) and (4.11) and using the monotonicity of the function Q we obtain

$$\begin{aligned}
&\frac{1}{\overline{q}^{-1}(q(h(t)))}\\
&\le \frac{-\mu_n'(t)}{\overline{q}^{-1}(q(1))NC_N^{1/N}\mu_n(t)^{1-\frac{1}{N}}}K^{-1}\left(\frac{\displaystyle\int_{\{|u_n|>t\}} |f_n|dx}{\overline{q}^{-1}(q(1))NC_N^{1/N}\mu_n(t)^{1-\frac{1}{N}}}\right).
\end{aligned}$$

Using the inequality in Orlicz spaces

$$q(t) \le t\overline{q}^{-1}(q(t)) \le 2q(t) \quad \text{for all } t \ge 0$$

and the fact that $0 < h(t) \le 1$, we obtain (4.10). □

step 3: Uniform L^∞-estimation. From (4.2), using the rearrangement technics, we prove that there exists a constant c_∞ such that

$$\|u_n\|_\infty \le c_\infty. \tag{4.12}$$

Proof . If we assume (3.6), by Hölder's inequality one has

$$\int_{\{|u_n|>t\}} |f_n|dx \leq \|f\|_N \mu_n(t)^{1-\frac{1}{N}}.$$

Then for almost every $t > \|\psi^+\|_\infty$, inequality (4.10) becomes

$$h(t) \leq \frac{2q(1)(-\mu_n'(t))}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}\mu_n(t)^{1-\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}}\right).$$

Then, integrating between $\|\psi^+\|_\infty$ and s, we get

$$\int_{\|\psi^+\|_\infty}^s h(t)dt \leq \frac{2q(1)}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}}\right)\int_{\|\psi^+\|_\infty}^s \frac{-\mu_n'(t)}{\mu_n(t)^{1-\frac{1}{N}}}dt.$$

So, one has

$$\begin{aligned} H(s) &\leq \int_0^{\|\psi^+\|_\infty} h(t)dt \\ &+\frac{2q(1)}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}}\right)\int_{\|\psi^+\|_\infty}^s \frac{-\mu_n'(t)}{\mu_n(t)^{1-\frac{1}{N}}}dt. \end{aligned}$$

Hence, a change of variables yields

$$H(s) \leq \|\psi^+\|_\infty + \frac{2q(1)}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}}\right)\int_{\mu_n(s)}^{|\Omega|} \frac{dt}{t^{1-\frac{1}{N}}}.$$

By (2.1) we get

$$H(u_n^*(\sigma)) \leq \|\psi^+\|_\infty + \frac{2q(1)}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}}\right)\int_{\sigma}^{|\Omega|} \frac{dt}{t^{1-\frac{1}{N}}}.$$

So that

$$H(u_n^*(0)) \leq \|\psi^+\|_\infty + \frac{2q(1)|\Omega|^{\frac{1}{N}}}{\overline{q}^{-1}(q(1))C_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}}\right).$$

Thanks to (2.2) and the fact that $\lim_{s\to+\infty} H(s) = +\infty$, we conclude that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Moreover, if we denote by H^{-1} the inverse function of H, one has:

$$\|u_n\|_\infty \leq H^{-1}\left(\|\psi^+\|_\infty + \frac{2q(1)|\Omega|^{\frac{1}{N}}}{\overline{q}^{-1}(q(1))C_N^{\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_N}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}}\right)\right). \tag{4.13}$$

We now assume that (3.7) is verified. Then, using again Hölder's inequality we have

$$\int_{\{|u_n|>t\}} |f_n|dx \leq \|f\|_m \mu_n(t)^{1-\frac{1}{m}}.$$

For almost every $t > \|\psi^+\|_\infty$, inequality (4.10) becomes

$$h(s) \leq \frac{2q(1)}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}}\frac{-\mu_n'(t)}{\mu_n(t)^{1-\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}\mu_n(t)^{\frac{1}{m}-\frac{1}{N}}}\right).$$

Integrating between $\|\psi^+\|_\infty$ and s, we get

$$\begin{aligned} &H(s) \leq \|\psi^+\|_\infty \\ &+\frac{2q(1)}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}}\int_{\|\psi^+\|_\infty}^s \frac{-\mu_n'(t)}{\mu_n(t)^{1-\frac{1}{N}}} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}\mu_n(t)^{\frac{1}{m}-\frac{1}{N}}}\right)dt. \end{aligned}$$

Then, a change of variables gives

$$H(s) \leq \\ \|\psi^+\|_\infty + \frac{2q(1)}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \int_{\mu_n(s)}^{|\Omega|} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}\sigma^{\frac{1}{m}-\frac{1}{N}}}\right)\frac{d\sigma}{\sigma^{1-\frac{1}{N}}}.$$

By virtue of (2.1) we get

$$H(u_n^*(\tau)) \leq \\ \|\psi^+\|_\infty + \frac{2q(1)}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \int_{\tau}^{|\Omega|} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}\sigma^{\frac{1}{m}-\frac{1}{N}}}\right)\frac{d\sigma}{\sigma^{1-\frac{1}{N}}}.$$

Then, by (2.2) we obtain

$$H(\|u_n\|_\infty) \leq \\ \|\psi^+\|_\infty + \frac{2q(1)}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}} \int_{0}^{|\Omega|} K^{-1}\left(\frac{c_0\|f\|_m}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}\sigma^{\frac{1}{m}-\frac{1}{N}}}\right)\frac{d\sigma}{\sigma^{1-\frac{1}{N}}}.$$

A change of variables gives

$$H(\|u_n\|_\infty) \leq \|\psi^+\|_\infty + \frac{2q(1)c_0^r\|f\|_m^r}{(\overline{q}^{-1}(q(1)))^{r+1}N^rC_N^{\frac{r+1}{N}}} \int_\lambda^{+\infty} rt^{-r-1}K^{-1}(t)dt,$$

where $\lambda = \dfrac{c_0\|f\|_m}{\overline{q}^{-1}(q(1))NC_N^{\frac{1}{N}}|\Omega|^{\frac{1}{rN}}}$. Then, by an integration by parts we obtain that

$$H(\|u_n\|_\infty) \leq \\ \|\psi^+\|_\infty + \frac{2q(1)c_0^r\|f\|_m^r}{(\overline{q}^{-1}(q(1)))^{r+1}N^rC_N^{\frac{r+1}{N}}}\left(\frac{K^{-1}(\lambda)}{\lambda^r} + \int_{K^{-1}(\lambda)}^{+\infty}\left(\frac{s}{q(s)}\right)^r ds\right).$$

The assumption made on H guarantees that the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$. Indeed, denoting by H^{-1} the inverse function of H, one has

$$\|u_n\|_\infty \leq \\ H^{-1}\left(\|\psi^+\|_\infty + \frac{2q(1)c_0^r\|f\|_m^r}{(\overline{q}^{-1}(q(1)))^{r+1}N^rC_N^{\frac{r+1}{N}}}\left(\frac{K^{-1}(\lambda)}{\lambda^r} + \int_{K^{-1}(\lambda)}^{+\infty}\left(\frac{s}{q(s)}\right)^r ds\right)\right). \tag{4.14}$$

Consequently, in both cases the sequence $\{u_n\}$ is uniformly bounded in $L^\infty(\Omega)$, so that in the sequel, we will denote by c_∞ the constant appearing either in (4.13) or in (4.14), that is :

$$\|u_n\|_\infty \leq c_\infty. \tag{4.15}$$

□

Step 4: Estimation in $W_0^1L_\varphi(\Omega)$. Using $v_n = u_n - \eta\phi_\lambda(u_n - \psi^+)$, where $\eta = \mathrm{e}^{-\lambda(c_\infty + \|\psi^+\|_\infty)^2}$, as test function in (4.1), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\cdot\nabla(u_n - \psi^+)\phi_\lambda'(u_n - \psi^+)\,dx \\ + \int_\Omega g_n(x, u_n, \nabla u_n)\phi_\lambda(u_n - \psi^+)\,dx \\ \leq \int_\Omega f_n\phi_\lambda(u_n - \psi^+)\,dx. \tag{4.16}$$

Let now ν be large enough. By (3.4) and Cauchy-Swhartz's inequality one has

$$-a(x, u_n, \nabla u_n)\cdot\nabla\psi^+ \geq -\frac{1}{\nu}a(x, u_n, \nabla u_n)\cdot\nabla u_n - a(x, u_n, \nu\nabla\psi^+)\cdot\nabla\psi^+ \\ - \overline{\varphi}_x^{-1}(\varphi(x, h(|u_n|)))\frac{\nu - 1}{2\nu}\frac{|a(x, u_n, \nu\nabla\psi^+)|}{\overline{\varphi}_x^{-1}(\varphi(x, h(|u_n|)))\frac{\nu-1}{2}}|\nabla u_n|.$$

Then, Young's inequality enables us to get

$$\begin{aligned}-a(x,u_n,\nabla u_n)\cdot\nabla\psi^+ \geq & -\frac{1}{\nu}a(x,u_n,\nabla u_n)\cdot\nabla u_n - a(x,u_n,\nu\nabla\psi^+)\cdot\nabla\psi^+\\ & -\overline{\varphi}_x^{-1}(\varphi(x,h(|u_n|)))\frac{\nu-1}{2\nu}\overline{\varphi}\Big(x,\frac{|a(x,u_n,\nu\nabla\psi^+)|}{\overline{\varphi}_x^{-1}(\varphi(x,h(|u_n|)))\frac{\nu-1}{2}}\Big)\\ & -\overline{\varphi}_x^{-1}(\varphi(x,h(|u_n|)))\frac{\nu-1}{2\nu}\varphi(x,|\nabla u_n|).\end{aligned}$$

Let us define the positive real number depending on x, $\rho(x) := \overline{\varphi}_x^{-1}(\varphi(x,h(c_\infty)))\frac{\nu-1}{2\nu}$ and the function γ_n by

$$\begin{aligned}\gamma_n(x) \;:=\; & a(x,u_n,\nu\nabla\psi^+)\cdot\nabla\psi^+\\ & +\overline{\varphi}_x^{-1}(\varphi(x,h(0)))\frac{\nu-1}{2\nu}\overline{\varphi}\Big(x,\frac{|a(x,u_n,\nu\nabla\psi^+)|}{\overline{\varphi}_x^{-1}(\varphi(x,h(c_\infty)))\frac{\nu-1}{2}}\Big).\end{aligned}$$

For each $n\in\mathbb{N}$ the function γ_n belongs to $L^1(\Omega)$. Thus we have

$$a(x,u_n,\nabla u_n)\cdot\nabla(u_n-\psi^+) \geq \rho(x)\varphi(x,|\nabla u_n|)-\gamma_n(x).$$

Then, by lemma 3.2 $\rho(x)\geq\rho^* := \overline{q}^{-1}(q(h(c_\infty)))\dfrac{\nu-1}{2\nu}$, and we have

$$a(x,u_n,\nabla u_n)\cdot\nabla(u_n-\psi^+) \geq \rho(x)\varphi(x,|\nabla u_n|)-\gamma_n(x) \geq \rho^*\varphi(x,|\nabla u_n|)-\gamma_n(x).$$

Being β continuous, thanks to (4.12) the sequence $\{\beta(u_n)\}$ is uniformly bounded. Thus, there exists a constant β_0 such that

$$\|\beta(u_n)\|_\infty \leq \beta_0. \tag{4.17}$$

In view of (3.5), we can rewrite (4.16) as

$$\begin{aligned}&\int_\Omega \varphi(x,|\nabla u_n|)\Big[\rho^*\phi_\lambda'(u_n-\psi^+)-\beta_0|\phi_\lambda(u_n-\psi^+)|\Big]\,dx\\ &\leq \int_\Omega |f_n||\phi_\lambda(u_n-\psi^+)|\,dx+\int_\Omega \gamma_n\phi_\lambda'(u_n-\psi^+)\,dx.\end{aligned}$$

Applying now lemma 2.5 with $c=\beta_0$, $d=\rho^*$ and $\lambda=\left(\dfrac{\beta_0}{2\rho^*}\right)^2$, we get

$$\begin{aligned}&\int_\Omega \varphi(x,|\nabla u_n|)\,dx\\ &\leq \frac{2}{\rho^*}\Big(\|f\|_{m^*}|\Omega|^{1-\frac{1}{m^*}}\phi_\lambda(c_\infty+\|\psi^+\|_\infty)+\|\gamma_n\|_{L^1(\Omega)}\phi_\lambda'(c_\infty+\|\psi^+\|_\infty)\Big),\end{aligned}\tag{4.18}$$

where m^* stands for either N or m according as we assume (3.6) or (3.7). Hence, by Poincaré- type inequalities in Musielak spaces from [8], the sequence $\{u_n\}$ is bounded in $W_0^1L_\varphi(\Omega)$. Therefore, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and a function u in $W_0^1L_\varphi(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } W_0^1L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi,\Pi E_{\overline{\varphi}}) \tag{4.19}$$

and the compact embedding (see Lemma 3.5 and Remark 3.6 of [28]) implies

$$u_n \to u \quad \text{in } E_\varphi(\Omega) \text{ strongly and a.e. in } \Omega. \tag{4.20}$$

Step 5: Almost everywhere convergence of the gradients. Let us begin with the following lemma which will be used later.

Lemma 4.5. The sequence $\{a(x,T_n(u_n),\nabla u_n)\}$ is bounded in $(L_{\overline{\varphi}}(\Omega))^N$.

Proof . We will use the dual norm of $(L_{\overline{\varphi}}(\Omega))^N$. Let $\phi \in (E_\varphi(\Omega))^N$ such that $\|\phi\|_\varphi = 1$. By (3.4) we have

$$\Big(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \frac{\phi}{k_4})\Big) \cdot \Big(\nabla u_n - \frac{\phi}{k_4}\Big) \geq 0.$$

By using (3.3), (4.12) and Young's inequality we get

$$\begin{aligned}
&\int_\Omega a(x, T_n(u_n), \nabla u_n)\phi dx \\
&\leq k_4 \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx - k_4 \int_\Omega a(x, T_n(u_n), \frac{\phi}{k_4}) \cdot \nabla u_n \, dx \\
&\quad + \int_\Omega a(x, T_n(u_n), \frac{\phi}{k_4}) \cdot \phi dx \\
&\leq k_4 \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx \\
&\quad + k_4 C_1 c_\infty \|f\|_{m^*} |\Omega|^{1-\frac{1}{m^*}} + k_4(1 + k_1 + k_3) \frac{C_1 c_\infty \|f\|_{m^*} |\Omega|^{1-\frac{1}{m^*}}}{\overline{q}^{-1}(q(h(c_\infty)))} \\
&\quad + (1 + k_4) \int_\Omega \overline{\varphi}(x, a_0(x)) \, dx + k_1(1 + k_4) \int_\Omega \overline{\varphi}(x, \overline{P}_x^{-1}(\varphi(x, k_2 c_\infty))) \, dx \\
&\quad + (k_3(1 + k_4) + (1 + k_1 + k_3)) \int_\Omega \varphi(x, 1) \, dx.
\end{aligned}$$

Since $a_0(x) \in E_{\overline{\varphi}}(\Omega)$, $\int_\Omega \overline{\varphi}(x, a_0(x))dx < +\infty$ and we have $\int_\Omega \varphi(x, 1)dx < +\infty$. To estimate the integral $\int_\Omega \overline{\varphi}(x, \overline{P}_x^{-1}(\varphi(x, k_2 c_\infty)))$ recall that $P \prec\prec \varphi \Leftrightarrow \overline{\varphi} \prec\prec \overline{P}$ and use the fact that (see[6])

$$\overline{\varphi} \prec\prec \overline{P} \Rightarrow \forall \varepsilon > 0, \exists h \in L^1(\Omega) : \quad \overline{\varphi}(x, t) \leq \overline{P}(x, \varepsilon t) + h(x).$$

Thus, taking $\varepsilon \leq 1$ and using that $\overline{P}$ is increasing, we get

$$\begin{aligned}
\int_\Omega \overline{\varphi}(x, \overline{P}_x^{-1}(\varphi(x, k_2 c_\infty))) \, dx &\leq \int_\Omega \overline{P}(x, \varepsilon \overline{P}_x^{-1}(\varphi(x, k_2 c_\infty))) \, dx + \int_\Omega h(x) \, dx \\
&\leq \int_\Omega \overline{P}(x, \overline{P}_x^{-1}(\varphi(x, k_2 c_\infty))) \, dx + \int_\Omega h(x) \, dx \\
&\leq \int_\Omega \varphi(x, k_2 c_\infty) \, dx + \int_\Omega h(x) \, dx \\
&< \infty.
\end{aligned}$$

To end the proof, we show that $\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx$ can be estimated independently of n. To do this, we have from (3.3), Young's inequality, the $\triangle_2$-condition on φ and (4.18),

$$\begin{aligned}
&\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx \leq \int_\Omega |a(x, T_n(u_n), \nabla u_n)||\nabla u_n| \, dx \\
&\leq \int_\Omega [a_0(x)|\nabla u_n| + k_1 |\nabla u_n| \overline{P}_x^{-1}(\varphi(x, k_2 T_n(u_n)) + k_3 |\nabla u_n| \overline{\varphi}_x^{-1}(\varphi(x, k_4 |\nabla u_n|))] \, dx \\
&\leq \int_\Omega [\overline{\varphi}(x, a_0(x)) + \varphi(x, |\nabla u_n|) + \varphi(x, k_1 |\nabla u_n|) + \overline{\varphi}(x, \overline{P}_x^{-1}(\varphi(x, k_2 T_n(u_n))) + \varphi(x, k_3 |\nabla u_n|) \\
&\quad + \overline{\varphi}(x, \overline{\varphi}_x^{-1}(\varphi(x, k_4 |\nabla u_n|)))] \, dx \\
&\leq \int_\Omega \overline{\varphi}(x, a_0(x)) \, dx + (1 + k_1' + k_3' + k_4') \int_\Omega \varphi(x, |\nabla u_n|) \, dx + \int_\Omega \overline{\varphi}(x, \overline{P}_x^{-1}(\varphi(x, k_2 c_\infty))) \, dx \\
&< \infty.
\end{aligned}$$

□

Thanks to (4.12), (4.19) and (4.20) we obtain, $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$, so lemma 2.2 gives that there exists a sequence $\{v_j\}$ in $\mathfrak{D}(\Omega)$ such that $v_j \to u$ in $W_0^1 L_\varphi(\Omega)$ as $j \to \infty$ for the modular convergence and almost everywhere in Ω.

For $s > 0$, we denote by χ_j^s the characteristic function of the set

$$\Omega_j^s = \{x \in \Omega : |\nabla v_j(x)| \leq s\}$$

and by χ^s the characteristic function of the set $\Omega^s = \{x \in \Omega : |\nabla u(x)| \leq s\}$. We have $\|u_n\|_\infty \leq c_\infty$, since β is continuous the sequence $\{\beta(u_n)\}$ is bounded, then there exists a constant β_0 such that

$$\|\beta(u_n)\|_\infty \leq \beta_0. \tag{4.21}$$

Choosing $v = u_n - \eta\phi_\lambda(u_n - v_j)$, where $\eta = \exp(-\lambda(N+2)^2 c_\infty^2)$, as test function in (4.1), for n large enough, we obtain

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \cdot (\nabla u_n - \nabla v_j)\phi_\lambda^{'}(u_n - v_j)dx + \int_\Omega g_n(x, u_n, \nabla u_n)\phi_\lambda(u_n - v_j)dx \\ &\leq \int_\Omega f_n \phi_\lambda(u_n - v_j)dx. \end{aligned} \tag{4.22}$$

Denote by $\epsilon_i(n, j)$, $(i = 0, 1, \dots)$, various sequences of real numbers which tend to 0 when n and $j \to \infty$, i.e.

$$\lim_{j\to\infty} \lim_{n\to\infty} \epsilon_i(n, j) = 0.$$

From (4.12) and (4.20), we have $\phi_\lambda(u_n - v_j) \to \phi_\lambda(u - v_j)$ *weakly** in $L^\infty(\Omega)$, it follows that

$$\int_\Omega f_n \phi_\lambda(u_n - v_j) \to \int_\Omega f \phi_\lambda(u - v_j) \quad \text{as} \quad n \to \infty.$$

we have as $j \to \infty$, $u - v_j \to 0$ *weakly** in $L^\infty(\Omega)$, then $\int_\Omega f \phi_\lambda(u - v_j) \to 0$.
And we get for the right-hand side of (4.22), we have

$$\int_\Omega f_n(u_n - v_j)dx = \epsilon_0(n, j). \tag{4.23}$$

The first integral in the left-hand side of (4.22) is written as

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \cdot (\nabla u_n - \nabla v_j)\phi_\lambda^{'}(u_n - v_j)\, dx \\ &= \int_\Omega \big(a(x, u_n, \nabla u_n) - a(x, u_n, \nabla v_j \chi_j^s)\big) \cdot \big(\nabla u_n - \nabla v_j \chi_j^s\big)\, dx \\ &\quad + \int_\Omega a(x, u_n, \nabla v_j \chi_j^s) \cdot (\nabla u_n - \nabla v_j \chi_j^s)\phi_\lambda^{'}(u_n - v_j)dx \\ &\quad - \int_{\Omega\setminus\Omega_j^s} a(x, u_n, \nabla u_n) \cdot \nabla v_j \phi_\lambda^{'}(u_n - v_j)\, dx \end{aligned} \tag{4.24}$$

We will pass to the limit over n and j, for s fixed, in the second and the third terms of the right-hand side of (4.24). By Lemma 4.5, we deduce that there exists $\xi_0 \in (L_{\overline{\varphi}}(\Omega))^N$ and up to a subsequence $a(x, T_n(u_n), \nabla u_n) \rightharpoonup \xi_0$ weakly in $(L_{\overline{\varphi}}(\Omega))^N$ for $\sigma(\Pi L_{\overline{\varphi}}, \Pi E_\varphi)$. Since $\nabla v_j \chi_{\Omega\setminus\Omega_j^s} \in (E_\varphi(\Omega))^N$, we have by letting $n \to \infty$,

$$\int_{\Omega\setminus\Omega_j^s} a(x, u_n, \nabla u_n) \cdot \nabla v_j \phi_\lambda^{'}(u_n - v_j)\, dx \to \int_{\Omega\setminus\Omega_j^s} \xi_0 \cdot \nabla v_j \phi_\lambda^{'}(u_n - v_j)\, dx.$$

Using the modular convergence of v_j, we get as $j \to \infty$

$$\int_{\Omega\setminus\Omega_j^s} \xi_0 \cdot \nabla v_j \phi_\lambda^{'}(u_n - v_j)\, dx \to \int_{\Omega\setminus\Omega^s} \xi_0 \cdot \nabla u\, dx \quad \text{as} \quad j \to \infty.$$

Hence, we have proved that

$$\int_{\Omega\setminus\Omega_j^s} a(x, u_n, \nabla u_n) \cdot \nabla v_j \phi_\lambda^{'}(u_n - v_j)\, dx = \int_{\Omega\setminus\Omega^s} \xi_0 \cdot \nabla u\, dx + \epsilon_1(n, j). \tag{4.25}$$

For the second term, as $n\to\infty$, we have

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot\left(\nabla u_n-\nabla v_j\chi_j^s\right)\phi_\lambda^{'}(u_n-v_j)\,dx\\&\to\int_\Omega a(x,u,\nabla v_j\chi_j^s)\cdot\left(\nabla u-\nabla v_j\chi_j^s\right)\phi_\lambda^{'}(u-v_j)\,dx,\end{aligned}$$

since $a(x,u_n,\nabla v_j\chi_j^s)\to a(x,u,\nabla v_j\chi_j^s)$ strongly in $(E_{\overline{\varphi}}(\Omega))^N$ as $n\to\infty$ by lemma 2.3 and $\nabla u_n\rightharpoonup\nabla u$ weakly in $(L_\varphi(\Omega))^N$ by (4.19). And since $\nabla v_j\chi_j^s\to\nabla u\chi^s$ strongly in $(E_\varphi(\Omega))^N$ as $j\to\infty$, we obtain

$$\int_\Omega a(x,u,\nabla v_j\chi_j^s)\cdot\left(\nabla u-\nabla v_j\chi_j^s\right)\phi_\lambda^{'}(u-v_j)\,dx\to 0$$

as $j\to\infty$. So that

$$\int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot\left(\nabla u_n-\nabla v_j\chi_j^s\right)\phi_\lambda^{'}(u_n-v_j)\,dx=\epsilon_2(n,j). \tag{4.26}$$

Then, from (4.23), (4.25) and (4.26), we obtain

$$\begin{aligned}&\int_\Omega\left(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\right)\cdot\left(\nabla u_n-\nabla v_j\chi_j^s\right)\phi_\lambda^{'}(u_n-v_j)\,dx\\&\quad+\int_\Omega g_n(x,u_n,\nabla u_n)\cdot\phi_\lambda(\nabla u_n-\nabla v_j)\,dx\\&=\int_{\Omega\setminus\Omega^s}\xi_0\cdot\nabla u\,dx+\epsilon_3(n,j).\end{aligned} \tag{4.27}$$

Let us define $\delta_0:=\dfrac{\beta_0}{\overline{\varphi}_x^{-1}(\varphi(x,h(c_\infty)))}$, thanks to lemma 3.2

$$\delta_0\le\delta_0^*:=\frac{\beta_0}{\overline{q}^{-1}(q(h(c_\infty)))}$$

From (3.5), (3.2) and (4.12) we get

$$\begin{aligned}&\Big|\int_\Omega g_n(x,u_n,\nabla u_n)\phi_\lambda(\nabla u_n-\nabla v_j)\,dx\Big|\\&\le\int_\Omega\beta(u_n)\varphi(x,|\nabla u_n|)|\phi_\lambda(\nabla u_n-\nabla v_j)|\,dx\\&\le\int_\Omega\frac{\beta(u_n)}{\overline{\varphi}_x^{-1}(\varphi(x,h(|u_n|)))}a(x,u_n,\nabla u_n)\cdot\nabla u_n|\phi_\lambda(u_n-v_j)|\,dx\\&\le\delta_0^*\int_\Omega\left(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\right)\cdot\left(\nabla u_n-\nabla v_j\chi_j^s\right)\\&\quad\times|\phi_\lambda(u_n-v_j)|\,dx\\&\quad+\delta_0^*\int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot\left(\nabla u_n-\nabla v_j\chi_j^s\right)|\phi_\lambda(u_n-v_j)|\,dx\\&\quad+\delta_0^*\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v_j\chi_j^s|\phi_\lambda(u_n-v_j)|\,dx.\end{aligned} \tag{4.28}$$

Arguing as above, we obtain

$$\delta_0^*\int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot\left(\nabla u_n-\nabla v_j\chi_j^s\right)|\phi_\lambda(u_n-v_j)|\,dx=\epsilon_4(n,j)$$

and

$$\delta_0^*\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v_j\chi_j^s|\phi_\lambda(u_n-v_j)|\,dx=\epsilon_5(n,j).$$

Which reduces (4.28) to

$$
\begin{aligned}
&|\int_\Omega g_n(x,u_n,\nabla u_n)\phi_\lambda(\nabla u_n-\nabla v_j)\,dx| \\
&\le \delta_0^*\int_\Omega \big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\big)\cdot\big(\nabla u_n-\nabla v_j\chi_j^s\big)\,|\phi_\lambda(u_n-v_j)|\,dx \\
&\quad+\epsilon_6(n,j).
\end{aligned}
$$

Combining this last inequality with (4.27) to have

$$
\begin{aligned}
&\int_\Omega \big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\big)\cdot\big(\nabla u_n-\nabla v_j\chi_j^s\big) \\
&\quad\times(\phi_\lambda'(u_n-v_j)-\delta_0^*|\phi_\lambda(u_n-v_j)|)\,dx \\
&\le\int_{\Omega\setminus\Omega^s}\xi_0\cdot\nabla u\,dx+\epsilon_7(n,j).
\end{aligned}
$$

Applying now lemma 2.5, with $d=1$, $c=\delta_0^*$ and $\lambda=\left(\frac{\delta_0^*}{2d}\right)^2$, and we get

$$
\begin{aligned}
&\int_\Omega \big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\big)\cdot\big(\nabla u_n-\nabla v_j\chi_j^s\big) \\
&\le 2\int_{\Omega\setminus\Omega^s}\xi_0\cdot\nabla u\,dx+2\epsilon_7(n,j).
\end{aligned}
\tag{4.29}
$$

On the other hand

$$
\begin{aligned}
&\int_\Omega (a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u\chi^s))\cdot(\nabla u_n-\nabla u\chi^s)\,dx \\
&=\int_\Omega \big(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla v_j\chi_j^s)\big)\cdot\big(\nabla u_n-\nabla v_j\chi_j^s\big)\,dx \\
&\quad+\int_\Omega a(x,u_n,\nabla u_n)\cdot\big(\nabla v_j\chi_j^s-\nabla u\chi^s\big)\,dx \\
&\quad-\int_\Omega a(x,u_n,\nabla u\chi^s)\cdot(\nabla u_n-\nabla u\chi^s)\,dx \\
&\quad+\int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot\big(\nabla u_n-\nabla v_j\chi_j^s\big)\,dx.
\end{aligned}
$$

Proceeding as above, we obtain

$$
\int_\Omega a(x,u_n,\nabla u_n)\cdot\big(\nabla v_j\chi_j^s-\nabla u\chi^s\big)\,dx=\epsilon_8(n,j),
$$

and

$$
\int_\Omega a(x,u_n,\nabla u\chi^s)\cdot(\nabla u_n-\nabla u\chi^s)\,dx=\epsilon_9(n,j),
$$

and

$$
\int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot\big(\nabla u_n-\nabla v_j\chi_j^s\big)\,dx=\epsilon_{10}(n,j).
\tag{4.30}
$$

Then, by (4.29) we have

$$
\begin{aligned}
&\int_\Omega (a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u\chi^s))\cdot(\nabla u_n-\nabla u\chi^s)\,dx \\
&=\epsilon_{11}(n,j)+\int_{\Omega\setminus\Omega^s}\xi_0\cdot\nabla u dx.
\end{aligned}
$$

For $r \leq s$, we have

$$\begin{aligned}
0 &\leq \int_{\Omega^r} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\, dx \\
&\leq \int_{\Omega^s} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\, dx \\
&= \int_{\Omega^s} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u\chi^s)) \cdot (\nabla u_n - \nabla u\chi^s)\, dx \\
&\leq \int_{\Omega} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u\chi^s)) \cdot (\nabla u_n - \nabla u\chi^s)\, dx \\
&\leq \epsilon_{11}(n,j) + 2\int_{\Omega\setminus\Omega^s} \xi_0 \cdot \nabla u dx.
\end{aligned}$$

Passing to the limit superior over n and then over j, yields

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \int_{\Omega^r} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\, dx \\
&\leq \int_{\Omega\setminus\Omega^s} \xi_0 \cdot \nabla u dx.
\end{aligned}$$

Letting $s \to +\infty$ in the previous inequality, we conclude that

$$\lim_{n\to\infty} \int_{\Omega^r} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\, dx = 0. \tag{4.31}$$

Define A_n by

$$A_n = (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u)\,.$$

As a consequence of (4.31), one has $A_n \to 0$ strongly in $L^1(\Omega^r)$, extracting a subsequence, still denoted by $\{u_n\}$, we get $A_n \to 0$ a.e in Ω^r. Then, there exists a subset Ω_0 of Ω^r, of zero measure, such that: $A_n(x) \to 0$ for all $x \in \Omega^r \setminus \Omega_0$. Using (3.2), (3.3) and then lemma 3.2 we obtain for all $x \in \Omega^r \setminus \Omega_0$,

$$\begin{aligned}
A_n(x) &\geq \overline{\varphi}_x^{-1}(\varphi(x,h(c_\infty)))\varphi(x,|\nabla u_n(x)|) \\
&\qquad - c_1(x)\Big(1 + \overline{\varphi}_x^{-1}(\varphi(x,k_4|\nabla u_n(x)|)) + |\nabla u_n(x)|\Big) \\
&\geq \overline{q}^{-1}(q(h(c_\infty)))\varphi(x,|\nabla u_n(x)|) \\
&\qquad - c_1(x)\Big(1 + \overline{\varphi}_x^{-1}(\varphi(x,k_4|\nabla u_n(x)|)) + |\nabla u_n(x)|\Big)
\end{aligned}$$

where c_∞ is the constant which appears in (4.12) and $c_1(x)$ is a constant which does not depend on n. For x fixed the sequence $\{\nabla u_n(x)\}$ is bounded in $\mathbb{R}^N$, else, if $\{\nabla u_n(x)\}$ is unbounded, there exists a subsequence still denoted by $\{\nabla u_n(x)\}$ which tends to $+\infty$ and we have $\varphi(x,|\nabla u_n(x)|) \to +\infty$ and recall the inequality

$$\varphi(x,t) \leq t\overline{\varphi}_x^{-1}(\varphi(x,t)) \leq 2\varphi(x,t),$$

for all $t \geq 0$, which implies that

$$\overline{\varphi}_x^{-1}(\varphi(x,k_4|\nabla u_n(x)|)) \leq \frac{2\varphi(x,k_4|\nabla u_n(x)|)}{k_4|\nabla u_n(x)|}.$$

Since φ satisfies the Δ_2-condition, there exists a positive constant k and a positive L^1-function c such that

$$\overline{\varphi}_x^{-1}(\varphi(x,k_4|\nabla u_n(x)|)) \leq \frac{2\varphi(x,k_4|\nabla u_n(x)|)}{k_4|\nabla u_n(x)|} \leq \frac{2k\varphi(x,|\nabla u_n(x)|)}{k_4|\nabla u_n(x)|} + c(x).$$

Consequently, the right term in each inequality goes to $+\infty$ as n goes to $+\infty$, which is a contradiction with $\lim\limits_{n\to\infty} A_n = 0$. Thus, $\{\nabla u_n(x)\}$ is bounded, then for a subsequence $\{u_{n'}(x)\}$, we have

$$\nabla u_{n'}(x) \to \xi \quad \text{in } \mathbb{R}^N,$$
$$(a(x,u(x),\xi) - a(x,u(x),\nabla u(x))) \cdot (\xi - \nabla u(x)) = 0.$$

Since $a(x,s,\xi)$ is strictly monotone, we have $\xi = \nabla u(x)$, and so $\nabla u_n(x) \to \nabla u(x)$ for the whole sequence. It follows that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega^r.$$

As a consequence, since r is arbitrary, we get

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{4.32}$$

Then, it follows

$$a(x, T_n(u_n), \nabla u_n) \rightharpoonup a(x,u,\nabla u) \quad \text{weakly in} \quad (L_{\overline{\varphi}}(\Omega))^N. \tag{4.33}$$

Step 6: Modular convergence of the gradients.

From (4.29), we get

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\cdot \nabla u_n \,dx \leq &\int_\Omega a(x,u_n,\nabla u_n)\cdot \nabla v_j \chi_j^s \,dx\\ &+ \int_\Omega a(x,u_n,\nabla v_j\chi_j^s)\cdot(\nabla u_n - \nabla v_j\chi_j^s)\,dx\\ &+ 2\int_{\Omega\setminus\Omega^s} \xi_0\cdot \nabla u\,dx + 2\epsilon_7(n,j).\end{aligned}$$

Using now (4.30), we obtain

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\cdot \nabla u_n \,dx \leq &\int_\Omega a(x,u_n,\nabla u_n)\cdot \nabla v_j \chi_j^s \,dx\\ &+ 2\int_{\Omega\setminus\Omega^s} \xi_0\cdot \nabla u\,dx + 2\epsilon_{12}(n,j).\end{aligned}$$

Passing to the limit sup over n and then over j, we get

$$\limsup_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx \leq \int_\Omega a(x,u,\nabla u)\cdot \nabla u \chi^s\,dx + 2\int_{\Omega\setminus\Omega^s}\xi_0\cdot\nabla u\,dx.$$

Let s go to ∞, we have

$$\limsup_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx \leq \int_\Omega a(x,u,\nabla u)\cdot\nabla u\,dx.$$

On the other hand we get, by using Fatou's lemma,

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla u\,dx \leq \liminf_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx$$

Finally, we obtain

$$\lim_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx = \int_\Omega a(x,u,\nabla u)\cdot\nabla u\,dx.$$

Then, lemma 2.4 yields

$$a(x,u_n,\nabla u_n)\cdot\nabla u_n \to a(x,u,\nabla u)\cdot\nabla u \quad \text{in} \quad L^1(\Omega) \tag{4.34}$$

Using the convexity of φ, we have

$$\varphi\Big(x, \frac{|\nabla u_n - \nabla u|}{2}\Big) \leq \frac{1}{2}\varphi(x,|\nabla u_n|) + \frac{1}{2}\varphi(x,|\nabla u|).$$

Taking into account that $\|u_n\|_\infty \leq c_\infty$, $\dfrac{\overline{\varphi}_x^{-1}(\varphi(x,h(|u_n|)))}{\overline{\varphi}_x^{-1}(\varphi(x,h(c_\infty)))} \geq 1$ and then using (3.2) yield

$$\begin{aligned}\varphi\Big(x,\frac{|\nabla u_n-\nabla u|}{2}\Big) &\leq \frac{1}{2\overline{\varphi}_x^{-1}(\varphi(x,h(c_\infty)))}\overline{\varphi}_x^{-1}(\varphi(x,h(|u_n|)))\varphi(x,|\nabla u_n|)\\ &\quad+\frac{1}{2\overline{\varphi}_x^{-1}(\varphi(x,h(c_\infty)))}\overline{\varphi}_x^{-1}(\varphi(x,h(|u|)))\varphi(x,|\nabla u|)\\ &\leq \frac{1}{2\overline{\varphi}_x^{-1}(\varphi(x,h(c_\infty)))}a(x,u_n,\nabla u_n)\cdot\nabla u_n\\ &\quad+\frac{1}{2\overline{\varphi}_x^{-1}(\varphi(x,h(c_\infty)))}a(x,u,\nabla u)\cdot\nabla u.\end{aligned}$$

Applying now lemma 3.2 which implies $\frac{1}{2\overline{\varphi}_x^{-1}(\varphi(x,h(c_\infty)))} \leq \frac{1}{2\overline{q}^{-1}(q(h(c_\infty)))}$ and we have

$$\varphi\Big(x, \frac{|\nabla u_n - \nabla u|}{2}\Big) \leq \frac{1}{2\overline{q}^{-1}(q(h(c_\infty)))} a(x,u_n,\nabla u_n)\cdot\nabla u_n + \frac{1}{2\overline{q}^{-1}(q(h(c_\infty)))} a(x,u,\nabla u)\cdot\nabla u.$$

Finally, since $a(x,u_n,\nabla u_n)\cdot\nabla u_n \to a(x,u,\nabla u)\cdot\nabla u$ in $L^1(\Omega)$, it follows

$$u_n \to u \quad \text{in} \quad W_0^1L_\varphi(\Omega) \quad \text{for the modular convergence}$$

Step 7: Equi-integrability of the non-linearities. Now, we shall prove that $g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u)$ strongly in $L^1(\Omega)$, we need to use Vitali's theorem. Thanks to (4.20) and (4.32) we have $g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u)$ a.e. in Ω, it suffices to show that the sequence $\{g_n(x,u_n,\nabla u_n)\}$ is uniformly equi-integrable in Ω. Let $E \subset \Omega$ be a measurable subset of Ω, from (3.5) and (4.21) we have

$$|g_n(x,u_n,\nabla u_n)| \leq \beta_0\varphi(x,|\nabla u_n|).$$

Using (3.2), (4.12) and then lemma 3.2, we get

$$\int_E |g_n(x,u_n,\nabla u_n)|\,dx \leq \int_E \frac{\beta_0}{\overline{\varphi}_x^{-1}(\varphi(x,h(c_\infty)))}|a(x,u_n,\nabla u_n)\cdot\nabla u_n|\,dx \leq \frac{\beta_0}{\overline{q}^{-1}(q(h(c_\infty)))}\int_E |a(x,u_n,\nabla u_n)\cdot\nabla u_n|\,dx$$

Finally, by virtue of the strong convergence of $\{|a(x,u_n,\nabla u_n)\cdot\nabla u_n|\}$ in $L^1(\Omega)$, so that

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \quad \text{strongly in} \quad L^1(\Omega) \tag{4.35}$$

Step 8: Passing to the limit. Let $v \in \mathbf{K}_\psi \cap L^\infty(\Omega)$. By (A_2) there is a sequence $\{v_j\} \subset \mathbf{K}_\psi \cap W_0^1E_\varphi(\Omega) \cap L^\infty(\Omega)$ such that $v_j \longrightarrow v$ for the modular convergence in $W_0^1L_\varphi(\Omega)$. For all $n > c_\infty$, using v_j as a test function in (4.1) yields

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla(u_n - v_j)dx + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v_j)dx \leq \int_\Omega f_n(u_n - v_j)dx.$$

Since $\nabla v_j \in (E_\varphi(\Omega))^N$, by (4.33) one has

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v_j dx \to \int_\Omega a(x,u,\nabla u)\cdot\nabla v_j dx$$

as $n \to \infty$. So that by (4.34) we get

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla(u_n - v_j)dx \to \int_\Omega a(x,u,\nabla u)\cdot\nabla(u - v_j)dx.$$

Using (4.12) and (4.35), passing to the limit as $n \to +\infty$

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla(u - v_j)dx + \int_\Omega g(x,u,\nabla u)(u - v_j)dx \leq \int_\Omega f(u - v_j)dx.$$

As we have, up to a subsequence still indexed by j, $v_j \to v$ *a.e.* in Ω and weakly for $\sigma(\Pi L_\varphi, \Pi L_{\overline{\varphi}})$, we can pass to the limit as $j \to \infty$ to obtain

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla(u - v)dx + \int_\Omega g(x,u,\nabla u)(u - v)dx \leq \int_\Omega f(u - v)dx.$$

By virtue of (4.20) we have $u \in \mathbf{K}_\psi \cap L^\infty(\Omega)$. This completes the proof of Theorem 3.3.

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