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On new generalization of Fejér type inequalities for double integrals

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Abstract

In this article, we first establish weighted identities based on twice partially differentiable mappings. Moreover, utilizing this equality, we derive the weighted Hermite–Hadamard type inequalities via co-ordinated convex mappings in a rectangle from the plane $\mathbb{R}^2$. More specifically, we establish new inequalities using the Hölder and power-mean inequalities. In addition, we obtain new results with special choices.

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1 Introduction

Integral inequalities have accelerated the solution process of many researches in mathematics today. In particular, research on Hermite–Hadamard, Trapezoid, Midpoint, Simpson type inequalities has an important place in the world of mathematics. These inequalities apply to pure mathematics and solving real-life problems. The inequalities obtained depending on the weighted function of these inequalities have generalized many studies.

The Hermite-Hadamard inequality discovered by C. Hermite and J. Hadamard see, e.g., [13], [27, p.137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that if $F : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\sigma_1, \sigma_2 \in I$ with $\sigma_1 < \sigma_2$, then

$$F\left(\frac{\sigma_1+\sigma_2}{2}\right) \leq \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} F(\kappa_1)d\kappa_1 \leq \frac{F(\sigma_1)+F(\sigma_2)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if F is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements

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and generalizations have been studied. For Hermite-Hadamard and other famous inequalities in the literature, see these references [10, 5, 16, 23, 22, 6, 4].

The weighted version of the inequalities (1.1), so-called Hermite-Hadamard-Fejér inequalities, was given by Fejér in [15] as follow:

Theorem 1.1. $F:[\sigma_1,\sigma_2]\to\mathbb{R}$, be a convex function, then the inequality

$$F\left(\frac{\sigma_1+\sigma_2}{2}\right)\int_{\sigma_1}^{\sigma_2} g(\kappa_1)d\kappa_1 \le \int_{\sigma_1}^{\sigma_2} F(\kappa_1)g(\kappa_1)d\kappa_1 \le \frac{F(\sigma_1)+F(\sigma_2)}{2}\int_{\sigma_1}^{\sigma_2} g(\kappa_1)d\kappa_1, \tag{1.2}$$

holds, where $g:[\sigma_1,\sigma_2]\to\mathbb{R}$ is non-negative, integrable, and symmetric about $\kappa_1=\frac{\sigma_1+\sigma_2}{2}$ (i.e. $g(\kappa_1)=g(\sigma_1+\sigma_2-\kappa_1)$).

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1.2. A function $F:\Delta\to\mathbb{R}$ is called co-ordinated convex on Δ, for all $(\kappa_1,u),(\kappa_2,\upsilon)\in\Delta$ and $\xi,\vartheta\in[0,1]$, if it satisfies the following inequality:

$$\begin{aligned} & F(\xi\kappa_1+(1-\xi)\ \kappa_2,\vartheta u+(1-\vartheta)\ \upsilon) \qquad (1.3)\\ \le\ & \xi\vartheta\ F(\kappa_1,u)+\xi(1-\vartheta)F(\kappa_1,\upsilon)+\vartheta(1-\xi)F(\kappa_2,u)+(1-\xi)(1-\vartheta)F(\kappa_2,\upsilon). \end{aligned}$$

The mapping F is a co-ordinated concave on Δ if the inequality (1.3) holds in reversed direction for all $\xi,\vartheta\in[0,1]$ and $(\kappa_1,u),(\kappa_2,\upsilon)\in\Delta$.

In [12], Dragomir proved the following inequalities which is Hermite-Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 1.3. Suppose that $F:\Delta\to\mathbb{R}$ is co-ordinated convex, then we have the following inequalities:

$$\begin{aligned} F\left(\frac{\sigma_1+\sigma_2}{2},\frac{\varsigma_1+\varsigma_2}{2}\right) &\le \frac{1}{2}\left[\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} F\left(\kappa_1,\frac{\varsigma_1+\varsigma_2}{2}\right)d\kappa_1+\frac{1}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2} F\left(\frac{\sigma_1+\sigma_2}{2},\kappa_2\right)d\kappa_2\right]\\ &\le \frac{1}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\sigma_1}^{\sigma_2}\int_{\varsigma_1}^{\varsigma_2} F(\kappa_1,\kappa_2)d\kappa_2 d\kappa_1 \qquad (1.4)\\ &\le \frac{1}{4}\left[\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} F(\kappa_1,\varsigma_1)d\kappa_1+\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} F(\kappa_1,\varsigma_2)d\kappa_1\right.\\ &+\left.\frac{1}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2} F(\sigma_1,\kappa_2)d\kappa_2+\frac{1}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2} F(\sigma_2,\kappa_2)d\kappa_2\right]\\ &\le \frac{F(\sigma_1,\varsigma_1)+F(\sigma_1,\varsigma_2)+F(\sigma_2,\varsigma_1)+F(\sigma_2,\varsigma_2)}{4}. \end{aligned}$$

The above inequalities are sharp. The inequalities in (1.4) hold in reverse direction if the mapping F is a co-ordinated concave mapping.

Over the years, many papers are dedicated on the generalizations and new versions of the inequalities (1.4) using the different type convex functions. For the other Hermite-Hadamard type inequalities for co-ordinated convex functions, please refer to ([2]-[11],[24],[25],[28]-[33]).

Moreover, Farid et al. established a weighted version of the inequalities (1.4) in [14]. Please see ([17]-[21], [32]) for other papers focused on Hermite-Hadamard-Fejér inequalities for co-ordinated convex functions.

The aim of this paper is to establish some weighed generalizations of Hermite-Hadamard and Simpson type integral inequalities. The results presented in this paper provide extensions of those given in [26] and [28].

The following inequalities are useful for our main results:

Let $\sigma_1 = (\sigma_{11}, \sigma_{12}, ..., \sigma_{1n})$ be a sequence of positive numbers and $r \neq 0$. Then the power mean $M_r(\sigma_1)$, of order r, is defined as follows:

$$M_r(\sigma_1) = \left(\frac{\sigma_{11}^r + \sigma_{12}^r + ... + \sigma_{1n}^r}{n}\right)^{\frac{1}{r}}.$$

Then we have the following power mean inequality

$$M_r(\sigma_1) \leq M_\vartheta(\sigma_1), \tag{1.5}$$

for any real numbers $r \leq \vartheta$.

On the other hand let $\sigma_1 = (\sigma_{11}, \sigma_{12}, ..., \sigma_{1n})$, $\sigma_2 = (\sigma_{21}, \sigma_{22}, ..., \sigma_{2n})$, $\varsigma_1 = (\varsigma_{11}, \varsigma_{12}, ..., \varsigma_{1n})$ and $\varsigma_2 = (\varsigma_{21}, \varsigma_{22}, ..., \varsigma_{2n})$ be four sequence of positive numbers. Then

$$\sum_{k=1}^{n} (\sigma_{1k} + \sigma_{2k} + \varsigma_{1k} + \varsigma_{2k})^\vartheta \leq \sum_{k=1}^{n} \sigma_{1k}^\vartheta + \sum_{k=1}^{n} \sigma_{2k}^\vartheta + \sum_{k=1}^{n} \varsigma_{1k}^\vartheta + \sum_{k=1}^{n} \varsigma_{2k}^\vartheta, \tag{1.6}$$

for $0 \leq \vartheta < 1$.

2 Weighted Hermite–Hadamard type Inequalities

In this section, we first prove a weighted identity for twice partially differentiable functions. Then, using this identity, we established some weighted Hermite-Hadamard type inequalities for co-ordinated convex mappings.

Firstly, let us start with some notations. Let υ and ρ be positive integers. We define the following operators;

$$\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi) = \left(\frac{\xi + \upsilon}{2\upsilon}\right)\sigma_1 + \left(\frac{\upsilon - \xi}{2\upsilon}\right)\sigma_2,$$

$$\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi) = \left(\frac{\xi + \upsilon}{2\upsilon}\right)\sigma_2 + \left(\frac{\upsilon - \xi}{2\upsilon}\right)\sigma_1,$$

$$\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta) = \left(\frac{\vartheta + \rho}{2\rho}\right)\varsigma_1 + \left(\frac{\rho - \vartheta}{2\rho}\right)\varsigma_2$$

and

$$\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta) = \left(\frac{\vartheta + \rho}{2\rho}\right)\varsigma_2 + \left(\frac{\rho - \vartheta}{2\rho}\right)\varsigma_1,$$

for $\xi \in [\sigma_1, \sigma_2]$ and $\vartheta \in [\varsigma_1, \varsigma_2]$.

Now we can give the following lemma:

Lemma 2.1. Let $\digamma : \Delta = [\sigma_1, \sigma_2] \times [\varsigma_1, \varsigma_2] \to \mathbb{R}$ be twice partially differentiable mapping on Δ°. Suppose that $g_1 : [\sigma_1, \sigma_2] \to [0, \infty)$ and $g_2 : [\varsigma_1, \varsigma_2] \to [0, \infty]$ are continuous mapping which are symmetric about $\frac{\sigma_1+\sigma_2}{2}$ and $\frac{\varsigma_1+\varsigma_2}{2}$, respectively. If $\frac{\partial^2 \digamma}{\partial\xi\partial\vartheta} \in L(\Delta)$, then for any positive integer υ and ρ. We have the following equality

$$\mathcal{F}_{g_1,g_2}(\sigma_1, \sigma_2; \varsigma_1, \varsigma_2) \tag{2.1}$$

$$= \frac{(\sigma_2 - \sigma_1)(\varsigma_2 - \varsigma_1)}{16\upsilon\rho} \int_0^\upsilon \int_0^\rho \left(\int_{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau) d\tau \right) \left(\int_{\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}^{\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)} g_2(\tau) d\tau \right)$$

$$\times\left[\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))-\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right.$$
$$\left.-\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)+\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right]d\vartheta d\xi,$$

where

$$\begin{aligned}
&\mathcal{F}_{g_1,g_2}(\sigma_1,\sigma_2;\varsigma_1,\varsigma_2)\\
=\;&\left(\int_{\sigma_1}^{\sigma_2}g_1(\tau)d\tau\right)\left(\int_{\varsigma_1}^{\varsigma_2}g_2(\tau)d\tau\right)\frac{F(\sigma_2,\varsigma_2)+F(\sigma_2,\varsigma_1)+F(\sigma_1,\varsigma_2)+F(\sigma_2,\varsigma_1)}{4}\\
&-\frac{1}{2}\left(\int_{\varsigma_1}^{\varsigma_2}g_2(\tau)d\tau\right)\int_{\sigma_1}^{\sigma_2}g_1(\kappa_1)\left[F(\kappa_1,\varsigma_2)+F(\kappa_1,\varsigma_1)\right]d\kappa_1\\
&-\frac{1}{2}\left(\int_{\sigma_1}^{\sigma_2}g_1(\tau)d\tau\right)\int_{\varsigma_1}^{\varsigma_2}g_2(\kappa_2)\left[F(\sigma_2,\kappa_2)+F(\sigma_1,\kappa_2)\right]d\kappa_2\\
&+\int_{\sigma_1}^{\sigma_2}\int_{\varsigma_1}^{\varsigma_2}g_1(\kappa_1)g_2(\kappa_2)F(\kappa_1,\kappa_2)d\kappa_2 d\kappa_1.
\end{aligned}$$

Proof . By the first integral on the right hand side of Lemma 2.1, we have

$$\begin{aligned}
I_1 &= \int_0^{\upsilon}\int_0^{\rho}\left[\left(\int_{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}g_1(\tau)d\tau\right)\left(\int_{\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}^{\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}g_2(\tau)d\tau\right)\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right]d\vartheta d\xi \qquad (2.2)\\
&= \int_0^{\upsilon}\int_{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}g_1(\tau)d\tau\left(\int_0^{\rho}\int_{\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}^{\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}g_2(\tau)d\tau\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))d\vartheta\right)d\xi.
\end{aligned}$$

Using the integration by pats, we get

$$\begin{aligned}
I_{11} &= \int_0^{\rho}\int_{\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}^{\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}g_2(\tau)d\tau\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))d\vartheta\\
&= \frac{2\rho}{\varsigma_2-\varsigma_1}\int_{\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}^{\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}g_2(\tau)d\tau\frac{\partial F}{\partial\xi}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\Bigg|_0^{\rho}\\
&\quad+\int_0^{\rho}\frac{\partial F}{\partial\xi}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\left[g_2(\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))+g_2(\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right]d\vartheta.
\end{aligned}$$

Since g_2 is symmetric with respect to $\frac{\varsigma_1+\varsigma_2}{2}$, we have

$$g_2(\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))=g_2(\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)),$$

and by changing variables with $\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)=\kappa_2$, we have

$$I_{11}=-\frac{2\rho}{\varsigma_2-\varsigma_1}\int_{\varsigma_1}^{\varsigma_2}g_2(\tau)d\tau\frac{\partial F}{\partial\xi}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\varsigma_2)+\frac{4\rho}{\varsigma_2-\varsigma_1}\int_{\frac{\varsigma_1+\varsigma_2}{2}}^{\varsigma_2}\frac{\partial F}{\partial\xi}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\kappa_2)g_2(\kappa_2)d\kappa_2. \qquad (2.3)$$

By putting the equality (2.3) in (2.2), we obtain

$$\begin{aligned} I_1 &= \int_0^{\upsilon} \left(\int_{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau \right) \left[-\frac{2\rho}{\varsigma_2-\varsigma_1} \int_{\varsigma_1}^{\varsigma_2} g_2(\tau)d\tau \frac{\partial F}{\partial \xi}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\varsigma_2) \right. \\ &\left. + \frac{4\rho}{\varsigma_2-\varsigma_1} \int_{\frac{\varsigma_1+\varsigma_2}{2}}^{\varsigma_2} g_2(\kappa_2) \frac{\partial F}{\partial \xi}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\kappa_2)d\kappa_2 \right] d\xi \\ &= -\frac{2\rho}{\varsigma_2-\varsigma_1} \left(\int_{\varsigma_1}^{\varsigma_2} g_2(\tau)d\tau \right) \int_0^{\upsilon} \left(\int_{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau \right) \frac{\partial F}{\partial \xi}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\varsigma_2)d\xi \\ &+ \frac{4\rho}{\varsigma_2-\varsigma_1} \int_{\frac{\varsigma_1+\varsigma_2}{2}}^{\varsigma_2} g_2(\kappa_2) \left[\int_0^{\upsilon} \left(\int_{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau \right) \frac{\partial F}{\partial \xi}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\kappa_2)d\xi \right] d\kappa_2. \end{aligned} \tag{2.4}$$

By using the integration by parts, we get

$$\begin{aligned} I_{12} &= \int_0^{\upsilon} \left(\int_{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau \right) \frac{\partial F}{\partial \xi}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\varsigma_2)d\xi \\ &= \frac{2\upsilon}{\sigma_2-\sigma_1} \left(\int_{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau \right) F(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\varsigma_2) \Bigg|_0^{\upsilon} \\ &+ \int_0^{\upsilon} F(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\varsigma_2)[g_1(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)) + g_1(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi))]d\xi. \end{aligned}$$

Since g_1 is symmetric with respect to $\frac{\sigma_1+\sigma_2}{2}$, we have

$$g_1(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)) = g_2((\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)),$$

and setting $\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi) = \kappa_1$, we have

$$I_{12} = -\frac{2\upsilon}{\sigma_2-\sigma_1} \int_{\sigma_1}^{\sigma_2} g_1(\tau)d\tau \; F(\sigma_2,\varsigma_2) + \frac{4\upsilon}{\sigma_2-\sigma_1} \int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_2} F(\kappa_1,\varsigma_2)g_1(\kappa_1)d\kappa_1. \tag{2.5}$$

Similarly, utilizing the integration by parts, we have

$$\begin{aligned} I_{13} &= \int_0^{\upsilon} \left(\int_{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau \frac{\partial F}{\partial \xi} \right) (\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\kappa_2)d\xi \\ &= \frac{2\upsilon}{\sigma_2-\sigma_1} \left(\int_{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau \right) F(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\kappa_2) \Bigg|_0^{\upsilon} \\ &+ \int_0^{\upsilon} F(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\kappa_2)[g_1(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)) + g_1(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi))]d\xi \end{aligned} \tag{2.6}$$

$$= -\frac{2\upsilon}{\sigma_2-\sigma_1}\left(\int_{\sigma_1}^{\sigma_2} g_1(\tau)d\tau\right) F(\sigma_2,\kappa_2) + 2\int_0^{\upsilon} F(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\kappa_2) g_1(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi))]d\xi$$

$$= -\frac{2\upsilon}{\sigma_2-\sigma_1}\left(\int_{\sigma_1}^{\sigma_2} g_1(\tau)d\tau\right) F(\sigma_2,\kappa_2) + \frac{4\upsilon}{\sigma_2-\sigma_1}\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_2} F(\kappa_1,\kappa_2) g_1(\kappa_1)d\kappa_1.$$

Therefore, by substituting the equalities (2.5) and (2.6) in (2.4), we establish

$$\begin{aligned}
I_1 &= \frac{4\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\sigma_1}^{\sigma_2} g_1(\tau)d\tau \int_{\varsigma_1}^{\varsigma_2} g_2(\tau)d\tau \; F(\sigma_2,\varsigma_2) \\
&\quad -\frac{8\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\varsigma_1}^{\varsigma_2} g_2(\tau)d\tau \int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_2} g_1(\kappa_1)F(\kappa_1,\varsigma_2)d\kappa_1 \\
&\quad -\frac{8\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\sigma_1}^{\sigma_2} g_1(\tau)d\tau \int_{\frac{\varsigma_1+\varsigma_2}{2}}^{\varsigma_2} g_2(\kappa_2)F(\sigma_2,\kappa_2)d\kappa_2 \\
&\quad +\frac{16\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_2}\int_{\frac{\varsigma_1+\varsigma_2}{2}}^{\varsigma_2} g_1(\kappa_1)g_2(\kappa_2)F(\kappa_1,\kappa_2)d\kappa_2 d\kappa_1.
\end{aligned}$$

Similarly, one can obtain

$$\begin{aligned}
I_2 &= \int_0^{\upsilon}\int_0^{\rho}\left[\left(\int_{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau\right)\left(\int_{\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}^{\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)} g_2(\tau)d\tau\right)\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right]d\vartheta d\xi \\
&= -\frac{4\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\sigma_1}^{\sigma_2} g_1(\tau)d\tau \int_{\varsigma_1}^{\varsigma_2} g_2(\tau)d\tau \; F(\sigma_2,\varsigma_1) \\
&\quad +\frac{8\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\varsigma_1}^{\varsigma_2} g_2(\tau)d\tau \int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_2} g_1(\kappa_1)F(\kappa_1,\varsigma_1)d\kappa_1 \\
&\quad +\frac{8\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\sigma_1}^{\sigma_2} g_1(\tau)d\tau \int_{\frac{\varsigma_1+\varsigma_2}{2}}^{\varsigma_2} g_2(\kappa_2)F(\sigma_2,\kappa_2)d\kappa_2 \\
&\quad -\frac{16\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_2}\int_{\varsigma_1}^{\frac{\varsigma_1+\varsigma_2}{2}} g_1(\kappa_1)g_2(\kappa_2)F(\kappa_1,\kappa_2)d\kappa_2 d\kappa_1,
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_0^{\upsilon}\int_0^{\rho}\left[\left(\int_{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau\right)\left(\int_{\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}^{\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)} g_2(\tau)d\tau\right)\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right]d\vartheta d\xi \\
&= -\frac{4\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\sigma_1}^{\sigma_2} g_1(\tau)d\tau \int_{\varsigma_1}^{\varsigma_2} g_2(\tau)d\tau \; F(\sigma_1,\varsigma_2)
\end{aligned}$$

$$+\frac{8\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\varsigma_1}^{\varsigma_2}g_2(\tau)d\tau\int_{\sigma_1}^{\frac{\sigma_1+\sigma_2}{2}}g_1(\kappa_1)F(\kappa_1,\varsigma_2)d\kappa_1$$
$$+\frac{8\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\sigma_1}^{\sigma_2}g_1(\tau)d\tau\int_{\frac{\varsigma_1+\varsigma_2}{2}}^{\varsigma_2}g_2(\kappa_2)F(\sigma_1,\kappa_2)d\kappa_2$$
$$-\frac{16\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_2}\int_{\varsigma_1}^{\frac{\varsigma_1+\varsigma_2}{2}}g_1(\kappa_1)g_2(\kappa_2)F(\kappa_1,\kappa_2)d\kappa_2d\kappa_1,$$

and

$$\begin{aligned}I_4 &= \int_0^{\upsilon}\int_0^{\rho}\left[\left(\int_{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}g_1(\tau)d\tau\right)\left(\int_{\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}^{\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}g_2(\tau)d\tau\right)\frac{\partial^2F}{\partial\xi\partial\vartheta}(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right]d\vartheta d\xi\\
&= \frac{4\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\sigma_1}^{\sigma_2}g_1(\tau)d\tau\int_{\varsigma_1}^{\varsigma_2}g_2(\tau)d\tau\ F(\sigma_1,\varsigma_1)\\
&\quad-\frac{8\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\varsigma_1}^{\varsigma_2}g_2(\tau)d\tau\int_{\sigma_1}^{\frac{\sigma_1+\sigma_2}{2}}g_1(\kappa_1)F(\kappa_1,\varsigma_1)d\kappa_1\\
&\quad-\frac{8\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\sigma_1}^{\sigma_2}g_1(\tau)d\tau\int_{\varsigma_1}^{\frac{\varsigma_1+\varsigma_2}{2}}g_2(\kappa_2)F(\sigma_1,\kappa_2)d\kappa_2\\
&\quad+\frac{16\upsilon\rho}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\sigma_1}^{\frac{\sigma_1+\sigma_2}{2}}\int_{\varsigma_1}^{\frac{\varsigma_1+\varsigma_2}{2}}g_1(\kappa_1)g_2(\kappa_2)F(\kappa_1,\kappa_2)d\kappa_2d\kappa_1.\end{aligned}$$

This completes the proof. □

Corollary 2.2. If we choose $\upsilon=1$ and $\rho=1$ in Lemma 2.1, we have,

$$\mathcal{F}_{g_1,g_2}(\sigma_1,\sigma_2;\varsigma_1,\varsigma_2) \tag{2.7}$$
$$\begin{aligned}&= \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16}\int_0^1\int_0^1\left(\int_{\phi^1_{1,\sigma_1,\sigma_2}(\xi)}^{\psi^1_{1,\sigma_1,\sigma_2}(\xi)}g_1(\tau)d\tau\right)\left(\int_{\phi^2_{1,\varsigma_1,\varsigma_2}(\vartheta)}^{\psi^2_{1,\varsigma_1,\varsigma_2}(\vartheta)}g_2(\tau)d\tau\right)\\
&\quad\times\left[\frac{\partial^2F}{\partial\xi\partial\vartheta}(\phi^1_{1,\sigma_1,\sigma_2}(\xi),\phi^2_{1,\varsigma_1,\varsigma_2}(\vartheta))-\frac{\partial^2F}{\partial\xi\partial\vartheta}\left(\phi^1_{1,\sigma_1,\sigma_2}(\xi),\psi^2_{1,\varsigma_1,\varsigma_2}(\vartheta)\right)\right.\\
&\quad\left.-\frac{\partial^2F}{\partial\xi\partial\vartheta}\left(\psi^1_{1,\sigma_1,\sigma_2}(\xi),\phi^2_{1,\varsigma_1,\varsigma_2}(\vartheta)\right)+\frac{\partial^2F}{\partial\xi\partial\vartheta}\left(\psi^1_{1,\sigma_1,\sigma_2}(\xi),\psi^2_{1,\varsigma_1,\varsigma_2}(\vartheta)\right)\right].\end{aligned}$$

Corollary 2.3. If we choose $g_1(\tau)=1$ and $g_2(\tau)=1$ in Lemma 2.1, we have,

$$\frac{F(\sigma_2,\varsigma_2)+F(\sigma_2,\varsigma_1)+F(\sigma_1,\varsigma_2)+F(\sigma_2,\varsigma_1)}{4} \tag{2.8}$$

$$-\frac{1}{2(\sigma_2-\sigma_1)}\int_{\sigma_1}^{\sigma_2} g_1(\kappa_1)\left[F(\kappa_1,\varsigma_2)+F(\kappa_1,\varsigma_1)\right]d\kappa_1-\frac{1}{2(\varsigma_2-\varsigma_1)}\int_{\varsigma_1}^{\varsigma_2} g_2(\kappa_2)\left[F(\sigma_2,\kappa_2)+F(\sigma_1,\kappa_2)\right]d\kappa_2$$

$$+\frac{1}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\sigma_1}^{\sigma_2}\int_{\varsigma_1}^{\varsigma_2} g_1(\kappa_1)g_2(\kappa_2)F(\kappa_1,\kappa_2)d\kappa_2 d\kappa_1$$

$$= \frac{1}{16}\left[\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))-\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right.$$

$$\left.-\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)+\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right].$$

Theorem 2.4. Let $F:\Delta=[\sigma_1,\sigma_2]\times[\varsigma_1,\varsigma_2]\to\mathbb{R}$ be twice partially differentiable mapping on Δ°. Suppose that $g_1:[\sigma_1,\sigma_2]\to[0,\infty)$ and $g_2:[\varsigma_1,\varsigma_2]\to[0,\infty]$ are continuous mapping which are symmetric about $\frac{\sigma_1+\sigma_2}{2}$ and $\frac{\varsigma_1+\varsigma_2}{2}$, respectively. If $\frac{\partial^2 F}{\partial\xi\partial\vartheta}\in L(\Delta)$ and $\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\right|^q$, $q\geq 1$, is a co-ordinated convex, then for any positive integer υ and ρ we have the following inequality

$$\left|\mathcal{F}_{g_1,g_2}(\sigma_1,\sigma_2;\varsigma_1,\varsigma_2)\right|$$

$$\leq \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{4\upsilon\rho}\left[\int_0^{\upsilon}\int_0^{\rho}\left(\int_{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau\right)\left(\int_{\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}^{\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)} g_2(\tau)d\tau\right)d\vartheta d\xi\right]$$

$$\times\left[\frac{\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q}{4}\right]^{\frac{1}{q}}.$$

Proof . Taking modulus in Lemma 2.1, we have

$$\left|\mathcal{F}_{g_1,g_2}(\sigma_1,\sigma_2;\varsigma_1,\varsigma_2)\right|$$

$$\leq \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16\upsilon\rho}\int_0^{\upsilon}\int_0^{\rho}\left(\int_{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau\right)\left(\int_{\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}^{\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)} g_2(\tau)d\tau\right)$$

$$\times\left[\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right|\right.$$

$$\left.+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right|\right]$$

$$= \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16\upsilon\rho}\left[\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|d\vartheta d\xi\right.$$

$$+\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|d\vartheta d\xi$$

$$+\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right|d\vartheta d\xi$$

$$\left.+\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right|d\vartheta d\xi\right],$$

where

$$\Phi(\xi)=\int_{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau \text{ and } \Psi(\vartheta)=\int_{\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}^{\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)} g_2(\tau)d\tau. \tag{2.9}$$

By utilizing power-mean integral inequality and the inequality (1.5), we obtain

$$\begin{aligned}
&\left|\mathcal{F}_{g_1,g_2}(\sigma_1,\sigma_2;\varsigma_1,\varsigma_2)\right| \qquad (2.10)\\
\leq\ & \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16\upsilon\rho}\left(\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)d\vartheta d\xi\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|^q d\vartheta d\xi\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|^q d\vartheta d\xi\right)^{\frac{1}{q}}\\
&+\left(\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right|^q d\vartheta d\xi\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right|^q d\vartheta d\xi\right)^{\frac{1}{q}}\right]\\
\leq\ & \frac{4^{1-\frac{1}{q}}(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16\upsilon\rho}\left(\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)d\vartheta d\xi\right)^{1-\frac{1}{q}}\\
&\times\left[\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|^q d\vartheta d\xi\right.\\
&+\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|^q d\vartheta d\xi\\
&+\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right|^q d\vartheta d\xi\\
&\left.+\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right|^q d\vartheta d\xi\right]^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\right|^q$ is a co-ordinated convex, we get

$$\begin{aligned}
&\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|^q \qquad (2.11)\\
=\ & \left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\frac{\xi+\upsilon}{2\upsilon}\sigma_2+\frac{\upsilon-\xi}{2\upsilon}\sigma_1,\frac{\vartheta+\rho}{2\rho}\varsigma_2+\frac{\rho-\vartheta}{2\rho}\varsigma_1\right)\right|^q
\end{aligned}$$

$$\leq \frac{(\xi+\upsilon)(\vartheta+\rho)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+\frac{(\xi+\upsilon)(\rho-\vartheta)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q$$
$$+\frac{(\upsilon-\xi)(\vartheta+\rho)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+\frac{(\upsilon-\xi)(\rho-\vartheta)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q.$$

Similarly we have

$$\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|^q \tag{2.12}$$
$$\leq \frac{(\xi+\upsilon)(\vartheta+\rho)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+\frac{(\xi+\upsilon)(\rho-\vartheta)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q$$
$$+\frac{(\upsilon-\xi)(\vartheta+\rho)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q+\frac{(\upsilon-\xi)(\rho-\vartheta)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q,$$

$$\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right|^q \tag{2.13}$$
$$\leq \frac{(\xi+\upsilon)(\vartheta+\rho)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+\frac{(\xi+\upsilon)(\rho-\vartheta)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q$$
$$+\frac{(\upsilon-\xi)(\vartheta+\rho)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+\frac{(\upsilon-\xi)(\rho-\vartheta)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q$$

and

$$\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right|^q \tag{2.14}$$
$$\leq \frac{(\xi+\upsilon)(\vartheta+\rho)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q+\frac{(\xi+\upsilon)(\rho-\vartheta)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q$$
$$+\frac{(\upsilon-\xi)(\vartheta+\rho)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+\frac{(\upsilon-\xi)(\rho-\vartheta)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q.$$

If we substitute the inequalities (2.11)-(2.14) in (2.10), then we obtain

$$|\mathcal{F}_{g_1,g_2}(\sigma_1,\sigma_2;\varsigma_1,\varsigma_2)|$$
$$\leq \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{4^{1+\frac{1}{q}}\upsilon\rho}\left(\int_0^\upsilon\int_0^\rho \Phi(\xi)\Psi(\vartheta)d\vartheta d\xi\right)^{1-\frac{1}{q}}$$
$$\times\left[\int_0^\upsilon\int_0^\rho \Phi(\xi)\Psi(\vartheta)\left[\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q\right]d\vartheta d\xi\right]^{\frac{1}{q}}$$

which completes the proof. □

Corollary 2.5. If we choose $q = 1$ in Theorem 2.4, we have,

$$|\mathcal{F}_{g_1,g_2}(\sigma_1,\sigma_2;\varsigma_1,\varsigma_2)| \tag{2.15}$$

$$\leq \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16\upsilon\rho}\left[\int_0^{\upsilon}\int_0^{\rho}\left(\int_{\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)}^{\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau\right)\left(\int_{\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)}^{\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)} g_2(\tau)d\tau\right)d\vartheta d\xi\right]$$
$$\times\left[\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|\right].$$

Corollary 2.6. If we choose $\upsilon = 1$ and $\rho = 1$ in Theorem 2.4, we have,

$$|\mathcal{F}_{g_1,g_2}(\sigma_1,\sigma_2;\varsigma_1,\varsigma_2)| \tag{2.16}$$
$$\leq \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{4}\left[\int_0^{1}\int_0^{1}\left(\int_{\psi^1_{1,\sigma_1,\sigma_2}(\xi)}^{\phi^1_{1,\sigma_1,\sigma_2}(\xi)} g_1(\tau)d\tau\right)\left(\int_{\psi^2_{1,\varsigma_1,\varsigma_2}(\vartheta)}^{\phi^2_{1,\varsigma_1,\varsigma_2}(\vartheta)} g_2(\tau)d\tau\right)d\vartheta d\xi\right]$$
$$\times\left[\frac{\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q}{4}\right]^{\frac{1}{q}}.$$

Corollary 2.7. If we choose $g_1(\tau) = 1$ and $g_2(\tau) = 1$ in Theorem 2.4, we have,

$$\left|\frac{F(\sigma_2,\varsigma_2)+F(\sigma_2,\varsigma_1)+F(\sigma_1,\varsigma_2)+F(\sigma_2,\varsigma_1)}{4}\right. \tag{2.17}$$
$$-\frac{1}{2(\sigma_2-\sigma_1)}\int_{\sigma_1}^{\sigma_2} g_1(\kappa_1)\left[F(\kappa_1,\varsigma_2)+F(\kappa_1,\varsigma_1)\right]d\kappa_1-\frac{1}{2(\varsigma_2-\varsigma_1)}\int_{\varsigma_1}^{\varsigma_2} g_2(\kappa_2)\left[F(\sigma_2,\kappa_2)+F(\sigma_1,\kappa_2)\right]d\kappa_2$$
$$\left.+\frac{1}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\sigma_1}^{\sigma_2}\int_{\varsigma_1}^{\varsigma_2} g_1(\kappa_1)g_2(\kappa_2)F(\kappa_1,\kappa_2)d\kappa_2 d\kappa_1\right|$$
$$\leq \frac{1}{4}\left[\frac{\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q}{4}\right]^{\frac{1}{q}}.$$

Theorem 2.8. Let $F : \Delta = [\sigma_1,\sigma_2]\times[\varsigma_1,\varsigma_2] \to \mathbb{R}$ be twice partially differentiable mapping on Δ°. Suppose that $g_1 : [\sigma_1,\sigma_2] \to [0,\infty)$ and $g_2 : [\varsigma_1,\varsigma_2] \to [0,\infty]$ are continuous mapping which are symmetric about $\frac{\sigma_1+\sigma_2}{2}$ and $\frac{\varsigma_1+\varsigma_2}{2}$, respectively. If $\frac{\partial^2 F}{\partial\xi\partial\vartheta} \in L(\Delta)$ and $\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\right|^q$ is a co-ordinated convex , then for any positive integer υ and ρ we have the following inequality,

$$|\mathcal{F}_{g_1,g_2}(\sigma_1,\sigma_2;\varsigma_1,\varsigma_2)|$$
$$\leq \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16^{1+\frac{1}{q}}\upsilon\rho}\left[\int_0^{\upsilon}\int_0^{\rho}\left[\Phi(\xi)\Psi(\vartheta)\right]^p d\vartheta d\xi\right]^{\frac{1}{p}}$$
$$\times\left\{\left[9\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}}\right.$$
$$+\left[3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+9\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}}$$
$$+\left[3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+9\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}}$$

$$+\left[\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+9\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}}\Bigg\}$$

$$\leq \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16^{\frac{1}{q}}(\upsilon\rho)^{\frac{1}{p}}}\left[\int_0^{\upsilon}\int_0^{\rho}[\Phi(\xi)\Psi(\vartheta)]^p\,d\vartheta d\xi\right]^{\frac{1}{p}}$$
$$\times\left[\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$ for $q>1$.

Proof . By Lemma 2.1, we have

$$\begin{aligned}
&|\mathcal{F}_{g_1,g_2}(\sigma_1,\sigma_2;\varsigma_1,\varsigma_2)| \qquad (2.18)\\
\leq\ & \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16\upsilon\rho}\left[\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|d\vartheta d\xi\right.\\
&+\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|d\vartheta d\xi\\
&+\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right|d\vartheta d\xi\\
&\left.+\int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\left(\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right)\right|d\vartheta d\xi\right]\\
=\ & \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16\upsilon\rho}[K_1+K_2+K_3+K_4],
\end{aligned}$$

where $\Phi(\xi)$ and $\Psi(\vartheta)$ are defined as in (2.9).

Using well-known Hölder inequality and the co-ordinated convexity of $\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\right|^q$,

$$\begin{aligned}
K_1 =\ & \int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|d\vartheta d\xi \qquad (2.19)\\
\leq\ & \left[\int_0^{\upsilon}\int_0^{\rho}[\Phi(\xi)\Psi(\vartheta)]^p\,d\vartheta d\xi\right]^{\frac{1}{p}}\left[\int_0^{\upsilon}\int_0^{\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|^q d\vartheta d\xi\right]^{\frac{1}{q}}\\
\leq\ & \left[\int_0^{\upsilon}\int_0^{\rho}[\Phi(\xi)\Psi(\vartheta)]^p\,d\vartheta d\xi\right]^{\frac{1}{p}}\left[\int_0^{\upsilon}\int_0^{\rho}\left(\frac{(\xi+\upsilon)(\vartheta+\rho)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q\right.\right.\\
&+\frac{(\xi+\upsilon)(\rho-\vartheta)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+\frac{(\upsilon-\xi)(\vartheta+\rho)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q\\
&\left.\left.+\frac{(\upsilon-\xi)(\rho-\vartheta)}{4\upsilon\rho}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right)d\vartheta d\xi\right]^{\frac{1}{q}}
\end{aligned}$$

$$= \left[\int_0^{\upsilon}\int_0^{\rho}[\Phi(\xi)\Psi(\vartheta)]^p\,d\vartheta d\xi\right]^{\frac{1}{p}}\left[\frac{9\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+\frac{3\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q\right.$$
$$\left.+\frac{3\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+\frac{\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}},$$

Similarly, we one can establish

$$K_2 = \int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\phi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta))\right|d\vartheta d\xi \tag{2.20}$$
$$\leq \left[\int_0^{\upsilon}\int_0^{\rho}[\Phi(\xi)\Psi(\vartheta)]^p\,d\vartheta d\xi\right]^{\frac{1}{p}}\left[\frac{9\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+\frac{3\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q\right.$$
$$\left.+\frac{3\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q+\frac{\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q\right]^{\frac{1}{q}},$$

$$K_3 = \int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\phi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right|d\vartheta d\xi \tag{2.21}$$
$$\leq \left[\int_0^{\upsilon}\int_0^{\rho}[\Phi(\xi)\Psi(\vartheta)]^p\,d\vartheta d\xi\right]^{\frac{1}{p}}\left[\frac{9\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+\frac{3\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right.$$
$$\left.+\frac{3\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+\frac{\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q\right]^{\frac{1}{q}},$$

and

$$K_4 = \int_0^{\upsilon}\int_0^{\rho}\Phi(\xi)\Psi(\vartheta)\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}\psi^1_{\upsilon,\sigma_1,\sigma_2}(\xi),\psi^2_{\rho,\varsigma_1,\varsigma_2}(\vartheta)\right|d\vartheta d\xi \tag{2.22}$$
$$\leq \left[\int_0^{\upsilon}\int_0^{\rho}[\Phi(\xi)\Psi(\vartheta)]^p\,d\vartheta d\xi\right]^{\frac{1}{p}}\left[\frac{9\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q+\frac{3\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q\right.$$
$$\left.+\frac{3\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+\frac{\upsilon\rho}{16}\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q\right]^{\frac{1}{q}}.$$

If we substitute the inequalities (2.19)-(2.22) in (2.18), we get

$$\left|\mathcal{F}_{g_1,g_2}(\sigma_1,\sigma_2;\varsigma_1,\varsigma_2)\right|$$
$$\leq \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16^{1+\frac{1}{q}}\upsilon\rho}\left[\int_0^{\upsilon}\int_0^{\rho}[\Phi(\xi)\Psi(\vartheta)]^p\,d\vartheta d\xi\right]^{\frac{1}{p}}$$
$$\times\left\{\left[9\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}}\right.$$

$$+\left[9\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q+\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q\right]^{\frac{1}{q}}$$

$$+\left[9\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q\right]^{\frac{1}{q}}$$

$$+\left[9\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q\right]^{\frac{1}{q}}\Bigg\}.$$

This completes the first inequality. The proof of second inequality is obvious from the inequality (1.6) and the fact that $9^{\frac{1}{q}}+2.3^{\frac{1}{q}}+1\leq 16$. □

Corollary 2.9. If we choose $\upsilon=1$ and $\rho=1$ in Theorem 2.8, we have

$$\begin{aligned}
&\left|\mathcal{F}_{g_1,g_2}(\sigma_1,\sigma_2;\varsigma_1,\varsigma_2)\right|\\
\leq\ &\frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16^{1+\frac{1}{q}}}\left[\int_0^1\int_0^1\left[\Phi^*(\xi)\Psi^*(\vartheta)\right]^p d\vartheta d\xi\right]^{\frac{1}{p}}\\
&\times\Bigg\{\left[9\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+3\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+3\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}}\\
&+\left[3\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+9\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+3\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}}\\
&+\left[3\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+9\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+3\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}}\\
&+\left[\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+3\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+3\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+9\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}}\Bigg\}\\
\leq\ &\frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16^{\frac{1}{q}}}\left[\int_0^1\int_0^1\left[\Phi^*(\xi)\Psi^*(\vartheta)\right]^p d\vartheta d\xi\right]^{\frac{1}{p}}\\
&\times\left[\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|\right],
\end{aligned}$$

where $\Phi^*(\xi)=\int_{\psi^1_{1,\sigma_1,\sigma_2}(\xi)}^{\phi^1_{1,\sigma_1,\sigma_2}(\xi)}g_1(\tau)d\tau$ and $\Psi^*(\vartheta)=\int_{\psi^2_{1,\varsigma_1,\varsigma_2}(\vartheta)}^{\phi^2_{1,\varsigma_1,\varsigma_2}(\vartheta)}g_2(\tau)d\tau$.

Corollary 2.10. If we choose $g_1(\tau)=1$ and $g_2(\tau)=1$ in Theorem 2.8, we have,

$$\begin{aligned}
&\Bigg|\frac{F(\sigma_2,\varsigma_2)+F(\sigma_2,\varsigma_1)+F(\sigma_1,\varsigma_2)+F(\sigma_2,\varsigma_1)}{4}\\
&-\frac{1}{2(\sigma_2-\sigma_1)}\int_{\sigma_1}^{\sigma_2}g_1(\kappa_1)\left[F(\kappa_1,\varsigma_2)+F(\kappa_1,\varsigma_1)\right]d\kappa_1-\frac{1}{2(\varsigma_2-\varsigma_1)}\int_{\varsigma_1}^{\varsigma_2}g_2(\kappa_2)\left[F(\sigma_2,\kappa_2)+F(\sigma_1,\kappa_2)\right]d\kappa_2\\
&+\frac{1}{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}\int_{\sigma_1}^{\sigma_2}\int_{\varsigma_1}^{\varsigma_2}g_1(\kappa_1)g_2(\kappa_2)F(\kappa_1,\kappa_2)d\kappa_2 d\kappa_1\Bigg|\\
\leq\ &\frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16^{1+\frac{1}{q}}\upsilon^{\frac{1}{q}}\rho^{\frac{1}{q}}(p+1)^{\frac{2}{p}}}
\end{aligned}$$

$$\times\left[9\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}}$$

$$+\left[3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+9\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}}$$

$$+\left[3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+9\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}}$$

$$+\left[\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|^q+3\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|^q+9\upsilon\rho\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|^q\right]^{\frac{1}{q}}$$

$$\leq \frac{(\sigma_2-\sigma_1)(\varsigma_2-\varsigma_1)}{16^{\frac{1}{q}}(p+1)^{\frac{2}{p}}}$$

$$\times\left[\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_2)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_2,\varsigma_1)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_2)\right|+\left|\frac{\partial^2 F}{\partial\xi\partial\vartheta}(\sigma_1,\varsigma_1)\right|\right].$$

3 Conclusion

In this article, we derive the weighted Hermite-Hadamard inequalities resulting from double integrals. New results can be obtained with different special function choices. Going forward, researchers can derive new inequalities with different fractional integrals of these inequalities. In addition, new inequalities can be created with the help of different types of convexity in the literature

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