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Study of a mathematical model of an epidemic via dynamic programming approach

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Abstract

We use some recent developments in Dynamics Programming Method to obtain a rigorous solution of the epidemic model formulated in [19] as an unsolved problem. In fact, this problem is proposed in the context of using Pontryagin's Maximum Principle. We use a certain refinement of Cauchy's Method of characteristics for stratified Hamilton-Jacobi equations to describe a large set of admissible trajectories and identify a domain on which the value function exists and is generated by a certain admissible control. The optimality is justified by using of one of the well-known verification theorems as an argument for sufficient optimality conditions.

Keywords: Optimal control, Dynamic programming, Maximum principle, Differential inclusion, Hamiltonian flow, Value function
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1 Introduction

Mathematical models have long been used in infectious disease epidemiology to understand the dynamics of epidemics in populations and predicting outcomes of effective control strategies. The importance of this theme comes to the fore with the unfortunate COVID-19 pandemic and the need to discover its crypts by introducing it as dynamic models amenable to study, by using recent results in control theory. In this proposed study, we have considered a simplified (classical) version of an epidemic model developed by Trélat in [19] to examine the effect of vaccination strategy. In general, vaccination is an important control measure to reduce the size of infected populations. Many researchers have studied how to predict and evaluate the effectiveness of various vaccination strategies with a mathematical model [1, 2, 6, 14, 15, 16, 18]. We have to remark that, usually in such models the size of population is assumed to be constant. Additionally, there are some papers in which an epidemic control means finding such parameters or initial conditions of the model at which the number of suspected individuals is reduced (for example, by imposing total vaccination). Mathematical models of epidemics, have developed over time and become an important tool in describing the dynamics of the spread of infectious diseases and influence on dynamics of different preventive methods, such as vaccination, treatment, quarantine,...etc. In [2] was introduced the optimal pair of vaccination and treatment strategies that would maximize the recovered population and minimize both the infected and susceptible

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population, and at the same time minimize the costs of applying the vaccination and treatment strategies. While, in [15] the optimal triplet of vaccination, treatment and quarantine strategies were used. Therefore, these models can be reformulated as certain differential games models (see for instance, [6]). There are two main methods in Optimal Control Theory: the maximum principle and dynamic programming approach (DP). In epidemiology, the classical version of Pontryagin's maximum principle was used, for instance in [2, 14]. The classical version of (DP) was used in [8] using the so-called viscosity solution, which is considered as an approximation solution of the HJB equation. While, the generalized (DP) described in [10, 11], was used in [3] will be our essential tool in what follows.

The aim of this paper is to apply step by step manner the theoretical dynamic programming algorithm, described in [10, 11], as to combine these results with numerical procedures, to obtain a more rigorous and theoretically complete solution of the unsolved problem formulated as Example 7.3.11 in [19]. In fact, in [19], this problem is proposed to answer only a certain question, within the framework of the use of the principle of the maximum in [4, 5, 7, 17], but not to study it in the rigorous manner in contrast to what we do below.

Solving this problem with the dynamic programming method described in [10, 11] has the advantage that, on one hand, we can determine all the admissible trajectories of the problem and, on the other hand, the hypotheses to be verified are much more natural and easier to verify, considering the elements of the theory of Hamilton-Jacobi equations as well as the recent results from the so-called Non-Smooth Analysis in [5, 10].

Working as in [3, 12, 13]; more precisely, we first use a certain extension of Cauchy's Method of characteristics for stratified Hamiltonian-Jacobi equations to identify a much larger set of admissible (possibly optimal) trajectories which are defined as solutions of certain state constrained differential systems; as rigorous criterion for choosing the optimal trajectories we are using the associated value function which in turn, has some monotonicity properties along these admissible trajectories. The characteristic flow makes it possible to identify a subset of the set of initial states over which optimal control and the corresponding value function are described, while the optimality is proved using a suitable Elementary Verification Theorem, according to which a sufficient optimality condition, for the admissible control is the verification of a certain differential inequality [5, 10, 9].

The paper is organized as follows: after the introduction, we present in Section 2 the position of the problem, its dynamic programming formulation, and the characterization of the Hamiltonian. Section 3 gives the generalized stratified Hamiltonian field. In Section 4, we describe the partial Hamiltonian flow whose trajectories have terminal segments on each of the strata. Section 5 shows the existence of the corresponding value function which defines a certain admissible and possibly optimal control of the considered control problem. Finally, some concluding remarks are provided in Section 6.

2 Position of the problem

In [19] it has been considered a population of N individuals affected by an epidemic, that is to be controlled by vaccination. For simplicity, it is assumed that, an individual who has been ill and cared can fall ill again. This leads us to solve the optimal control problem of *minimizing* the cost functional:

$$\begin{cases} \mathcal{C}(u) = \int_0^T [x(t) + \beta u(t)]\, dt, \\ x' = \alpha x(N-x) - u(t)x,\ x(0) = x_0 \in [0, N] \\ u(t) \in [0, a],\ t \in [0, T],\ T \text{ fixed}, \end{cases} \tag{2.1}$$

where, $\beta > 0$, $a > 0$ are constant; the functions involved have the following virology significance:

$x(t)$: the number of infected individuals at time $t \in [0, T]$;

$u(t)$: the vaccination rate (which is actually the control function) at time $t \in [0, T]$;

$\alpha > 0$: the contamination rate.

The model described in (2.1) is a description that treats an epidemic disease. The aim is to study the control of the effect of vaccination on the population in order to reduce the risk of contamination and stop the spread of the epidemic.

2.1 The dynamic programming formulation

In order to use the Dynamic Programming Approach in [10, 11], we reformulate the problem in (2.1) using standard notations in Optimal Control Theory and embedding this problem in a set of problems associated to each initial point in the phase-space as in [3, 12, 13]. Thus, we obtain the following standard Lagrange autonomous optimal control problem:

Problem 2.1. Given T, $\beta > 0$, find:

$$\inf_{u(.)} \mathcal{C}\left(y, u\left(.\right)\right), \ \forall y \in Y_0 \tag{2.2}$$

subject to:

$$\begin{array}{l} \mathcal{C}\left(y, u\left(.\right)\right) = g\left(x\left(T\right)\right) + \int_0^T f_0\left(x\left(t\right), u\left(t\right)\right) dt, \\ x'(t) = f(x(t), u(t)),\ u\left(t\right) \in U\left(x(t)\right)\ a.e.\ ([0,T]),\ x\left(0\right) = y, \\ x\left(t\right) \in Y_0,\ \forall t \in [0,T),\ x\left(T\right) \in Y_1,\ T \text{ fixed} \end{array} \tag{2.3}$$

defined by the following data:

$$\begin{array}{l} f\left(x,u\right) = \alpha x\left(N - x\right) - ux,\ f_0\left(x,u\right) = x + \beta u, \\ U\left(x\right) = U = \left[0,a\right],\ a > 0,\ g\left(\xi\right) = 0,\ \forall \xi \in Y_1, \\ Y_0 = \left(0,N\right),\ Y_1 \subset cl\left(Y_0\right). \end{array} \tag{2.4}$$

2.2 Characterization of the Hamiltonian

The first step of the Dynamic Programming Approach consists in the characterization of the true Hamiltonian of the problem. The pseudo-Hamiltonian $\mathcal{H}(x,p,u) =< p, f(x,u) > +f_0(x,u)$ is given in our case by:

$$\mathcal{H}(x,p,u) = \alpha p x\left(N - x\right) + x - \left(px - \beta\right)u. \tag{2.5}$$

The Hamiltonian and the corresponding multifunction of minimum points are given by the formulas:

$$\begin{array}{l} H\left(x,p\right) = \min\limits_{u\in U} \mathcal{H}(x,p,u) = \alpha p x\left(N - x\right) + x + \min\limits_{u\in U}\left[-\left(px - \beta\right)u\right] \\ \widehat{U}\left(x,p\right) = \left\{u \in U;\ \mathcal{H}(x,p,u) = H\left(x,p\right)\right\},\ \ (x,p) \in Z = dom\left(H\left(.,.\right)\right). \end{array} \tag{2.6}$$

Using the well known fact that:

$$\min_{u\in U}\left[-\left(px - \beta\right)u\right] = \phi\left(x,p\right) = \begin{cases} a\left(\beta - px\right), & \text{if } h\left(x,p\right) = xp > \beta \\ 0, & \text{if } h\left(x,p\right) \le \beta, \end{cases} \tag{2.7}$$

therefore, the Hamiltonian function as well as the corresponding multifunction of minimum points turn out to be defined on Z by:

$$\begin{array}{l} H\left(x,p\right) = \alpha p x\left(N - x\right) + x + \phi\left(x,p\right),\ \ (x,p) \in Z \\ \widehat{U}\left(x,p\right) = \left\{\widehat{u}\left(x,p\right)\right\} = \begin{cases} \{a\}, & \text{if } h\left(x,p\right) > \beta \\ \{0\}, & \text{if } h\left(x,p\right) < \beta \\ U = [0,a], & \text{if } h\left(x,p\right) = \beta \end{cases} \end{array} \tag{2.8}$$

First, we remark that the Hamiltonian $H(.,.)$ in (2.8) as well as its domain Z are $\mathcal{C}^1$-stratified by the stratification $S_H = \{Z_+,\ Z_-,\ Z_0\}$ defined by:

$$\begin{array}{l} Z_+ = \left\{(x,p) \in Z;\ h\left(x,p\right) > \beta\right\} \\ Z_- = \left\{(x,p) \in Z;\ h\left(x,p\right) < \beta\right\} \\ Z_0 = \left\{(x,p) \in Z;\ h\left(x,p\right) = \beta\right\}. \end{array} \tag{2.9}$$

If we denote by: $H_\pm(.,.) = H(.,.)|_{Z_\pm}$, $H_0(.,.) = H(.,.)|_{Z_0}$ then, from (2.7) and (2.9) it follows:

$$\begin{array}{ll} H_+\left(x,p\right) = \alpha p x\left(N - x\right) + x + a\left(\beta - px\right), & \text{if } \ (x,p) \in Z_+ \\ H_-\left(x,p\right) = \alpha p x\left(N - x\right) + x, & \text{if } \ (x,p) \in Z_- \\ H_0\left(x,p\right) = \alpha \beta\left(N - x\right) + x, & \text{if } \ (x,p) \in Z_0. \end{array} \tag{2.10}$$

Next, we need to compute the set of terminal transversality values defined in the general case by:

$$Z^* = \{(\xi, q) \in \overline{Z},\ \xi \in Y_1,\ H(\xi,q) = 0,\ < q, \overline{\xi} >= Dg(\xi)\overline{\xi}\ \forall\ \overline{\xi} \in T_\xi Y_1\}. \tag{2.11}$$

In order to characterize the set of terminal transversality values Z^* defined above, we prove the following result.

Lemma 2.2. The set of terminal transversality values Z^* in our case is given by the formulas:

$$\begin{array}{l} Z^* = Z_+^* \cup Z_-^* \cup Z_0^*,\ \Delta(q) = [q(\alpha N - a) + 1]^2 + 4a\alpha\beta q > 0 \\ Z_+^* = \{(\xi(q), q);\ q > 0\} \subset Z_+,\ \xi(q) = \frac{q(\alpha N - a) + 1 + \sqrt{\Delta(q)}}{2\alpha q} \\ Z_-^* = \left\{ \left(N + \frac{1}{\alpha q}, q \right);\ q < \frac{1}{N}\left(\beta - \frac{1}{\alpha}\right) \right\} \subset Z_- \\ Z_0^* = \left\{ \left(\frac{\alpha\beta N}{\alpha\beta - 1}, \frac{1}{N}\left(\beta - \frac{1}{\alpha}\right) \right) \right\} \subset Z_0,\ \text{if } \alpha\beta - 1 \neq 0. \end{array} \tag{2.12}$$

Proof. Since, T is fixed then, the tangent space $T_\xi Y_1 = \{0\}$ and from (2.11) it follows that, $q\overline{\xi} = 0$, $\forall \overline{\xi} \in T_\xi Y_1$ therefore, $q \in \mathbb{R}$. On the other hand, due to the fact that, $H(\xi, q) = 0$, $\xi > 0$ then, the following cases can be extracted:

Case 1: If $(\xi, q) \in Z_+$ then, $h(\xi, q) = \xi q > \beta > 0$ hence, $q > 0$. While, from the fact that, $H_+(\xi, q) = 0$, $\xi > 0$, we obtain the following second order equation:

$$-\alpha q \xi^2 + [q(\alpha N - a) + 1]\xi + a\beta = 0,\ \xi > 0,$$

which has as a positive root that $\xi = \xi(q)$, $q > 0$ given in (2.12).

Case 2: If $(\xi, q) \in Z_-$ then, $h(\xi, q) = \xi q < \beta$ and from (2.10) it follows that, $H_-(\xi, q) = \xi[\alpha q(N - \xi) + 1] = 0$, since, $\xi > 0$ then, $\xi = N + \frac{1}{\alpha q}$. Moreover, using the fact that, $h(\xi, q) = \xi q = qN + \frac{1}{\alpha} < \beta$ hence, $q < \frac{1}{N}\left(\beta - \frac{1}{\alpha}\right)$.

Case 3: If $(\xi, q) \in Z_0$, since, $H_0(\xi, q) = \alpha\beta(N - \xi) + \xi = \alpha\beta N - (\alpha\beta - 1)\xi = 0$ then, $\xi = \frac{\alpha\beta N}{\alpha\beta - 1}$, if $\alpha\beta - 1 \neq 0$. Moreover, using the fact that, $h(\xi, q) = \xi q = q\frac{\alpha\beta N}{\alpha\beta - 1} = \beta$, and therefore, $q = \frac{1}{N}\left(\beta - \frac{1}{\alpha}\right)$. This completes the proof. ■

3 Generalized Hamiltonian and characteristic flow

The first main computational operation consists in the backward integration (for $t \leq 0$), of the Hamiltonian inclusion:

$$(x', p') \in d_S^{\#} H(x, p),\ (x(0), p(0)) = z = (\xi, q) \in Z^*, \tag{3.1}$$

defined by the generalized Hamiltonian field $d_S^{\#} H(.,.)$:

$$\begin{array}{l} d_S^{\#} H(x, p) = \{(x', p') \in T_{(x,p)} Z;\ x' \in f(x, \widehat{U}(x, p)), \\ < x', \overline{p} > - < p', \overline{x} >= DH(x, p)(\overline{x}, \overline{p}),\ \forall\ (\overline{x}, \overline{p}) \in T_{(x,p)} Z\}. \end{array} \tag{3.2}$$

As it is specified in the algorithm in [10, 11], for each terminal point $z = (\xi, q) \in Z^*$ one should identify the maximal solutions: $X^*(.) = (X(.), P(.)) : I(z) := (t^-(z), 0] \to Z$, of the Hamiltonian inclusion in (3.1) that satisfy the following conditions:

$$\begin{array}{l} X(t) \in Y_0,\ \forall\ t \in I_0(z) = (t^-(z), 0) \\ H(X(t), P(t)) = 0,\ \forall\ t \in I(z) \\ X'(t) = f(X(t), u(t)),\ u(t) \in \widehat{U}(X^*(t))\ a.e.\ I_0(z). \end{array} \tag{3.3}$$

In the case in which there exist more such solutions for the same terminal point $z = (\xi, q) \in Z^*$, one should parameterize by $\lambda \in \Lambda(z)$ the set of these solutions to obtain a generalized Hamiltonian flow: $X^*(.,.) = (X(.,.), P(.,.)) : B = \{(t, b);\ t \in I(b),\ b \in A\} \to Z;\ A = graph(\Lambda(.)),\ b = (z, \lambda)$. We recall also the fact that for each $(t, b) \in B_0 = \{(t, b) \in B;\ t \neq 0\}$ the Hamiltonian flow $X^*(.,.)$ defines the controls and, respectively, trajectories:

$$u_{t,b}(s) = u_b(t + s),\ x_{t,b}(s) = X(t + s, b),\ s \in [0, -t], \tag{3.4}$$

which are admissible with respect to the initial point $y = X(t, b) \in Y_0$, and for which the value of the cost functional in (2.3) is given by the function $V(.,.)$ defined by:

$$V(t, b) = g(\xi) + \int_0^t < P(\sigma, b), X'(\sigma, b) > d\sigma,\ \text{if } b = (z, \lambda), \tag{3.5}$$

and which, together with the Hamiltonian flow $X^*(.,.)$ defines the generalized characteristic flow $C^*(.,.) = (X^*(.,.), V(.,.))$; using the definition of the Hamiltonian $H(.,.)$ and the second condition in (3.3) one has $< P(\sigma, b), X'(\sigma, b) >= -f_0(X(\sigma, b), \widehat{u}(X^*(\sigma, b)))$; it follows that in our case the function $V(.,.)$ is given by:

$$V(t, b) = -\int_0^t [X(\sigma, b) + \beta\widehat{u}(X^*(\sigma, b))]\, d\sigma,\ (t, b) \in B,\ b = (z, \lambda). \tag{3.6}$$

Moreover, it follows from (2.9) that, the generalized Hamiltonian field $d_S^{\#}H(.,.)$ is given by the formulas:

$$d_S^{\#}H(x,p) = \begin{cases} d_S^{\#}H_{\pm}(x,p), \text{ if } (x,p) \in Z_{\pm} \\ d_S^{\#}H_0(x,p), \text{ if } (x,p) \in Z_0. \end{cases} \tag{3.7}$$

Since the manifolds Z_+, $Z_- \subset Z$, are open subsets, the Hamiltonian fields $d_S^{\#}H_{\pm}(.,.)$ coincide with classical Hamiltonian vector fields:

$$d_S^{\#}H_{\pm}(x,p) = \left\{ \left(\frac{\partial H_{\pm}}{\partial p}(x,p), -\frac{\partial H_{\pm}}{\partial x}(x,p) \right) \right\}, \quad (x,p) \in Z_{\pm}, \tag{3.8}$$

which are easy to calculate and will be described and studied later, while on the 1-dimensional singular stratum $Z_0 \subset Z$ the corresponding Hamiltonian field is more difficult to compute.

3.1 The Hamiltonian field on the singular stratum Z_0

The characterization of the Hamiltonian field $d_S^{\#}H_0(.,.)$ on the singular stratum Z_0 is proved in the following result.

Lemma 3.1. For any $(x,p) \in Z_0$, one has $d_S^{\#}H_0(x,p) = \emptyset$.

Proof. In order to compute the generalized Hamiltonian field $d_S^{\#}H_0(.,.)$, we note first that, according to a certain classical results as in [10], the tangent space to the 1-dimensional manifold Z_0 are given by:

$$\begin{aligned} T_{(x,p)}Z_0 &= \{(\overline{x},\overline{p}) \in \mathbb{R}\times\mathbb{R};\ Dh(x,p)(\overline{x},\overline{p}) = 0\} \\ &= \{(\overline{x},\overline{p}) \in \mathbb{R}\times\mathbb{R};\ p\overline{x} + x\overline{p} = 0\}, \end{aligned}$$

on the stratum Z_0 one has $DH_0(x,p)(\overline{x},\overline{p}) = (1-\alpha\beta)\,\overline{x}$ hence a vector $(x',p') \in d_S^{\#}H_0(x,p)$ is fully characterized by the properties:

$$x'\overline{p} - [p' + (1-\alpha\beta)]\,\overline{x} = 0,\ \forall \overline{x},\ \overline{p} \in \mathbb{R}$$

it follows that, at each point $(x,p) \in Z_0$ one has:

$$x' = 0,\ p' = \alpha\beta - 1 \neq 0. \tag{3.9}$$

Using the fact that, $(x',p') \in T_{(x,p)}Z_0$ then, from (3.9) we deduce that, $px' + xp' = xp' = 0$, which leads to a contradiction. This completes the proof. ■

Summarizing, the Hamiltonian field in (3.2) is given by the formulas:

$$d_S^{\#}H(x,p) = \begin{cases} d_S^{\#}H_{\pm}(x,p), & \text{if } (x,p) \in Z_{\pm} \\ \emptyset, & \text{if } (x,p) \in Z_0, \end{cases} \tag{3.10}$$

where $d_S^{\#}H_{\pm}(.,.)$ are the Hamiltonian fields in formula (3.8) which will be described and studied in what follows.

3.2 The Hamiltonian system on the open stratum Z_-

On the open stratum Z_- in (2.9) for which $h(x,p) = xp < \beta$, the differential inclusion in (3.1) coincides with the smooth Hamiltonian system:

$$\begin{cases} x' = \alpha x(N-x) \\ p' = \alpha p(2x-N) - 1. \end{cases} \tag{3.11}$$

Standard results from differential equations theory show that the general solution of the system in (3.11) is given by the formulas:

$$\begin{aligned} x^-(t) &= \tfrac{Nce^{\alpha Nt}}{ce^{\alpha Nt}-1},\ k,\ c \in \mathbb{R},\ t < 0 \\ p^-(t) &= \tfrac{1}{\alpha Nc}\left(c - e^{-\alpha Nt}\right) + k\left(ce^{\alpha Nt} - 1\right)^2 e^{-\alpha Nt}. \end{aligned} \tag{3.12}$$

3.3 The Hamiltonian system on the open stratum Z_+

On the open stratum Z_+ in (2.9) for which $h(x,p) = xp > \beta$, the differential inclusion in (3.1) coincides with the smooth Hamiltonian system:

$$\begin{cases} x' = x\left[\alpha\left(N - x\right) - a\right] \\ p' = p\left[\alpha\left(2x - N\right) + a\right] - 1, \end{cases} \tag{3.13}$$

whose general solution is given by the formulas:

$$\begin{cases} x^+(t) = \frac{\alpha N - a}{\alpha} \frac{ce^{(\alpha N - a)t}}{ce^{(\alpha N - a)t} - 1}, \; k, \; c \in \mathbb{R}, \; t < 0 \\ p^+(t) = \frac{ce^{(\alpha N - a)t} - 1}{ce^{(\alpha N - a)t}} \left[\frac{1}{\alpha N - a} + k\left(ce^{(\alpha N - a)t} - 1\right)\right]. \end{cases} \tag{3.14}$$

4 Construction of the Hamiltonian flow

4.1 The Hamiltonian flow ending on the stratum Z_-

In this section, we describe the partial Hamiltonian flow whose trajectories have terminal segments on the stratum Z_-. Considering the general solution in (3.12), an admissible trajectory $X_-^*(.,z) = (X^-(.,z), P^-(.,z)), \; z \in Z_-^*$ of system (3.11) should satisfy the terminal conditions from the set of transversality terminal points Z_-^* in (2.12) and also the fact that $X_-^*(t,z) \in Z_- \; \forall \, t < 0$. From the terminal condition in (2.12) it follows that $c = \alpha q N + 1$, $k = \frac{1}{\alpha N(\alpha q N + 1)}$, $q < \frac{1}{N}\left(\beta - \frac{1}{\alpha}\right)$.

Therefore, we obtain the solution of the differential system in (3.11) in the form of a maximal flow $X_-^*(.,.,.) = (X^-(.,.,.), P^-(.,.))$ whose components are given by the formulas:

$$\begin{aligned} X^-(t,q) &= \frac{N(\alpha q N + 1)e^{\alpha N t}}{(\alpha q N + 1)e^{\alpha N t} - 1}, \; q < \frac{1}{N}\left(\beta - \frac{1}{\alpha}\right), \; t < 0 \\ P^-(t,q) &= \frac{1}{\alpha N(\alpha q N + 1)}\left(-e^{-\alpha N t} + \alpha q N + 1\right) + \\ &\quad \frac{1}{\alpha N(\alpha q N + 1)}\left[(\alpha q N + 1)e^{\alpha N t} - 1\right]^2 e^{-\alpha N t}. \end{aligned} \tag{4.1}$$

From the dynamic programming algorithm in [10, 11] it follows that we must retain only the trajectories $X_-^*(.,z)$, $z = (\xi, q) \in Z_-^*$, that satisfy the conditions in (3.3). We note first that the second condition in (3.3) is automatically satisfied since $H_-(.,.)$ defined in (2.10) is a first integral of differential system (3.11) hence:

$$\begin{aligned} &h_-(t,q) = h\left(X_-^*(t,q)\right) = X^-(t,q)\,P^-(t,q) < \beta, \; \forall \, t < 0 \\ &H_-(X_-^*(t,q)) = 0, \; X^-(t,q) \in (0,N), \; q < \frac{1}{N}\left(\beta - \frac{1}{\alpha}\right). \end{aligned} \tag{4.2}$$

The admissible trajectories must satisfy also the conditions:

$$\begin{aligned} &X_-^*(t,q) = (X^-(t,q), P^-(t,q)) \in Z_-, \; \forall \, t \in (\tau^-(q), 0) \\ &X^-(t,q) \in Y_0 = (0,N), \; q < \frac{1}{N}\left(\beta - \frac{1}{\alpha}\right), \end{aligned} \tag{4.3}$$

on the maximal intervals $I^-(q) = (\tau^-(q), 0)$, $q < \frac{1}{N}\left(\beta - \frac{1}{\alpha}\right)$, hence the extremity $\tau^-(.)$ is defined by:

$$\begin{aligned} &\tau^-(q) = \max\left\{\tau_1^-(q), \tau_2^-(q)\right\}, \\ &\tau_1^-(q) = \inf\left\{\tau < 0; \; X^-(t,q) \in (0,N), \forall \, t \in (\tau, 0)\right\} \\ &\tau_2^-(q) = \inf\left\{\tau < 0; \; h_-(t,q) < \beta, \forall \, t \in (\tau, 0)\right\}. \end{aligned} \tag{4.4}$$

Trying to obtain an explicit formula for the extremity $\tau^-(.)$, we note that the first expression in (4.1) allows an explicit formula for the extremity $\tau_1^-(.)$ defined in (4.4). To this end, if we denote by, $y = (\alpha q N + 1)e^{\alpha N t}$ then, it follows from (4.1) that:

$$X^-(t,q) \in (0,N) \Rightarrow \frac{Ny}{y-1} > 0, \; \frac{Ny}{y-1} < N,$$

and this implies that, $Ny(y-1) > 0$, $y < 1$ hence, $y = (\alpha q N + 1)e^{\alpha N t} < 0$, $q < -\frac{1}{\alpha N}$, $\forall t \in (-\infty, 0)$ and we obtain:

$$\begin{aligned} &\tau_1^-(q) = -\infty, \; q < -\frac{1}{\alpha N} < \frac{1}{N}\left(\beta - \frac{1}{\alpha}\right) \\ &\tau^-(q) = \tau_2^-(q). \end{aligned} \tag{4.5}$$

One may note here that, geometrically, the trajectories $X^{-}(.,q)$, $q<-\frac{1}{\alpha N}$, are the curves in Figure 1 and cover the domain $Y_0^{-}\subset Y_0$ such that:

$$\begin{aligned} Y_0^{-} &= \{X^{-}(t,q);\ (t,q)\in B^{-}\}, \\ B^{-} &= \left\{(t,q);\ t\in(\tau^{-}(q),0),\ q<-\tfrac{1}{\alpha N}\right\}. \end{aligned} \tag{4.6}$$

Several other similar results lead to the conjecture below, which can only be justified by numerical tests (as shown in Figure 2)

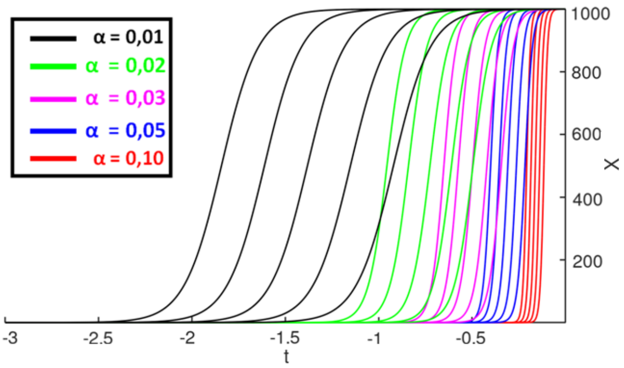

Figure 1: $X^{-}(.,q)\,, q<-\frac{1}{\alpha N}$

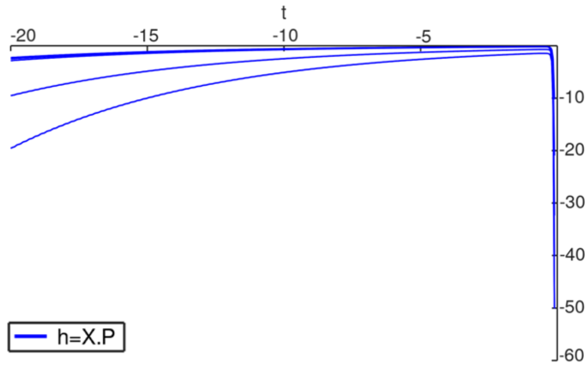

Figure 2: $h_{-}(.,q)\,, q<-\frac{1}{\alpha N}$

Conjecture 4.1. The function $h_{-}(.,.)$ defined in (4.2) satisfies the following condition:

$$h_{-}(t,q)<0,\ \forall t\in(-\infty,0)\,,\ q<-\frac{1}{\alpha N}. \tag{4.7}$$

We note here that, the Hamiltonian flow $X_{-}^{*}(.,.)=(X^{-}(.,.)\,,P^{-}(.,.))$ described in (4.1) does not depend on the choice of the parameter $\beta>0$ so, from (4.7) it follows that, $h_{-}(t,q)-\beta<-\beta<0$, $\forall t\in(-\infty,0)\,,\ q<-\frac{1}{\alpha N}$, and therefore, the extremity $\tau_2^{-}(.)$ in (4.4) is given by:

$$\tau_2^{-}(q)=-\infty. \tag{4.8}$$

4.2 The Hamiltonian flow ending on the stratum Z_+

As in the previous section, we shall describe the partial Hamiltonian flow whose trajectories are ending on the stratum Z_+. Considering the general solution in (3.14), an admissible trajectory $X_+^{*}(.,z)=(X^{+}(.,z),P^{+}(.,z))$, $z\in Z_+^{*}$ of system (3.13) should satisfy the terminal conditions from the set of transversality terminal points Z_+^{*} in (2.12) and also the fact that $X_+^{*}(t,z)\in Z_+\ \forall\ t<0$. From the terminal condition in (2.12) it follows that:

$$\begin{cases} c=c(q)=\frac{\alpha\xi(q)}{\alpha(\xi(q)-N)+a}=\frac{q(\alpha N-a)+1+\sqrt{\Delta(q)}}{-q(\alpha N-a)+1+\sqrt{\Delta(q)}},\ q>0 \\ k=k(q)=\frac{1}{c(q)-1}\left[\frac{qc(q)}{c(q)-1}+\frac{1}{\alpha N-a}\right], \end{cases} \tag{4.9}$$

where, $\xi(.)\,,\Delta(.)$ denote the functions given in (2.12).

Therefore, we obtain the solution of the differential system in (3.13) in the form of a maximal flow $X_+^{*}(.,.)=(X^{+}(.,.),P^{+}(.,.))$ whose components are given by the formulas:

$$\begin{cases} X^{+}(t,q)=\frac{\alpha N-a}{\alpha}\frac{c(q)e^{(\alpha N-a)t}}{c(q)e^{(\alpha N-a)t}-1},\ q>0,\ t<0 \\ P^{+}(t,q)=\frac{c(q)e^{(\alpha N-a)t}-1}{c(q)e^{(\alpha N-a)t}}\left[\frac{1}{\alpha N-a}+k(q)\left(c(q)e^{(\alpha N-a)t}-1\right)\right], \end{cases} \tag{4.10}$$

where the maximal interval $I^{+}(.)$ is of the same form as in (4.4) Instead, the extremity $\tau_2^{+}(.)$ is defined in this case as:

$$\begin{aligned} &\tau_2^{+}(q)=\inf\{\tau<0;\ h_{+}(t,q)>\beta,\ \forall\ t\in(\tau,0)\},\ q>0 \\ &h_{+}(t,q)=h\left(X_+^{*}(t,q)\right)=\tfrac{1}{\alpha}\left[1+(\alpha N-a)\,k(q)\left(c(q)\,e^{(\alpha N-a)t}-1\right)\right]. \end{aligned} \tag{4.11}$$

In order to characterize the partial Hamiltonian flow $X_+^{*}(.,.)$ given in (4.10), we prove the following result.

Lemma 4.2. There exists an extremity $\tau_1^+(.) : (0, q^+] \to (-\infty, 0)$, $q^+ = \frac{1}{2a} - \frac{1}{2\alpha N} + \frac{\beta}{N}$, given by:

$$\tau_1^+(q) = -\frac{1}{\alpha N - a} \ln\left[\frac{\alpha\xi(q)}{\alpha\xi(q) - \alpha N + a}\right], \tag{4.12}$$

such that:

$$X^+(t,q) > N, \ \forall t \in \left(\tau_1^+(q), 0\right). \tag{4.13}$$

Proof. First, from (4.9) it follows that:

$$c(q) - 1 = \frac{2q(\alpha N - a)\left[q(\alpha N - a) + \sqrt{\Delta(q)} + 1\right]}{2\left[q(\alpha N - a) + \sqrt{\Delta(q)} + 2a\alpha\beta q + 1\right]} > 0, \ \forall q > 0,$$

and therefore:

$$c(q) > 1, \ \forall q > 0. \tag{4.14}$$

Moreover, it follows from (4.10) and (4.14) that, $X^+(t,q) > 0 \Leftrightarrow e^{(\alpha N - a)t} > \frac{1}{c(q)} \in (0,1)$ this implies that, $(\alpha N - a)t > -\ln(c(q))$ and, we deduce that, there exists an extremity $t > \tau_1^+(q)$ given as in (4.12) and such that, $\tau_1^+(q) < 0$, $\forall q > 0$. While, from (4.10) one has:

$$\frac{\partial X^+}{\partial t}(t,q) = -\frac{(\alpha N - a)^2}{\alpha}\left[\frac{c(q)e^{(\alpha N - a)t}}{(c(q)e^{(\alpha N - a)t} - 1)^2}\right] < 0, \ \forall t \in (\tau_1^+(q), 0), \ q > 0,$$

hence, the component $X^+(., q)$, $q > 0$ is strictly decreasing on $(\tau_1^+(q), 0)$, $q > 0$.

In addition we have:

$$X^+(0,q) = \frac{\alpha N - a}{\alpha}\frac{c(q)}{c(q) - 1} = \frac{q(\alpha N - a) + 1 + \sqrt{\triangle(q)}}{2\alpha q}, q > 0,$$

from here, we obtain:

$$\begin{aligned} X^+(0,q) - N &= \tfrac{1 + \sqrt{\triangle(q)} - q(\alpha N + a)}{2\alpha q} \\ &= \tfrac{2(1 + \sqrt{\triangle(q)}) - 4\alpha a N q[q - (\frac{1}{2a} - \frac{1}{2\alpha N} + \frac{\beta}{N})]}{2\alpha q[1 + \sqrt{\triangle(q)} + q(\alpha N + a)]}, \end{aligned}$$

and therefore:

$$\begin{aligned} &X^+(0,q) - N > 0 \Leftrightarrow -4\alpha a N q[q - (\tfrac{1}{2a} - \tfrac{1}{2\alpha N} + \tfrac{\beta}{N})] \geq 0 \\ &X^+(t,q) - N > X^+(0,q) - N > 0, \ \forall t \in (\tau_1^+(q), 0), \ q \in (0, q^+], \end{aligned}$$

which means that, $X^+(t,q) \notin Y_0 = (0, N)$, $\forall t \in (\tau_1^+(q), 0)$, $q \in (0, q^+]$ (as shown in Figure 3). This completes the proof. ∎

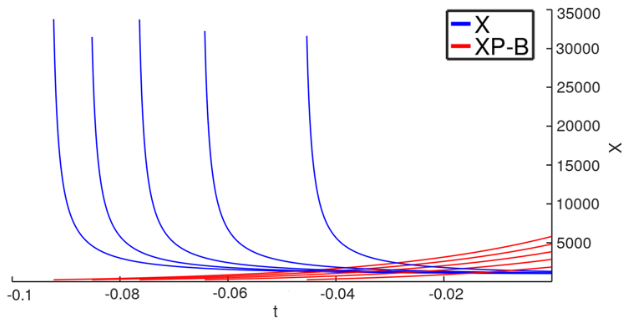

Figure 3: $X^+(.,q)$, $q \in (0, q^+]$

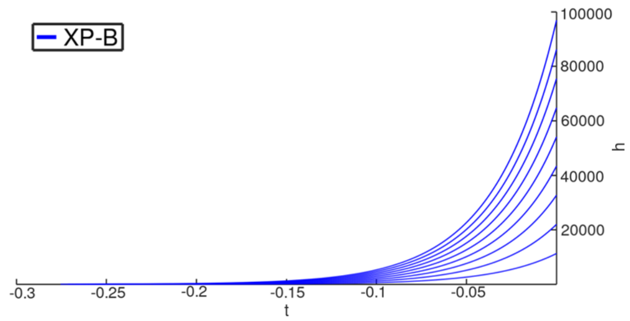

Figure 4: $h_+(.,q) - \beta$, $q > q^+$

Moreover, due to the complexity of formulas (4.10) the analytical characterization of the case $q > q^+$ seems impossible however, certain information on the admissible trajectories $X^+(.,q), q > q^+$ may be illustrated via the following conjecture, which can only be justified by numerical tests (as shown in Figures 4-5).

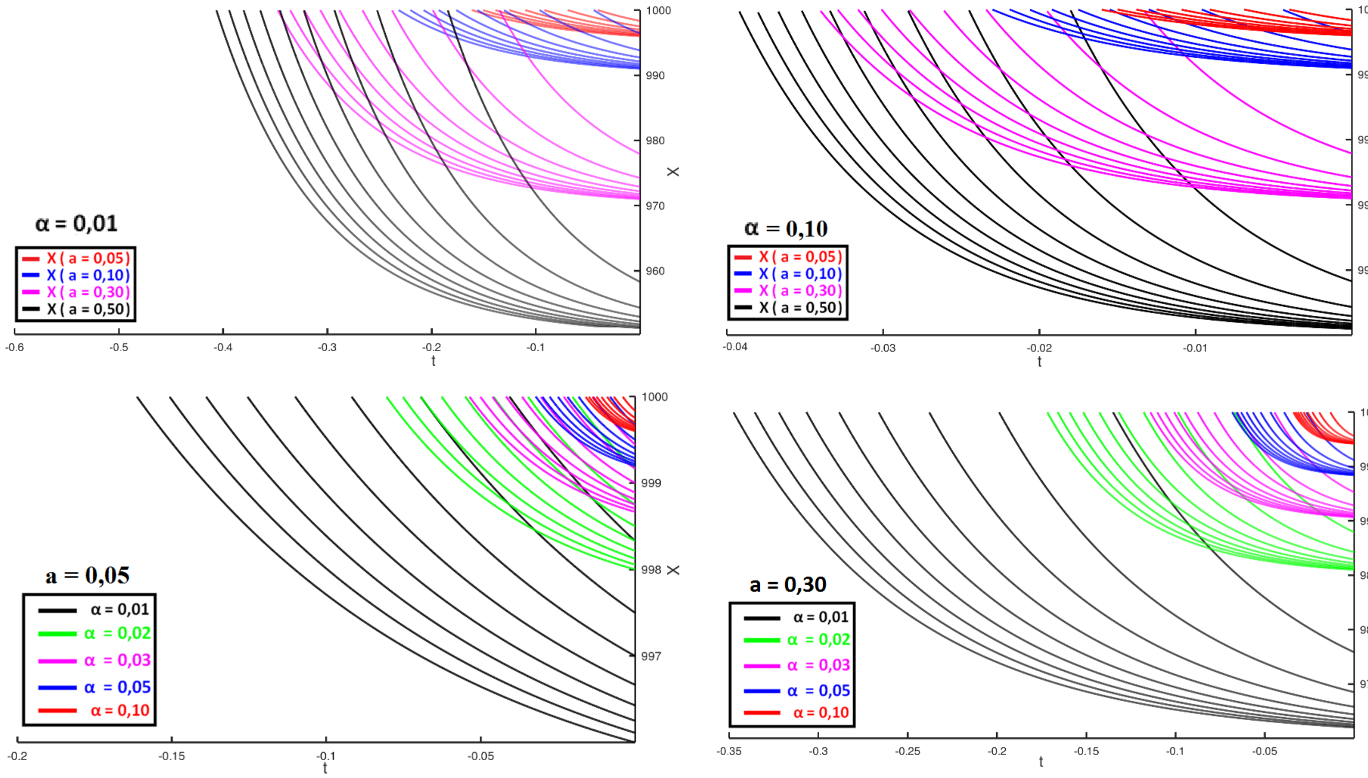

Figure 5: $X^+(.,q)\,, q > q^+$

Conjecture 4.3. The following properties are satisfied:

$$\begin{array}{l} X^+(t,q) \in (0,N)\,,\ \forall t \in \left(\tau_1^+(q)\,,0\right),\ q > q^+ \\ h_+(t,q) > \beta,\ \forall t \in (-\infty, 0)\,. \end{array} \tag{4.15}$$

Therefore, the extremity $\tau^+(.)$ defined as in (4.4) and (4.11) is given by:

$$\tau^+(q) = \tau_1^+(q)\,,\ \tau_2^+(q) = -\infty,\ q > q^+. \tag{4.16}$$

One may note here that, geometrically, the trajectories $X^+(.,q)$, $q > q^+$, are the curves in Figure 5 and cover the domain $Y_0^+ \subset Y_0$ such that:

$$\begin{array}{l} Y_0^+ = \{X^+(t,q)\,;\ (t,q) \in B^+\}\,, \\ B^+ = \left\{(t,q)\,;\ t \in \left(\tau_1^+(q), 0\right),\ q > q^+\right\}. \end{array} \tag{4.17}$$

On the other hand, besides, the complexity of the adjoint victors $P^{\pm}(.,.)$ as well as the guidance functions $h_{\pm}(.,.)$ does not allow an explicit expression for the extremities $\tau_2^{\pm}(.)$. However, certain information on the conditions in (4.3) may be obtained from numerical tests and the images of the trajectories $X^-(.,q)\,, q < -\frac{1}{\alpha N}$, $X^+(.,q)\,, q > q^+$. We develop an implementation with *GNU Octave 6.1.0* and we present some simulations for the graphs of these trajectories under the following data:

$$\begin{array}{l} a \in \{0.05,\ 0.1,\ 0.3,\ 0.5\}\,,\ N = 10^3,\ \beta = 0.5 \\ \alpha \in \{0.01,\ 0.02,\ 0.03,\ 0.05,\ 0.1\}\,. \end{array} \tag{4.18}$$

Remark 4.4. The statements in *Lemma* 4.2 show that, the trajectories $X^+(.,q)$, $q \in (0, q^+]$ cannot be admissible in the sense that, these trajectories are outside their domain, therefore, they are eliminated from the start in the study of Problem 2.1. In the following, we will deal only with the partial Hamiltonian flows $X_-^*(.,q)$, $q < -\frac{1}{\alpha N}$ and $X_+^*(.,q)$, $q > q^+$ given by the formulas in (4.1) and (4.10).

Thus, the Hamiltonian systems in (3.11) and (3.13) generate the generalized characteristic flows $C^*_-(.,.) = (X^*_-(.,.), V(.,.))$ and $C^*_+(.,.) = (X^*_+(.,.), V(.,.))$ described in (3.6), (4.1) and (4.10) and which, according to the well known classical results [10, 11] satisfy the basic differential relation for any $(t,q) \in B^{\pm}$:

$$DV(t,q).(\bar{t},\bar{q}) =< P(t,q), DX^-(t,q).(\bar{t},\bar{q}) >,\ \forall (\bar{t},\bar{q}) \in T_{(t,q)}B^{\pm}, \tag{4.19}$$

where $T_{(t,q)}B^{\pm}$ denotes the tangent space at the point $(t,q) \in B^{\pm}$.

An essential step in using the general algorithm in [10, 11] consists in the fact that the value of the cost functional in (2.3) is given by the function $V(.,.)$ given in (3.6), having as formula:

$$V(t,q) = \begin{cases} -\frac{1}{\alpha}\ln\left[\frac{(\alpha qN+1)e^{\alpha Nt}-1}{\alpha qN}\right], & (t,q) \in B^- \\ -\frac{1}{\alpha}\ln\left[\frac{c(q)e^{(\alpha N-a)t}-1}{c(q)-1}\right] - a\beta t, & (t,q) \in B^+. \end{cases} \tag{4.20}$$

5 Value function and optimal trajectories

As indicated in the theoretical algorithm in [10, 11], the natural candidate for value function and optimal controls in *Problem* 2.1 are the extreme ones, defined by the next minimization process:

$$\begin{array}{l} W(x) = \begin{cases} g(x) = 0, & \text{if } x \in Y_1 \\ W_0(x) = \inf\limits_{X(t,q)=x,\ (t,q)\in B} V(t,q), & \text{if } x \in Y_0 \end{cases} \\ \widehat{B}(x) = \{(t,q) \in B;\ X(t,q) = x,\ V(t,q) = W_0(x)\} \\ \widetilde{U}(x) = \overline{U}(\widehat{B}(x)),\ \overline{U}(t,q) = \{u_q(t);\ u_q(.) \in \overline{\mathcal{U}}(q)\}, \end{array} \tag{5.1}$$

where $\overline{\mathcal{U}}(q)$ denotes the set of control mappings that satisfy (3.3); one may note that:

$$\overline{U}(t,q) \subseteq \widehat{U}(X^*(t,q)),\ \forall\ (t,q) \in B, \tag{5.2}$$

and also that if $X(.,.)$ is invertible at $(t,q) \in B$ with inverse $\widehat{B}(x) = (X(.,.))^{-1}(x)$, then one has:

$$W_0(x) = V(\widehat{B}(x)), \tag{5.3}$$

moreover, it follows from (4.19) that if, in addition, the function $W_0(.)$ is differentiable at the point $y \in Int(Y_0)$, then its derivative is given by:

$$DW_0(x) = \widetilde{P}(x) = P(\widehat{B}(x)) \tag{5.4}$$

and verifies the relations:

$$\begin{array}{l} DW_0(x)f(x,\overline{u}) + f_0(x,\overline{u}) = 0,\ \forall \overline{u} \in \widetilde{U}(x) \\ \widetilde{U}(x) = \{\widetilde{u}(x)\} = \widehat{U}(x,\widetilde{P}(x)), \end{array} \tag{5.5}$$

and $\widetilde{U}(.)$, is the corresponding candidate for optimal control; moreover, from (2.10) and (3.3) it follows that in this case $W_0(.)$ verifies the basic equation:

$$\min_{u\in U(x)}[DW_0(x)f(x,u) + f_0(x,u)] = 0. \tag{5.6}$$

The fact that the Hamiltonian flows $X^*_{\pm}(.,.)$ described above and, the corresponding value functions $W_0^{\pm}(.)$ defined as in (5.1), may characterize a partial solution of the problem on their domains $Y_0^{\pm} \subset Y_0$ is proved in the following main ingredient quasi-elementary result:

Lemma 5.1. The mappings $X^{\pm}(.,.) : B^{\pm} \to Y_0^{\pm}$ defined in (4.1) and (4.10) is a diffeomorphism whose inverses $\widehat{B}^{\pm}(.)$ is described by:

$$\widehat{B}^{\pm}(x) = (\widehat{t}^{\pm}(x), \widehat{q}^{\pm}(x)),\ x \in Y_0^{\pm}. \tag{5.7}$$

Proof. If $x \in Y_0^-$, then it follows from (4.1) that, a point $(t,q) \in B^-$ for which $X^-(t,q) = x$ is characterized by the equation:

$$(\alpha qN + 1)\,e^{\alpha Nt} = \eta(x) = \frac{x}{x-N}, \tag{5.8}$$

whence it results the existence and uniqueness of inverse $\widehat{B}^{-}(.)$, which checks the properties:

$$\widetilde{t}^{-}(x,q) = \frac{1}{\alpha N}\ln\left[\frac{x}{(\alpha q N+1)(x-N)}\right],\ \widehat{t}^{-}(x) = \widetilde{t}^{-}\left(x,\widehat{q}^{-}(x)\right). \tag{5.9}$$

Further, if $x \in Y_0^{+}$, the proof of this statement is done similarly as in the previous case; thus, it follows from (4.10) that, a point $(t,q) \in B^{+}$ for which $X^{+}(t,q) = x$ is characterized by the equation:

$$c(q)\,e^{(\alpha N-a)t} = \psi(x) = \frac{\alpha x}{\alpha(x-N)+a}, \tag{5.10}$$

which leads to the existence and uniqueness of the inverse $\widehat{B}^{+}(.)$, of the form in (5.7) checking the properties:

$$\widetilde{t}^{+}(x,q) = \frac{1}{\alpha N-a}\ln\left[\frac{\alpha x}{c(q)\left[\alpha(x-N)+a\right]}\right],\ \widehat{t}^{+}(x) = \widetilde{t}^{+}\left(x,\widehat{q}^{+}(x)\right). \tag{5.11}$$

This completes the proof. ■

The results in *Lemma* 5.1 show that the characteristic flows $C^{*}_{\pm}(.,.)$ described in (4.1), (4.10) and (4.20) are invertible in the sense of (5.3) and define the smooth partial proper value function, since from (4.20) and (5.3) it follows that:

$$\begin{aligned} W_0(x) &= \begin{cases} W_0^{-}(x) = V\left(\widehat{t}^{-}(x),\widehat{q}^{-}(x)\right), & x \in Y_0^{-} = (0,N) \\ W_0^{+}(x) = V\left(\widehat{t}^{+}(x),\widehat{q}^{+}(x)\right), & x \in Y_0^{+} \end{cases} \\ &= \begin{cases} \frac{1}{\alpha}\ln\left(1-\frac{x}{N}\right), & x \in Y_0^{-} \\ \frac{1}{\alpha}\ln\left[\frac{\left[c\left(\widehat{q}^{+}(x)\right)-1\right]\left[\alpha(x-N)+a\right]}{\alpha N-a}\right] - \beta a\widehat{t}^{+}(x), & x \in Y_0^{+}. \end{cases} \end{aligned} \tag{5.12}$$

which may be naturally extended by $W(\xi) = 0,\ \forall\ \xi \in Y_1$ to the corresponding terminal set defined in (2.4).

Moreover, from (2.8) and (5.5) it follows that the corresponding admissible controls are given by:

$$\begin{aligned} &\widetilde{u}(x) = \widehat{u}\left(x,\widetilde{P}(x)\right) = \begin{cases} \widetilde{u}^{-}(x) = 0, & x \in Y_0^{-} \\ \widetilde{u}^{+}(x) = a, & x \in Y_0^{+} \end{cases} \\ &\widetilde{P}^{\pm}(x) = P^{\pm}\left(\widehat{B}^{\pm}(x)\right). \end{aligned} \tag{5.13}$$

The main result in this section is the following.

Theorem 5.2. (1). The function $W_0(.)$ defined in (5.12) is a solution of the equation in (5.6) on the corresponding domain $Y_0^{-} \cup Y_0^{+}$; moreover, it is the value function in the sense of (5.1) of the corresponding admissible controls in (5.13).

(2). The corresponding admissible controls $\widetilde{u}(.)$ in (5.13) are optimal for the restriction on Their domain $Y_0^{-} \cup Y_0^{+}$.

Proof. (1) The fact that $W_0(.)$ in (5.12) is a solution of basic equation (5.6) on its domain, $Y_0^{-} \cup Y_0^{+}$ follows from *Lemma* 5.1 and the classical theory of smooth Hamiltonian-Jacobi equations [5, 10, 11] using the basic differential relations in (4.19), and from (2.4), (3.3), (5.4), (5.6) and (5.13) one has:

$$\begin{aligned} &\min_{u\in[0,a]}\left[DW_0^{\pm}(x)f(x,u) + f_0(x,u)\right] = \min_{u\in[0,a]}\mathcal{H}(x,\widetilde{P}^{\pm}(x),u) \\ &= \mathcal{H}(x,\widetilde{P}^{\pm}(x),\widetilde{u}^{\pm}(x)) = H_{\pm}\left(X^{\pm}\left(\widehat{B}^{\pm}(x)\right),\widetilde{P}^{\pm}(x)\right) = 0. \end{aligned}$$

(2). Since the value function $W_0(.)$ in (5.12) is differentiable (or rather, is of class $\mathcal{C}^1$), the optimality of the controls $\widetilde{u}(.)$ in (5.13) and therefore of the corresponding trajectories:

$$\widetilde{x}_y(t) = y,\ \forall\ t \in [0,T],\ y \in Y_0^{-} \cup Y_0^{+} \tag{5.14}$$

follows from the so called *Elementary Verification Theorem* [5, 9, 10], according to which a *sufficient optimality condition*, for the admissible controls $\widetilde{u}(.)$ in (5.13) is the verification of the differential inequality:

$$DW_0(x)f(x,\widetilde{u}) + f_0(x,\widetilde{u}) \geq 0,\ \forall\ \widetilde{u} \in \widetilde{U}(x),\ x \in Y_0^{-} \cup Y_0^{+}. \tag{5.15}$$

Case 1. If $y \in Y_0^-$, It follows from (2.4), (5.12) and (5.13) that:

$$f\left(x, \widetilde{u}^-(x)\right) = \alpha x (N - x) - \widetilde{u}^-(x)\, x$$
$$f_0\left(x, \widetilde{u}^-(x)\right) = x + \beta \widetilde{u}^-(x) = x$$
$$DW_0^-(x) = -\tfrac{1}{\alpha}\tfrac{1}{N-x},$$

and therefore:

$$DW_0^-(x) f(x, \widetilde{u}^-(x)) + f_0(x, \widetilde{u}^-(x)) = -\frac{1}{\alpha}\frac{\alpha x (N-x)}{N-x} + x = -x + x = 0,$$

which proves inequality (5.15)..

Case 2. If $y \in Y_0^+$, differential inequality (5.15) is already proved in the proof of statement 1 above. Hence the optimality of the controls $\widetilde{u}(.)$. This completes the proof. ■

Remark 5.3. We note that in most cases, especially in the theory of necessary optimality conditions (the use of Pontriagin's Minimum Principle in its standard form [4, 5, 7, 17, 19]), an optimal control problem requires the solution $\widetilde{u}(.) \in U(x_0)$ corresponding to a fixed initial point $x_0 \in Y_0$. However, the Dynamic Programming approach is able to solve the family of problems corresponding to all initial points $y \in Y_0$ and possibly provide a "feedback optimal solution" as found in the present paper.

6 Conclusion

Finally, we are able to draw some conclusions related to the numerical tests performed as well as from the images of these trajectories illustrated above and we mention the following:

1. Figure 1 shows that, if the contamination rate taken as $\alpha = 0.1$ then, the number of infected individuals increases significantly and suddenly in a short time. While, the peak of epidemic it reached earlier at a critical point and so many people are exposed to disease. Instead, when the contamination rate varies between low values $\alpha \in \{0.01, 0.2, 0.03, 0.05\}$ then, the number of infected people grows more slowly and the peak of the epidemic it reached later. Naturally, as many people are exposed to the disease without vaccination, we see the growth of the infected population. Also, from Figure 1 it understood that, it was created an isolated (quarantined) environment for two reasons, to stop the spread and to gain the well-known herd immunity.
2. Figure 5 shows that, for a fixed low contamination rate $\alpha = 0.01$. If the vaccination rate seems low $a = 0.05$ then, the number of infected individuals decreases insignificantly to the values of Neighborhood of the critical point N. So, in this case, we don't get good results in the sense of decreasing the number of infected individuals. Therefore, small quantities of vaccines are used in vain so, this epidemic puts the economy in crisis. Conversely, if the vaccination rate varies between high values $a \in \{0.1, 0.3, 0.5\}$ then, the number of infected individuals decreases more slowly to the acceptable level in a long time. Further, for a fixed higher contamination rate $\alpha = 0.1$, as in the previous case for the vaccination rate values, we get the same results, but the procedure happens in a short time.
3. Also, from Figure 5 it is found that, for the vaccination rate fixed at $a = 0.05$. We remark that, if the contamination rate is low $\alpha = 0.01$ then the number of infected individuals decreases almost abruptly to the point that marks the end of the spread of the epidemic. Therefore, all those infected are recovered. Moreover, if the contamination rate varies between $\alpha \in \{0.02, 0.03, 0.05, 0.1\}$, then, the number of infected individuals decreases gradually to insignificant values in a short time, which marks the further spread of the epidemic. When the vaccination rate is fixed at $a = 0.3$ then, for a minimum contamination rate such as $\alpha = 0.01$ we remark that, the number of infected people decreases more slowly to some significant minimum values. With the same fixed vaccination rate $a = 0.3$, if $\alpha \in \{0.02, 0.03, 0.05, 0.1\}$, it can be seen that, the number of infected individuals decreases more slowly in a long time at some insignificant values but higher than those of the previous case.
4. Due to statements (2) and (3) it can be concluded that, the gradual administration in a small size of the vaccine does nothing but slow down the spread of the epidemic. Therefore, for the effective control of the epidemic, it is essential to vaccinate at the highest rate for the first few days and then gradually lower the rate. Perhaps the most important result for the vaccination model is that, it is optimal to start vaccination immediately and to be vaccinated for the entire epidemic.
5. Numerical results demonstrated an excellent fit with the results presented in [2, 8, 15], which were obtained for a particular case when the infection rate is significantly low, vaccination is unnecessary, no matter how many people are susceptible, unless there are many infective individuals.

In this work, we considered the epidemic model as an optimal control problem. First, the DP method has been used which gives sufficient conditions of optimality. We developed an implementation with GNU Octave 6.1.0 software, we have traced the evolutions of the states constraints considered in the problem. The results found show that, the DP is the most effective tool for the complete resolution of concrete problems and provides accurate results.

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