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A study on analytic resolvent semilinear integro-differential equations with control functions

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Abstract

The goal of this research is to look at some of the sufficient conditions for approximate controllability in nonlinear resolvent integro-differential evolution control systems. We have considered that nonlinear term is satisfying Lipschitz continuity. To show the key results, we employ Gronwall's inequality, semigroup theory, and the resolvent operators. The main results have been discussed under two sets of assumptions. Application of common fixed point theorems such as Banach, Schauder, Sadovskii, etc. is avoided as discussed earlier by several researchers in the available literature. Finally, one case study based on the proposed problem is discussed in order to verify the theoretical findings.

Keywords: Approximate controllability, Gronwall's inequality, Integro-differential system, Resolvent operators
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1 Introduction

In a range of areas, including nuclear reactor dynamics and thermoelasticity, the system's memory effect must be considered. The impact of history is ignored when differential equations, which incorporate functions at any specific time and space, are employed to model such systems. As a result, an integro-differential system is constructed by adding an integration term to the differential system to integrate the memory holdings in these frameworks. Integral-differential systems have been used extensively in viscoelastic mechanics, fluid dynamics, thermoelastic contact, control theory, heat conduction, industrial mathematics, finance mathematics, biological models, and other areas.

Controllability is the key to every mathematical control theory and technological fields since it's associated to pole assignment, observer design, second degree optimum control and other concepts. Exact and approximate controllability are the two primary ideas of controllability that can be distinguished in infinite dimensional systems. This is due to the truth that there are linear subspaces in infinite dimensions spaces that are not closed. The debate about controllability and optimal control for fractional and integer order frameworks has sparked a lot of study, see [3, 7, 9, 10, 8, 12, 19, 30, 31, 25, 26, 35, 34, 22, 23, 11, 24, 18, 4, 21, 32].

Grimmer began by demonstrating the existence of integro-differential systems in [2, 7, 9, 10, 8] using resolvent operators. The resolvent operator using fixed point methodology is easiest and most suitable choice as method for solving intergrodifferential equations. For further information on resolvent operators, readers should consult

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[2, 5, 6, 7, 9, 10, 8, 25, 26] and the references cited therein. Using facts about resolvent operators and the regularity of evolution integro-differential systems, the author has presented the existence of nonlocal analytic resolvent operator integro-differential equations in [25, 26]. In [33], the author used facts about resolvent operators and Bohnenblust-fixed Karlin's point approach to study the approximate controllability of resolvent type integro-differential systems. The authors have recently proven the controllability and existence outcomes for the integro-differential system using theories of resolvent operator and several fixed-point theorems in [5, 6, 35, 34]. This article is devoted to the sufficient conditions for approximate controllability of the nonlinear resolvent integro-differential evolution control systems. We have considered that nonlinear term is satisfying Lipschitz continuity. For obtaining the main results, we employ Gronwall's inequality, semigroup theory, and the resolvent operators theory. The article avoid the hefty estimations as discussed earlier by several researchers using the fixed point theorems in the available literature.

Consider the nonlinear resolvent type integro-differential equations with control as follow.

$$\frac{d}{d\varpi}\chi(\varpi) = A\Big[\chi(\varpi) + \int_0^{\varpi} B(\varpi-\hbar)\chi(\hbar)d\hbar\Big] + Bv(\varpi) + E(\varpi,\chi(\varpi)),\ \varpi \in V = [0,c], \tag{1.1}$$

$$\chi(0) = \chi_0. \tag{1.2}$$

In the above, A is the infinitesimal generator of a C_0-semigroup $\mathcal{S}(\varpi)$, $\varpi > 0$ in a Hilbert space X. $v(\cdot) \in L^2(V,U)$, a Hilbert space of admissible control functions. Additionally, the linear operator $B : U \longrightarrow X$ is bounded and $E : V \times X \longrightarrow X$. $B(\varpi) \in \mathcal{B}(X)$, $\varpi \in V$, $B(\varpi) : Y \longrightarrow Y$, $AB(\cdot)\chi(\cdot) \in L^1(V,X)$, here $\mathcal{B}(X)$ is the space of all bounded linear operators on X. Additionally, Y is also a Banach space formed from $D(A)$.

Motivations and Contributions:

- Using two alternative criteria, we investigate the some of the conditions sufficient for the approximate controllability of the system (1.1)-(1.2).
- Gronwall's inequality and the Lipchitz condition on nonlinearity are used to derive the results.
- The study aims to investigate the existence of approximate controllability of the systems under consideration(1.1)-(1.2) without use of fixed point theory.
- The suggested method is simple in terms of hefty estimations as compared to standard ways such as fixed point theory approach.
- Results are obtained with weaker condition (Lipschitz) on nonlinearity and can be extended for the delay differential equations.
- The results are advanced and weighted enough as contribution in control differential equations.

We have organized the key content of this article as follows:

1. The second section covers some basic theories on resolvent operators and control theory definitions.
2. In section 3, we illustrate the approximate controllability of proposed problem.
3. Section 4 demonstrates how hypotheses might be validated after they have been acquired.

2 Preliminaries

We provide some essential definitions, notations, and preliminary outcomes concerning resolvent family in this part. The resolvent set of a linear operator A is denoted by $\rho(A)$. $\exists\ M \geq 1,\ w \ni \|\mathcal{S}(\varpi)\| \leq Me^{w\varpi}$, $\varpi \geq 0$ for [7]. $C(V,X)$ is a Banach space equipped with $\|\chi\|_{\mathcal{C}} \equiv \sup_{\varpi\in V}\|\chi(\varpi)\|$, for $\chi \in \mathcal{C}$, is denoted by $\mathcal{C}$.

Definition 2.1. [7, 9] "A family of bounded linear operator $\mathcal{S}(\varpi) \in \mathcal{B}(X)$ for $\varpi \in V$ is called a resolvent operator for

$$\frac{d\chi}{d\varpi} = A\Big[\chi(\varpi) + \int_0^{\varpi} B(\varpi-\hbar)\chi(\hbar)d\hbar\Big]$$

if

(i) $\mathcal{S}(0) = I$ (the identity operator on X),

(ii) for all $\chi \in X$, $\mathcal{S}(\varpi)\chi$ is continuous for $\varpi \in V$,

(iii) $\mathcal{S}(\varpi) \in \mathcal{B}(Y)$, $\varpi \in V$. For $y \in Y$, $\mathcal{S}(\varpi)y \in C^1(V,X) \cap C^1(V,Y)$,

$$\begin{aligned}\frac{d}{d\varpi}\mathcal{S}(\varpi)y &= A\Big[\mathcal{S}(\varpi)y + \int_0^{\varpi} B(\varpi-\hbar)\mathcal{S}(\hbar)yd\hbar\Big]\\ &= \mathcal{S}(\varpi)Ay + \int_0^{\varpi}\mathcal{S}(\varpi-\hbar)AB(\hbar)yd\hbar,\ \varpi \in V.\text{”}\end{aligned}$$

Definition 2.2. The function $\chi \in \mathcal{C}$ is called the mild solution of (1.1)-(1.2) if $\chi(0) = \chi_0$, and $\exists\ E(\varpi, \chi(\varpi)) \in L^1(V,X)$ such that

$$\chi(\varpi) = \mathcal{S}(\varpi)\chi_0 + \int_0^{\varpi}\mathcal{S}(\varpi-\hbar)E(\hbar,\chi(\hbar))d\hbar + \int_0^{\varpi}\mathcal{S}(\varpi-\hbar)Bv(\hbar)d\hbar,\ \varpi \in V,$$

is satisfied.

Definition 2.3. [31] “The reachable set of (1.1)-(1.2) is given by

$K_c(E) = \{\chi(c) \in X : \chi(\varpi)$ represents mild solution of (1.1)-(1.2)$\}$.

In case $E \equiv 0$, then the system (1.1)-(1.2) reduces to corresponding linear system. The reachable set in this case is denoted by $K_c(0)$.”

Definition 2.4. [31] “ If $\overline{K_c(E)} = X$, then the semilinear control system is approximately controllable on $[0,c]$. Here $\overline{K_c(E)}$ represents the closure of $K_c(E)$. It is clear that, if $\overline{K_c(0)} = X$, then linear system is approximately controllable.”

We now present the operator $\Omega : L^2([0,c];X) \to L^2([0,c];X)$

$$[\Omega\chi](\varpi) = E(\varpi, \chi(\varpi)) \tag{2.1}$$

3 Controllability for Integro-differential system

3.1 Controllability results when $B = I$

The approximate controllability of the linear system is proven to reach from the semilinear system under specified nonlinear term constraints in this study. Obviously, $X = U$.

Let us assume that the subsequent linear system given by

$$\frac{d}{d\varpi}w(\varpi) = A\Big[w(\varpi) + \int_0^{\varpi}B(\varpi-\hbar)w(\hbar)d\hbar\Big] + u(\varpi),\ \varpi \in V = [0,c], \tag{3.1}$$

$$w(0) = \chi_0, \tag{3.2}$$

and the semilinear system

$$\frac{d}{d\varpi}\chi(\varpi) = A\Big[\chi(\varpi) + \int_0^{\varpi}B(\varpi-\hbar)\chi(\hbar)d\hbar\Big] + v(\varpi) + E(\varpi,\chi(\varpi)),\ \varpi \in V = [0,c], \tag{3.3}$$

$$\chi(0) = \chi_0, \tag{3.4}$$

We need to present the following assumptions in order to prove the main results of (3.3)-(3.4):

($\mathbf{H_1}$) The linear system (3.1)-(3.2) is approximate controllable.

($\mathbf{H_2}$) Nonlinear function $E(\varpi,\chi(\varpi))$ satisfies the Lipschitz condition in z, i.e.

$$\|E(\varpi,\chi) - E(\varpi,u)\|_X \le l(\|\chi - u\|_X),\ l > 0,\ \forall\ \chi, u \in X,\ \varpi \in [0,c].$$

Theorem 3.1. Under the assumptions ($\mathbf{H_1}$)-($\mathbf{H_2}$), systems (3.3)-(3.4) is approximately controllable.

Proof . Let $w(\sigma)$, along with the control u, is the mild solution of (3.1)-(3.2). Assume that the semilinear system that follows is

$$\frac{d}{d\varpi}\chi(\varpi) = A\Big[\chi(\varpi) + \int_0^{\varpi} B(\varpi-\hbar)\chi(\hbar)d\hbar\Big] + E(\varpi,\chi(\varpi)) + u(\varpi) - E(\varpi,w(\varpi)),\ \varpi \in V = [0,c], \tag{3.5}$$

$$\chi(0) = \chi_0, \tag{3.6}$$

on comparing (3.3)-(3.4) and (3.5)-(3.6), It's very clear that the control function $v(\varpi)$ is given as

$$v(\varpi) = u(\varpi) - E(\varpi, w(\varpi)). \tag{3.7}$$

The mild solution of (3.1)-(3.2) is given as

$$w(\varpi) \quad = \mathcal{S}(\varpi)\chi_0 + \textstyle\int_0^{\varpi} \mathcal{S}(\varpi-\hbar)u(\hbar)d\hbar,\ \varpi \in V, \tag{3.8}$$

and the mild solution of (3.5)-(3.6) is given as

$$\chi(\varpi) = \mathcal{S}(\varpi)\chi_0 + \int_0^{\varpi} \mathcal{S}(\varpi-\hbar)[E(\hbar,\chi(\hbar)) + u(\hbar) - E(\hbar,w(\hbar))]d\hbar,\ \varpi \in V, \tag{3.9}$$

From (3.8) and (3.9), we get

$$w(\varpi) - \chi(\varpi) = \int_0^{\varpi} \mathcal{S}(\varpi-\hbar)\{E(\zeta,w(\zeta)) - E(\zeta,\chi(\zeta))\}d\zeta \tag{3.10}$$

Applying norm on both sides, one can get

$$\begin{aligned} \|w(\varpi) - \chi(\varpi)\|_X \leq & \int_0^{\varpi} \|\mathcal{S}(\varpi-\hbar)\|\ \|E(\zeta,w(\zeta)) - E(\zeta,\chi(\zeta))\|d\zeta \\ \leq & M\int_0^{\varpi} \|E(\zeta,w(\zeta)) - E(\zeta,\chi(\zeta))\|d\zeta. \end{aligned} \tag{3.11}$$

Using assumption (H_2), we get

$$\|w(\varpi) - \chi(\varpi)\|_X \leq Ml\int_0^{\varpi} \|w(\zeta) - \chi(\zeta)\|d\zeta \tag{3.12}$$

Referring to Gronwall's inequality, $w(\varpi) = \chi(\varpi)$, $\forall\ \varpi \in [0,c]$. As a consequence, the linear system's solution w along the control u is a semilinear system's solution χ along the control v, where $K_c(E) \supset K_c(0)$. Since $K_c(0)$ is dense in X (according to assumptions $(H1)$ and $K_c(E)$ is dense in X as well, resulting approximate controllability of (3.3)-(3.4) which completes the proof. □

3.2 Controllability results: when $B \neq I$

Now, when $B \neq I$ is verified with given set of specific conditions on A, B, and E, the approximate controllability is verified.

Assume that the subsequent linear system

$$\frac{d}{d\varpi}w(\varpi) = A\Big[w(\varpi) + \int_0^{\varpi} B(\varpi-\hbar)w(\hbar)d\hbar\Big] + Bu(\varpi),\ \varpi \in V = [0,c] \tag{3.13}$$

$$w(0) = \chi_0, \tag{3.14}$$

and the semilinear system

$$\frac{d}{d\varpi}\chi(\varpi) = A\Big[\chi(\varpi) + \int_0^{\varpi} B(\varpi-\hbar)\chi(\hbar)d\hbar\Big] + Bv(\varpi) + E(\varpi,\chi(\varpi)),\ \varpi \in V = [0,c], \tag{3.15}$$

$$\chi(0) = \chi_0, \tag{3.16}$$

We must take following assumptions into account in order to prove the fundamental purpose of this section, namely, the approximate controllability of (3.15)-(3.16):

($\mathbf{H_3}$) The linear system (3.13)-(3.14) is approximate controllable.
($\mathbf{H_4}$) $Range(\Omega) \subseteq \overline{Range(B)}$.

Theorem 3.2. If ($\mathbf{H_2}$)-($\mathbf{H_4}$) are fulfilled, then (3.15)-(3.16) is approximately controllable.

Proof . The mild solution of (3.13)-(3.14) in agreement with the control u is presented as

$$w(\varpi) \quad = \mathcal{S}(\varpi)\chi_0 + \int_0^{\varpi} \mathcal{S}(\varpi - \hbar)Bu(\hbar)d\hbar, \ \varpi \in V, \tag{3.17}$$

Assume that the subsequent semilinear system

$$\frac{d}{d\varpi}\chi(\varpi) = A\Big[\chi(\varpi) + \int_0^{\varpi} B(\varpi - \hbar)\chi(\hbar)d\hbar\Big] + E(\varpi, \chi(\varpi)) + Bu(\varpi) - E(\varpi, w(\varpi)), \ \varpi \in V = [0, c], \tag{3.18}$$

$$\chi(0) = \chi_0, \tag{3.19}$$

Since $\Omega\chi \in \overline{Range(B)}$ and for $\epsilon > 0$ there exist a control function $\nu \in L^2(V, U)$ such that

$$\|\Omega\chi - B\nu\|_X \leq \epsilon \tag{3.20}$$

Further, consider that $\chi(\varpi)$ be the mild solution of (1.1)-(1.2) corresponding to the control function $(u - \nu)$ given as

$$\chi(\varpi) = \mathcal{S}(\varpi)\chi_0 + \int_0^{\varpi} \mathcal{S}(\varpi - \hbar)\{B(u - \nu) + [\Omega z]\}(\hbar)d\hbar, \ \varpi \in V. \tag{3.21}$$

From (3.17) and (3.21), we have

$$\begin{aligned} w(\varpi) - \chi(\varpi) &= \int_0^{\varpi} \mathcal{S}(\varpi - \hbar)[B\nu - \Omega z](\hbar)d\hbar \\ &= \int_0^{\varpi} \mathcal{S}(\varpi - \hbar)[B\nu - \Omega w](\hbar)d\hbar + \int_0^{\varpi} \mathcal{S}(\varpi - \hbar)[\Omega w - \Omega z](\hbar)d\hbar \end{aligned}$$

Applying norm operator and using (3.20), we obtain

$$\begin{aligned} \|w(\varpi) - \chi(\varpi)\|_X &= \int_0^{\varpi} \|\mathcal{S}(\varpi - \hbar)\| \, \|B\nu(\hbar) - \Omega w(\hbar)\| d\hbar \\ &+ \int_0^{\varpi} \|\mathcal{S}(\varpi - \hbar)\| \, \|\Omega w(\hbar) - \Omega\chi(\hbar)\|_X d\hbar \\ &\leq M\Big(\int_0^{\varpi} d\hbar\Big)^{1/2}\Big(\int_0^{\varpi} \|B\nu(\hbar) - \Omega w(\hbar)\|^2 d\hbar\Big)^{1/2} \\ &+ M\int_0^{\varpi} \|\Omega w(\hbar) - \Omega\chi(\hbar)\|_X \\ &\leq M\sqrt{\varpi}\Big(\|\Omega w - B\nu\|_{L^2([0,c];X)}\Big) + M\int_0^{\varpi} \|[\Omega w](\hbar) - [\Omega\chi](\hbar)\|_X d\hbar \\ &\leq M\sqrt{c}\epsilon + Ml\int_0^{\varpi} \|w(\hbar) - \chi(\hbar)\|_X d\hbar. \end{aligned}$$

By referring Gronwall's inequality, one can obtain

$$\|w(\varpi) - \chi(\varpi)\|_X \leq M\sqrt{c}\epsilon \exp(Mlc).$$

Since RHS of the above inequality is dependent on $\epsilon > 0$ and ϵ is random, picking an appropriate control function ν makes $||w(\varpi) - \chi(\varpi)||_X$ arbitrary small. Clearly, the (1.1)-(1.2) reachable set is dense in the reachable set of (3.13)-(3.14), which is dense in X owing to assumption ($H3$). As a result, (3.13)-(3.14) approximate controllability implies that of the semilinear control system (1.1)-(1.2). □

4 Example

Let us assume that the following controllable

$$\frac{\partial}{\partial \varpi}\Big[\chi(\varpi, y)\Big] = \frac{\partial^2}{\partial y^2}\Big[\chi(\varpi, y) + \int_0^{\varpi} k(\varpi - \hbar)\chi(\hbar, y)d\hbar\Big] + \mu(\varpi, y) \tag{4.1}$$

$$+ \frac{\chi^2(\varpi, y)}{(1+\varpi)(1+\varpi^2)}, \quad 0 \le \varpi \le c,\ 0 \le y \le \pi, \tag{4.2}$$

$$\chi(\varpi, 0) = \chi(\varpi, \pi) = 0, \tag{4.3}$$

$$\chi(0, y) = \chi_0(y), \quad 0 \le y \le \pi. \tag{4.4}$$

In the above $\chi_0(y) \in X = L^2([0, \pi])$. $\mu : V \times [0, \pi] \longrightarrow [0, \pi]$ is a continuous function.

We may convert (4.2)-(4.4) into (1.1)-(1.2) by assuming A is given by $A\chi = \chi''$ with

$$D(A) = \{\chi \in X : \chi'' \in X \text{ and } \chi(0) = \chi(\pi) = 0\}.$$

Where A be a generator element of a strongly continuous semigroup $\mathcal{S}(\varpi)$, that is self-adjoint and analytic. The spectrum of A for the eigenvalues are $m^2, m \in \mathbb{N}$ with the corresponding normalized eigenvectors $\chi_m(y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(my)$ and the kernel not in general $k(\varpi - \hbar)$ is continuous, then $\exists\, k_1 > 0$ such that $|k(\varpi - \hbar)| \le k_1$. Therefore, the subsequent characteristic is fulfilled:

- If $x \in D(A)$, then $Ax = \sum_{m=1}^{\infty} m^2 \langle x, \chi_m\rangle \chi_m$.

Now let's assume the function $E : V \times X$ by

$$E(\varpi, x) = \frac{\chi^2(\varpi, x)}{(1+\varpi)(1+\varpi^2)},$$
$$Bu(\varpi, \xi) = \mu(\varpi, \xi),$$

Assume that the above functions satisfies the hypotheses conditions with $B = I$, as demonstrated. Since all the hypotheses have been confirmed, (4.2)-(4.4) is approximately controllable.

Remark 4.1. One can study the optimal control and asymptotic stability of the proposed system using the technique from [32] and [28] with suitable medications.

5 Conclusion

Using the theory of Gronwall's inequality, semigroup theory, and the resolvent operators, we investigated the approximate controllability for the proposed systems under suitable assumptions. The usage of well-known fixed point theorem methodologies is avoided in this paper. In future we will extend these results for Sobolev space with and without finite and infinite delay.

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