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On ap-sequential Henstock integral for interval valued functions

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Abstract

The aim of this paper is to introduce the notion of interval ap-Sequential Henstock integral (shortly, the $ap\text{-}ISH$). Some interesting properties of $ap\text{-}ISH$ are investigated.

Keywords: ap-Sequential Henstock integral, Sequential Henstock integral, Interval-valued functions, Guages
2020 MSC: 28B05, 28B10, 28B15, 46G10

1 Introduction

It is well known that the Henstock integral generalises the Riemann integral, and more powerful and simpler than the Lebesgue integral. This concept was introduced independently by R. Henstock and J. Kursweil in 1955 and 1957 respectively. It is also well known that Henstock integral which recovers a continuous function from its derivative is equivalent to the Denjoy and Perron integrals and is easier and more reliable than the Wiener, Feynmann and Lebesgue integrals (see, e.g. [1]-[12]). In 1967, Henstock [3] gave a Riemann definition of an integral which is equivalent to the Burkill integral that recovers a real function from its approximate derivative. This he called approximate continuous Henstock integral(br.ap-Henstock integral). Wu and Gong [12] established the concept of the Henstock (H) integrals of interval valued functions and Fuzzy number-valued functions and obtain some basic properties of the integral.

In 2016, Hamid, Elmuiz and Shiema [2] introduced the idea of the ap-Henstock-Stieltjes integral of interval-valued functions and Fuzzy number-valued functions which are extension of [12] and obtain a number of its interesting properties.

Paxton [9] developed an alternative sequential definition of the Henstock integral which he denotes as the Sequential Henstock (SH) integral; and then discussed the notion of the integral as generalizations of the Henstock(H) integral and established its properties. The authors in [6] proved equivalence of Henstock and certain Sequential Henstock integrals. They also proved dominated and bounded convergence theorems involving Sequential Henstock-Stieltjes integral.

In this paper, we introduce the notion of ap-interval Sequential Henstock integral which is an extension of ap-Henstock integral and discuss some of its' basic properties.

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2 Preliminaries and new results

Let E be a measurable set and let $c \in \mathbb{R}$. The density of E at x is defined by

$$d_x E = \lim_{h \to 0^+} \frac{\mu(E \bigcap (x-h, x+h))}{2h},$$

provided the limit exists. The point x is called a point of density of E if $d_x E = 1$. The set E^d represents the set of all points $x \in E$ such that x is a point of density of E.

A function $f : [a,b] \to \mathbb{R}$ is said to be approximate continuous at $c \in [a,b]$ if there exists a measurable set $E \in [a,b]$ given that $c \in E^d$ and $f|_E$ is continuous at c.

A sequence of approximate neighborhoods(or ap-nbd) of $t_{i_n} \in [a,b]$ is a measurable set $S_{t_{i_n}} \subseteq [a,b]$ containing t_{i_n} as a sequence of points of density. For every $t_{i_n} \in E \subseteq [a,b]$, choose an ap-nbd $S_{t_{i_n}} \subseteq [a,b]$ of t_{i_n}. Then we say that $S = \{S_{t_{i_n}} : t_{i_n} \in E\}$ is a choice on E. A tagged interval $(t_{i_n}, [c_{i_n}, d_{i_n}])$ is said to be subordinate to the choice $S = S_{t_{i_n}}$ if $c_{i_n}, d_{i_n} \in S_{t_{i_n}}$. Let $P_n = \{(t_{i_n}, [c_{i_n}, d_{i_n}]) : 1 \leq i \leq m, m \in \mathbb{N}\}$ be a finite collection of non-overlapping tagged intervals. If $(t_{i_n}, [c_{i_n}, d_{i_n}])$ is subordinate to a choice S for each $i_n (i = 1, ..., m)$, then we say that P_n is subordinate to S. If P_n is subordinate to S and $[a,b] = \bigcup_{i=1}^{n} [c_{i_n}, d_{i_n}]$, then we say that P_n is a tagged partition of $[a,b]$ that is subordinate to S.

Let $\mathbb{R}$ denotes the set of real numbers, $F(X)$ as an interval valued function, F^-, the left endpoint, F^+ as right endpoint, $\{\delta_n(x)\}_{n=1}^{\infty}$, as set of gauge functions, P_n, as set of partitions of subintervals of a compact interval $[a,b]$, X, as non empty interval in $\mathbb{R}$ and $d(X) = X^+ - X^-$, as width of the interval X and $\ll$ as much more smaller (see [5]).

A gauge on $[a,b]$ is a positive real-valued function $\delta : [a,b] \to \mathbb{R}^+$. This gauge is δ-fine if $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$ while sequence of tagged partition P_n of [a,b] is a finite collection of ordered pairs $P_n = \{(u_{(i-1)_n} u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ where $[u_{i-1}, u_i] \in [a,b]$, $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ and $a = u_0 < u_{i_1} <, ..., < u_{m_n} = b$.

Firstly, we recall the following concepts:

Definition 2.1 [9] A function $f : [a,b] \to \mathbb{R}$ is Henstock integrable to α on $[a,b]$ if there exists a number $\alpha \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a function $\delta(x) > 0$ such that

$$|S(f,P) - \alpha| < \varepsilon.$$

whenever $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^{n}$ is a $\delta(x) - fine$ partitions on $[a,b]$ and $S(f,P) = \sum_{i=1}^{n} f(t_i)(u_i - u_{(i-1)})$. We say that α is a Henstock integral of f on $[a,b]$ i.e $\alpha = H \int_a^b f$. We use $H_f[a,b]$ to denote the set of all Henstock integrable functions defined on $[a,b]$.

Definition 2.2 [2] (ap-Henstock integrable) A function $f : [a,b] \to \mathbb{R}$ is ap-Henstock integrable on $[a,b]$ if there exists a vector $\alpha \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a choice S on $[a,b]$ such that

$$|S(f,P) - \alpha| < \varepsilon,$$

whenever $P = \{([u_{i-1}, u_i], t_i)\}_{i=1}^{n}$ is a $S(x) - fine$ partitions on $[a,b]$. We say that α is an *ap*-Henstock integral of f on $[a,b]$ i.e $\alpha = ap\text{-}H \int_a^b f$. We use $ap\text{-}H_f[a,b]$ to denote the set of all ap-Henstock integrable functions defined on $[a,b]$.

Definition 2.3 [9] A function $f : [a,b] \to \mathbb{R}$ is Sequential Henstock integrable on $[a,b]$ if there exists a number $\alpha \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that

$$|S(f,P_n) - \alpha| < \varepsilon,$$

whenever $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ is a $\delta_n(x) - fine$ tagged partition on $[a,b]$ and $S(f,P_n) = \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n})$. We say that α is a Sequential Henstock integral of f on $[a,b]$ i.e $\alpha = SH \int_a^b f$. We use $SH_f[a,b]$ to denote the set of all Sequential Henstock integrable functions defined on $[a,b]$.

Definition 2.4 [2] An interval valued function $F : [a,b] \to I_{\mathbb{R}}$ is Henstock integrable on $[a,b]$ if there exists a number $I_0 \in I_{\mathbb{R}}$ such that for any $\varepsilon > 0$ there exists a positive gauge function $\delta(x) > 0$ on $[a,b]$ such that

$$d(\sum_{i=1}^{n \in \mathbb{N}} F(t_i)(u_i - u_{i-1}), I_o) < \varepsilon$$

whenever $\delta(x)-fine$ is a tagged partitions $P=\{(u_{i-1},u_i),t_i\}_{i=1}^n$ on $[a,b]$. We say that I_0 is the Henstock integral of F on $[a,b]$ with $(IH)\int_a^b F=I_0$ and $F\in IH[a,b]$. We use $IH_F[a,b]$ to denote the set of all interval Henstock integrable functions defined on $[a,b]$.

Definition 2.5 [10, 12] Let $I_{\mathbb{R}}=\{I=[I^-,I^+]$ is the closed bounded interval on the real line $\mathbb{R}\}$. For $X,Y\in I_{\mathbb{R}}$, we define
i. $X\le Y$ if and only if $X^-\le Y^-$ and $X^+\le Y^+$,
ii. $X+Y=Z$ if and only if $Z^-=X^-+Y^-$ and $Z^+=X^++Y^+$,
iii. $X.Y=\{x.y: x\in X, y\in Y\}$, where

$$(X.Y)^-=\min\{X^-Y^-,X^-Y^+,X^+Y^-,X^+Y^+\}$$

and

$$(X.Y)^+=\max\{X^-.Y^-,X^-.Y^+,X^+.Y^-,X^+.Y^+\}.$$

Define $d(X,Y)=\max(|X^--Y^-|,|X^+-Y^+|)$ as the distance between X and Y.

Here, we shall give the definition of interval Sequential Henstock integrals newly.

Definition 2.6 A function $f:[a,b]\to I_{\mathbb{R}}$ is interval Sequential Henstock integrable on $[a,b]$ if there exist a number $I_0\in I_{\mathbb{R}}$ such that for any $\varepsilon>0$ there exists a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ on [a,b] such that

$$d(\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n}-u_{(i-1)_n}),I_0)<\varepsilon.$$

whenever $P_n=\{(u_{(i-1)_n},u_{i_n}),t_{i_n}\}_{i=1}^{m_n}$ is a $\delta_n(x)-fine$ Sequential Henstock partitions on $[a,b]$. We say that I_0 is an interval Sequential Henstock integral of f on $[a,b]$ i.e $I_0=(SH)\int_a^b f$ and $f\in ISH[a,b]$. We use $ISH_f[a,b]$ to denote the set of all interval Sequential Henstock integrable functions defined on $[a,b]$.

Now, we will define newly the ap-interval Sequential Henstock integral and then discuss some of the properties of the integral.

Definition 2.7(ap-interval Sequential Henstock integral) A function $f:[a,b]\to I_{\mathbb{R}}$ is ap-interval Sequential Henstock integrable on $[a,b]$ if there exists a vector $I_0\in I_{\mathbb{R}}$ such that for any $\varepsilon>0$ there exists a sequence of positive choice functions $\{S_n(x)\}_{n=1}^{\infty}$ on [a,b] such that

$$d(\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n}-u_{(i-1)_n}),I_0)<\varepsilon.$$

whenever $P_n=\{([u_{(i-1)_n},u_{i_n}],t_{i_n})\}_{i=1}^{m_n}$ is a $S_n(x)-fine$ ap-interval Sequential Henstock partitions on $[a,b]$. We say that I_0 is a ap-interval Sequential Henstock integral of f on $[a,b]$ i.e $I_0=(ap\text{-}ISH)\int_a^b f$ and $f\in ap\text{-}ISH[a,b]$. We use $ap\text{-}ISH_f[a,b]$ to denote the set of all ap-interval Sequential Henstock integrable functions defined on $[a,b]$.

Example 2.8 Suppose the Dirichlet's function $f:[a,b]\to I_{\mathbb{R}}$ is defined by

$$f(x)=\begin{cases}1, & \text{if } x\in Q,\\ 0, & \text{if } x\notin Q,\end{cases}$$

Suppose that we define our choice function

$$S_n(x)=\begin{cases}\frac{\varepsilon}{2^n}, & \text{if } x\in Q\\ 1, & \text{if } x\notin Q,\end{cases}$$

So, we have our

$$\begin{aligned} S(f,P_n) &= \sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n}-u_{(i-1)_n})\\ &\le d(\sum_{i=\Pi} f(t_{i_n})(u_{i_n}-u_{(i-1)_n}))+d(\sum_{i=\tau} f(t_{i_n})(u_{i_n}-u_{(i-1)_n}))\\ &\le \varepsilon\sum_{i=1}^{\infty}\frac{\varepsilon}{2^n}=\varepsilon \end{aligned}$$

where ε is arbitrary. Hence the equality holds and

$$I_0 = \int_a^b f = 0.$$

Remark 2.9. It is clear that if f is a real valued function, then our Definition 1.6 implies the definition of Sequential Henstock integral introduced by [5]. We also have that Definition 1.7 extends all the other integrals mentioned in this paper.

The following lemma is useful in the proof of one of our Theorems:

Lemma 2.10 . [2] Let f, g be Sequential Henstock (SH)integrable functions on $[a, b]$, if $f \leq g$ is almost everywhere on $[a, b]$, then

$$\int_a^b f \leq \int_a^b g.$$

Remark 2.11. If $f = f^- = f^+$, then it is clear that Definition 1.8 implies the real-valued of the Sequential Henstock integral.

Theorem 2.12 If $f \in$ *ap-*$ISH[a, b]$, then the integral is unique.

Proof . Suppose the integral value is not unique and let $I_1 = (ap\text{-}ISH) \int_a^b f$ and $I_2 = (ap\text{-}ISH) \int_a^b f$ with $I_1 \neq I_2$. Let $\varepsilon > 0$ be given then there exists a sequence of positive choice functions $\{S_n^1(x)\}_{n=1}^\infty$ and $\{S_n^2(x)\}_{n=1}^\infty$ such that for each $S_n^1(x)$-fine tagged partitions P_n^1 of $[a, b]$ and for each $S_n^2(x)$-fine tagged partitions P_n^2 of $[a, b]$, we have

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_1) < \frac{\varepsilon}{2}$$

and

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_2) < \frac{\varepsilon}{2}$$

respectively. Define a positive choice function $S_n(x)$ on $[a, b]$ by $S_n(x) = \min\{S_n^1(x), S_n^2(x)\}$. Let P_n be any $S_n(x)$-fine tagged partition of $[a, b]$. Then by triangular inequality, we have

$$\begin{aligned} d(I_1, I_2) &= d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_1) + d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_2) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since for all $\varepsilon > 0$, there is a $S_n(x) > 0$ on [a,b] as above, then $I_1 = I_2$. $\square$

Theorem 2.13 Let $f \in$ *ap-*$ISH[a, b]$ if and only if $f^-, f^+ \in$ *ap-*$SH_{[a,b]}$ and

$$(ap\text{-}ISH) \int_a^b f = [(ap\text{-}SH) \int_a^b f^-, (ap\text{-}SH) \int_a^b f^+].$$

Proof . If $f \in$ *ap-*$ISH[a, b]$, from Definition 1.7 there is a unique interval number $I_0 = [I_0^-, I_0^+]$ with the property that for any $\varepsilon > 0$, there exists a sequence of positive choice functions $\{S_n(x)\}_{n=1}^\infty$ on $[a, b]$ such that

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_0) < \varepsilon.$$

whenever P_n is a $S_n(x)$-fine tagged partition of $[a, b]$. Observe that

$$d(\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}), I_0)$$

$$
\begin{aligned}
&= \max(|[\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n}-u_{(i-1)_n})]^- - I_0^-|, |[\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n}-u_{(i-1)_n})]^+ - I_0^+)|) < \varepsilon \\
&= \max(|\sum_{i=1}^{m_n\in\mathbb{N}} f^-(t_{i_n})(u_{i_n}-u_{(i-1)_n}) - I_0^-|, |\sum_{i=1}^{m_n\in\mathbb{N}} f^+(t_{i_n})(u_{i_n}-u_{(i-1)_n}) - I_0^+)|).
\end{aligned}
$$

Since $u_{i_n} - u_{(i-1)_n} \geq 0$, for $1 \leq i_n \leq m_n$, it follows that

$$
|\sum_{i=1}^{m_n\in\mathbb{N}} f^-(t_{i_n})(u_{i_n}-u_{(i-1)_n}) - I_0^-| < \varepsilon, |\sum_{i=1}^{m_n\in\mathbb{N}} f^-(t_{i_n})(u_{i_n}-u_{(i-1)_n}) - I_0^+)| < \varepsilon.
$$

for every $S_n(x)$-tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$. Thus, by Definition 1.7, we obtain $f^+, f^- \in ap\text{-}SH[a,b]$ and

$$
I_0^- = (ap\text{-}SH)\int_a^b f^-
$$

and

$$
I_0^+ = (ap\text{-}SH)\int_a^b f^+.
$$

Conversely, let $f^- \in ap\text{-}SH[a,b]$. Then there exist a unique $\beta_1 \in \mathbb{R}$ with the property that given $\varepsilon > 0$ then there exists a $\{S_n^1(x)\}_{n=1}^{\infty}$, such that

$$
|\sum_{i=1}^{m_n\in\mathbb{N}} f^-(t_{i_n})(u_{i_n}-u_{(i-1)_n}) - \beta_1| < \varepsilon,
$$

whenever P_n^1 is a $S_n^1(x)$-fine tagged partition of $[a,b]$.

Similarly, let $f^+ \in ap\text{-}SH[a,b]$. Then there exist a unique $\beta_2 \in \mathbb{R}$ with the property that given $\varepsilon > 0$ then there exists a $\{S_n^2(x)\}_{n=1}^{\infty}$, such that

$$
|\sum_{i=1}^{m_n\in\mathbb{N}} f^+(t_{i_n})(u_{i_n}-u_{(i-1)_n}) - \beta_2)| < \varepsilon,
$$

whenever P_n^2 is a $S_n^2(x)$-fine tagged partitions of $[a,b]$. We let $S_n(x) = \min(S_n^1(x), S_n^2(x))$ and $I_0 = [\beta_1, \beta_2]$, then if P_n is a $S_n(x) - fine$ tagged partition of $[a,b]$, we have

$$
d(\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})(u_{i_n}-u_{(i-1)_n}), I_0) < \varepsilon.
$$

Hence, $f : [a,b] \to I_{\mathbb{R}}$ is ap-Sequential Henstock integrable on $[a,b]$. This completes the proof. □

Example 2.14 Suppose that $[a,b] = [0,1]$. Q is a set of rational numbers in $[0,1]$ and $f \in ap\text{-}ISH[a,b]$ such that

$$
f(x) = \begin{cases} [0,1], & \text{if } x \in Q, \\ [1,2], & \text{if } x \notin [0,1]\backslash Q, \end{cases}
$$

Then

$$
(ap\text{-}ISH[a,b]\int_0^1 f = [(ap\text{-}ISH[a,b])\int_a^b f^-] + (ap\text{-}ISH[a,b]\int_a^b f^+) = [1,2].
$$

Theorem 2.15 If $f, g \in ap\text{-}ISH[a,b]$ and $\beta, \gamma \in \mathbb{R}$. Then $\beta f + \gamma g \in ap\text{-}ISH[a,b]$ and

$$
(ap\text{-}ISH)\int_a^b (\beta f + \gamma g) = \beta(ap\text{-}ISH)\int_a^b f + \gamma(ap\text{-}ISH)\int_a^b g.
$$

Proof . If $f, g \in ap\text{-}ISH[a,b]$, then $f^-, f^+, [g^-, g^+ \in ap\text{-}SH[a,b]$ by Theorem 2.13. Hence, we have $\gamma f^- + \xi g^-, \gamma f^- + \xi g^+, \gamma f^+ + \xi g^-, \gamma f^+ + \xi g^+ \in ap\text{-}SH[a,b]$.

Case 1. If $\gamma > 0$ and $\xi > 0$, then

$$\begin{aligned}(ap\text{-}SH)\int_a^b (\gamma f + \xi g)^- &= (ap\text{-}SH)\int_a^b (\gamma f^- + \xi g^-)\\ &= \gamma (ap\text{-}SH)\int_a^b f^- + \xi (ap\text{-}SH)\int_a^b g^-\\ &= \gamma((ap\text{-}ISH)\int_a^b f)^- + \xi((ap\text{-}ISH)\int_a^b g)^-\\ &= (\gamma(ap\text{-}ISH)\int_a^b f + \xi(ap\text{-}ISH)\int_a^b g)^-.\end{aligned}$$

Case 2. If $\gamma < 0$ and $\xi > 0$, then

$$\begin{aligned}(ap\text{-}SH)\int_a^b (\gamma f + \xi g)^- &= (ap\text{-}SH)\int_a^b (\gamma f^+ + \xi g^+)\\ &= \gamma (ap\text{-}SH)\int_a^b f^+ + \xi (ap\text{-}SH)\int_a^b g^+\\ &= \gamma((ap\text{-}ISH)\int_a^b f)^+ + \xi((ap\text{-}ISH)\int_a^b g)^+\\ &= (\gamma(ap\text{-}ISH)\int_a^b f + \xi(ap\text{-}ISH)\int_a^b g)^-.\end{aligned}$$

3) If $\gamma > 0$ and $\xi < 0$ (or $\gamma < 0$ and $\xi > 0$), then

$$\begin{aligned}(ap\text{-}SH)\int_a^b (\gamma f + \xi g)^- &= (ap\text{-}SH)\int_a^b (\gamma f^- + \xi g^+)\\ &= \gamma (ap\text{-}SH)\int_a^b f^- + \xi (ap\text{-}SH)\int_a^b g^+\\ &= \gamma((ap\text{-}ISH)\int_a^b f)^- + \xi((ap\text{-}ISH)\int_a^b g)^+\\ &= (\gamma(ap\text{-}ISH)\int_a^b f + \xi(ap\text{-}ISH)\int_a^b)^-.\end{aligned}$$

Similarly, for four cases above, we have

$$(ap\text{-}ISH)\int_a^b (\gamma f + \xi g)^+ = (\gamma(ap\text{-}ISH)\int_a^b f + \xi(ap\text{-}ISH)\int_a^b g)^+.$$

Hence, by Theorem 2.13, $\gamma f + \xi g \in ap\text{-}ISH[a,b]$ and

$$(ap\text{-}SH)\int_a^b (\gamma f + \xi g) = \gamma(ap\text{-}ISH)\int_a^b f + \xi(ap\text{-}ISH)\int_a^b g.$$

This completes the proof. □

Theorem 2.16 If $f, g \in ap\text{-}ISH[a,b]$ and $f(x) \leq g(x)$ nearly everywhere on $[a,b]$, then

$$(ap\text{-}ISH)\int_a^b f \leq (ap\text{-}ISH)\int_a^b g.$$

Proof . If $f(x) \leq g(x)$ nearly everywhere on $[a,b]$ and $f, g \in$ *ap-ISH*$[a,b]$, then $f^-, f^+, g^-, g^+ \in$ *ap-SH*$[a,b]$ with $f^- \leq f^+$, and $g^- \leq g^+$ nearly everywhere on $[a,b]$. By Lemma 2.5

$$(ap\text{-}SH)\int_a^b f^-(x) \leq (ap\text{-}SH)\int_a^b g^-(x)$$

and

$$(ap\text{-}SH)\int_a^b f^+ \leq (ap\text{-}SH)\int_a^b g^+(x).$$

Hence by Theorem 2.13, we have

$$(ap\text{-}ISH)\int_a^b f(x) \leq (ap\text{-}ISH)\int_a^b g(x).$$

This completes the proof. □

Theorem 2.17 Let $f, g \in$ *ap-ISH*$[a,b]$ and $d(f,g)$ is ap-Sequential Henstock (ap-SH) integrable on $[a,b]$, then

$$d((ap\text{-}ISH)\int_a^b f, (ap\text{-}ISH)\int_a^b g) \leq (ap\text{-}SH)\int_a^b d(f,g).$$

Proof . By metric definition, we have

$$d((ap\text{-}ISH)\int_a^b f, (ap\text{-}ISH)\int_a^b g)$$

$$\begin{aligned}
&= \max(|((ap\text{-}SH)\int_a^b f)^- - ((ap\text{-}SH)\int_a^b g)^-|, |((ap\text{-}SH)\int_a^b f)^+ - ((ap\text{-}SH)\int_a^b g)^+|) \\
&= \max(|(ap\text{-}SH)\int_a^b (f^- - g^-)|, |(ap\text{-}SH)\int_a^b (f^+ - g^+)|) \\
&\leq \max((ap\text{-}SH)\int_a^b |(f^- - g^-)|, (ap\text{-}SH)\int_a^b |(f^+ - g^+)|) \\
&\leq (ap\text{-}SH)\int_a^b \max(|(f^- - g^-)|, |(f^+ - g^+)|) \\
&\leq (ap\text{-}SH)\int_a^b d(f,g).
\end{aligned}$$

This completes the proof. □

From Theorem 2.13 and by Definition 1.8, we can easily obtain the following Theorem.

Theorem 2.18 Let $f \in$ *ap-ISH*$[a,c]$ and $f \in$ *ap-ISH*$[c,b]$, then $f \in$ *ap-ISH*$[a,b]$ and

$$(ap\text{-}ISH)\int_a^b f = (ap\text{-}ISH)\int_a^c f + (ap\text{-}ISH)\int_c^b f.$$

Proof . If $f \in$ *ap-ISH*$[a,c]$) and $f \in$ *ap-ISH*$[c,b]$) then by Theorem 2.2, $f^-, f^+ \in SH[a,c]$) and $f^-, f^+ \in SH[c,b]$). Hence, $f^-, f^+ \in SH[a,b]$) and

$$\begin{aligned}
(ap\text{-}SH)\int_a^b f^- &= (ap\text{-}SH)\int_a^c f^- + (ap\text{-}SH)\int_c^b f^- \\
&= ((ap\text{-}ISH)\int_a^c f + (ap\text{-}ISH)\int_c^b f)^-.
\end{aligned}$$

Similarly,

$$\begin{aligned}(ap\text{-}SH)\int_a^b f^+ &= (ap\text{-}SH)\int_a^c f^+ + (ap\text{-}SH)\int_c^b f^+\\ &= ((ap\text{-}ISH)\int_a^c f + (ap\text{-}ISH)\int_c^b f)^+.\end{aligned}$$

Hence by Theorem 2.13, $f \in ap\text{-}ISH[a,b]$ and

$$(ap\text{-}ISH)\int_a^b f = (ap\text{-}ISH)\int_a^c f + (ap\text{-}ISH)\int_c^b f.$$

□

3 Application of *ap-ISH* integral

Our newly introduced type of integral for interval-valued functions, may be useful in the further study of fuzzy-valued functions, interval optimization and interval-valued differential equations (see [**?** 6, 11]).

Holzmann et al. [4], Lang [8][8] as well as Kramer and Wedner [7] have successfully applied the techniques of interval analysis for approximate continuous functions to adaptive Gaussian quadrature (see [10]).

One other good application for consideration in the study of approximate Sequential Henstock integral is in the theory of trigonometric series and trigonometric integrals. One of the principle questions concerning trigonometric series is the question of recovering the coefficients of every convergent trigonometric series from its sum (see [4, 10]).

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