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Numerical solution of the linear inverse wave equation

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Abstract

In this paper, a numerical method is proposed for the numerical solution of a linear wave equation with initial and boundary conditions by using the cubic B-spline method to determine the unknown boundary condition. We apply the cubic B-spline for the spatial variable and the derivatives, which generate an ill-posed linear system of equations. In this regard, to overcome, this drawback, we employ the Tikhonov regularization (TR) method for solving the resulting linear system. It is proved that the proposed method has the order of convergence $O\Big((\Delta t)^2 + h^2\Big)$. Also, the conditional stability by using the Von-Neumann method is established under suitable assumptions. Finally, some numerical experiments are reported to show the efficiency and capability of the proposed method for solving inverse problems.

Keywords: Inverse wave problem, Existence and uniqueness, Stability analysis, Convergence analysis, Tikhonov regularization method
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1 Introduction

Inverse problems are encountered in many branches of engineering and science. In one particular branch, hyperbolic and parabolic initial and boundary value problems in one dimension have been studied by several authors [1, 5, 2, 3, 9]. Mathematically, the inverse problems belong to a class of problems called the ill-posed problems, i.e. small errors in the measured data can lead to large deviations in the estimated quantities. As a consequence, their solution does not satisfy the general requirement of existence, uniqueness, and stability under small changes to the input data.

Lin and Gilbert [9], presented a numerical algorithm for solving an undetermined coefficient problem for an inverse wave equation. The algorithm is based on an integral representation for the solution to the wave equation obtained by using transmutation. The stability of the inverse problem of determining a function $q(x)$ in a wave equation $\partial_t^2 u - \Delta u + q(x)u = 0$ presented in the bounded smooth domain in $\mathbb{R}^n$ from boundary observations [2]. Wu and Liu [15], considered an inverse problem for a one-dimensional integro-differential hyperbolic system, which comes from a simplified model of thermoelasticity. By using the fixed point theorem in suitable Sobolev spaces, the global in time existence and uniqueness results of this inverse problem are obtained.

The theory of B-spline functions has attracted attention in the literature for the numerical solution of linear and nonlinear boundary value problems in science and engineering. The B-spline scaling functions are used to find the

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approximate solution of the surface heat flux histories and temperature distribution in an inverse heat conduction problem [4]. The numerical solution of certain partial differential equations can be obtained using B-spline functions of various degrees. As examples, a combined finite difference and cubic B-spline approach was applied for the solution of the heat and wave equation [7], a collocation of modified cubic B-spline basis functions over the finite elements was developed in [11] for the solution of symmetric regularized long wave equations.

In this paper, we use the cubic B-spline method to solve a one-dimensional inverse problem, for a linear wave equation with initial and boundary conditions, using measurement data containing noise, as follows:

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad 0 < x < 1, \quad 0 < t < t_M \tag{1.1a}$$

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 \le x \le 1, \tag{1.1b}$$

$$u(0,t) = q(t), \qquad u(1,t) = h(t), \qquad 0 \le t \le t_M, \tag{1.1c}$$

and the overspecified condition

$$u(a,t) = k(t), \qquad 0 < a < 1, \qquad 0 \le t \le t_M, \tag{1.1d}$$

where $f(x)$ and $g(x)$ are known smooth functions on $[0,1]$, $h(t)$ and $k(t)$ are known smooth functions on $[0,t_M]$, where t_M represents the final time of interest for the time evolution of the problem, while $q(t)$ is unknown which remains to be determined from (1.1d). The measurements ensure that the inverse problem has a unique solution, but this solution is unstable hence the problem is ill-posed. This instability is overcome using the TR method with the generalized cross-validation (GCV) criterion for the choice of the regularization parameter. As well, the existence and uniqueness of the solution are also derived.

The plan of this paper is as follows: In Section 2, we will study the existence and uniqueness of the solution. In Section 3, we describe the properties of the cubic B-splines collocation method. In the following, in Section 4, we detail our presented method for solving the inverse wave problem (1.1). The conditional stability based on the Von-Neumann method is discussed in Sections 5. Some numerical examples are presented in Section 6 and finally concluding remarks are given in Section 7.

2 Existence and uniqueness

In this Section, we consider the problem of existence and uniqueness of solution for the inverse wave problem (1.1). Let us first consider the following initial and boundary value problem, in which all initial and boundary data are known.

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad 0 < x < 1, \quad 0 < t, \tag{2.1a}$$

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 \le x \le 1, \tag{2.1b}$$

$$u(0,t) = q(t), \qquad u(1,t) = p(t), \qquad 0 \le t. \tag{2.1c}$$

Here, we adopt the method used in [6, Section 2.4]. Introducing new independent variables ζ, η by the substitution

$$\zeta = x + t, \qquad \eta = x - t, \tag{2.2}$$

we transform the linear equation (2.1a) into the following equation:

$$(u_\zeta)_\eta = 0. \tag{2.3}$$

Since $(u_\zeta)_\eta = 0$ it follows that u_ζ is independent of η, say, $u_\zeta = F'(\zeta)$, and then $u = F(\zeta) + G(\eta)$, so that in the original variables we have

$$u = F(x+t) + G(x-t). \tag{2.4}$$

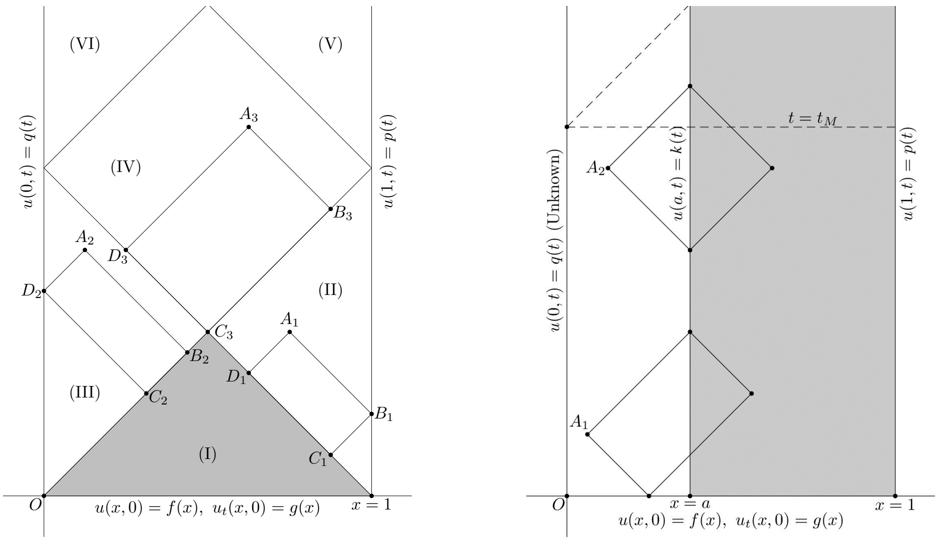
Impose the initial conditions (2.1b) and get

$$f(x) = F(x) + G(x), \qquad g(x) = F'(x) - G'(x).$$

Solving these two linear equations, we obtain

$$u(x,t) = \frac{1}{2}\big[f(x+t) + f(x-t)\big] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds. \tag{2.5}$$

We see from (2.5) that $u(x,t)$ is determined uniquely by the values of the initial functions f and g in the interval $[x-t, x+t]$ of the x-axis whose endpoints are cut by the characteristics through the point (x,t), e.g. region (I) in Figure 1a. For any $f, g \in \mathcal{C}^2(\mathbb{R})$, formula (2.4) represents a solution $u \in \mathcal{C}(\mathbb{R}^2)$ of (2.1a) with initial data (2.1b).



(a) Direct Problem (b) Inverse Problem

Figure 1: Wave Problem

After some calculations, one can see that any function u of the form (2.4) satisfies the following functional equation:

$$u(x,t) - u(x+\zeta, t+\zeta) - u(x-\eta, t+\eta) + u(x+\zeta-\eta, t+\zeta+\eta) = 0. \tag{2.6}$$

This means that, for any parallelogram $A_iB_iC_iD_i$ in the xt-plane bounded by four characteristic lines, see Figure 1a, the following holds:

$$u(A_i) + u(C_i) = u(B_i) + u(D_i). \tag{2.7}$$

We use (2.7) to solve the initial and boundary value problem (2.1) in the strip

$$0 < x < 1, \qquad t > 0.$$

We divide the strip into several regions by the characteristics through the corners and through that points of intersections of the characteristics with the boundaries, etc. as shown in Figure 1a.

In region (I) the solution u is determined by (2.5) from the initial data alone. In a point $A_1 = (x,t)$ in region (II) we form a parallelogram $A_1B_1C_1D_1$ in such a way that B_1 lies on the line $x = 1$ and C_1, D_1 lie in region (I), so that $u(B_1)$, $u(C_1)$, and $u(D_1)$ are known, and thus we get

$$u(A_1) = -u(C_1) + u(B_1) + u(D_1).$$

Similarly, we get $u(x,t)$ successively in all points of the regions (II), (III), ... If we want the solution of (2.1) to be of class $\mathcal{C}^2$ in the closure of the strip, the data f, g, p, and q have to fit together in the corners so that u and its first and second derivatives come out to be the same when computed either from f and g or from p and q. We clearly need the following compatibility conditions:

$$q(0) = f(0), \qquad q'(0) = g(0), \qquad q''(0) = f''(0), \tag{2.8}$$

$$p(1) = f(1), \qquad p'(1) = g(1), \qquad p''(1) = f''(1). \tag{2.9}$$

Now, consider the inverse problem (1.1) with overspecified condition (1.1d). Take $t_F = t_M + a$, and suppose that $p(t)$ and $k(t)$ are sufficiently smooth functions defined in $0 \le t \le t_F$. Problem (1.1) may be divided into two separate problems:

a direct problem:

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad a < x < 1, \quad 0 < t < t_F, \tag{2.10a}$$
$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad a \le x \le 1, \tag{2.10b}$$
$$u(a,t) = k(t), \qquad u(1,t) = p(t), \qquad 0 \le t \le t_F, \tag{2.10c}$$

and an inverse problem:

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad 0 < x < a, \quad 0 < t < t_M, \tag{2.11a}$$
$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 \le x \le a, \tag{2.11b}$$
$$u(0,t) = q(t), \qquad u(a,t) = k(t), \qquad 0 \le t \le t_M. \tag{2.11c}$$

Using the above argument, we obtain the solution of the direct problem (2.10) in the strip

$$a \le x \le 1, \qquad 0 < t < t_F.$$

For any point $A_i = (t,x)$ in the strip $0 \le x \le a$, $0 < t < t_M$, we can draw a parallelogram $A_iB_iC_iD_i$, as it is shown in Figure 1b, such that the two adjacent vertices B_i and D_i lie on the boundary $\{t = 0\} \cup \{x = a\}$ on which the initial data ($u = f(x)$) and the boundary data ($u(a,t) = k(t)$) are known, and the non-adjacent vertex C_i lie in the strip $a \le x \le 1$, $0 \le t \le t_F$ in which $u(x,t)$, the solution of direct problem (2.10), is known. Therefore, $u(A_i)$ can be obtained from the following equality:

$$u(A_i) = u(B_i) + u(D_i) - u(C_i).$$

We can summarize the above discussion in the following statement.

Theorem 2.1. Let $0 < a < 1$ and $t_M > 0$. Suppose that $f(x)$ and $g(x)$ are sufficiently smooth functions on $[0,1]$, and that $p(t)$, $k(t)$ are sufficiently smooth functions on $[0,t_F]$, where $t_F = t_M + a$, such that f, g, p, and k satisfy the consistency conditions

$$k(a) = f(a), \qquad k'(a) = g(a), \qquad k''(a) = f''(a),$$
$$p(1) = f(1), \qquad p'(1) = g(1), \qquad p''(1) = f''(1).$$

Then the inverse wave equation

$$\begin{aligned} &u_{tt}(x,t) = u_{xx}(x,t), && && 0 < x < 1, \quad 0 < t < t_M,\\ &u(x,0) = f(x), && u_t(x,0) = g(x), && 0 \le x \le 1,\\ &u(0,t) = q(t), && u(1,t) = p(t), && 0 \le t \le t_M,\\ &u(a,t) = k(t), && && 0 < a < 1, \quad 0 \le t \le t_M. \end{aligned}$$

has a unique solution (u,q).

3 Cubic B-spline functions

In this Section, we describe the uniform cubic B-spline on the finite interval $[0,1]$. For this purpose, we divide the interval $[0,1]$ into N-subintervals by the set of $N+1$ nodal points x_m, $0 \le m \le N$. This gives a partition $\pi : 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$ of $[0,1]$, where $\Delta x_m = x_m - x_{m-1}$, $\forall\ 1 \le m \le N$. The cubic B-splines are constructed for the partition

$$\Pi : x_{-2} < x_{-1} < x_0 = 0 < x_1 < \cdots < x_N = 1 < x_{N+1} < x_{N+2},$$

by using four fictitious nodes $x_{-2}, x_{-1}, x_{N+1}, x_{N+2}$. If we assume that $\Delta x_m = h$, $\forall\ -1 \le m \le N+2$, then the uniform cubic B-splines are defined by (see, [12])

$$B_m(x) = \frac{\Delta^4 F_x(x_{m-2})}{h^3},$$

where

$$F_x(x_m) = (x_m - x)^3_+ = \begin{cases} (x_m - x)^3, & x < x_m,\\ 0, & x \ge x_m, \end{cases}$$

and $\Delta^4 F_x(x_m)$ is the fourth forward difference with equally spaced nodes of third degree polynomial $F_x(x_m)$. After some simplification, we get

$$B_m(x) = \frac{1}{h^3}\begin{cases} (x - x_{m-2})^3, & x_{m-2} \le x \le x_{m-1}, \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, & x_{m-1} \le x \le x_m, \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, & x_m \le x \le x_{m+1}, \\ (x_{m+2} - x)^3, & x_{m+1} \le x \le x_{m+2}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

It can be easily see that the functions in $\{B_{-1}, B_0, \ldots, B_N, B_{N+1}\}$ are linearly independent on $[0, 1]$. By using splines defined in (3.1), the values of $B_m(x)$ and its derivatives at the nodes x_m's are given in Table 1.

Table 1: The values of $B_m(x)$ and its derivatives at the nodes x_m.

	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	0	1	4	1	0
$B'_i(x)$	0	$\frac{3}{h}$	0	$\frac{-3}{h}$	0
$B''_i(x)$	0	$\frac{6}{h^2}$	$\frac{-12}{h^2}$	$\frac{6}{h^2}$	0

4 Solution of the linear inverse wave equation (1.1)

In this Section, we first present our method based on the cubic B-spline functions for solving the linear inverse wave equation (1.1). To apply the proposed method, expressing $u(x, t)$ by using cubic B-spline functions. Let

$$U_n(x) = \sum_{i=-1}^{N+1} c_i^n B_i(x), \tag{4.1}$$

be the approximate solution of the initial and boundary value problem (1.1) at the n-th time level, where c_i^n is unknown time dependent quantities to be determined.

It is required that approximate solutions (4.1), satisfies equations (1.1a)-(1.1d) at the point $x = x_m$, $0 \le m \le N$. Setting equations (4.1) in (1.1a), it follows that

$$U_{tt}(x_m, t_n) = \sum_{i=-1}^{N+1} c_i^n B''_i(x_m).$$

Using the cubic B-spline functions and their derivatives up to second order which are determined in Table 1, we obtain

$$U_{tt}(x_m, t_n) = \frac{6}{h^2}\Big(c_{m-1}^n - 2c_m^n + c_{m+1}^n\Big). \tag{4.2}$$

In (4.2), the time derivative is discretized in a forward finite difference fashion

$$U_{tt}(x_m, t_n) = \frac{U_{n+1}(x_m) - 2U_n(x_m) + U_{n-1}(x_m)}{(\Delta t)^2}, \tag{4.3}$$

where Δt is the time step. So due to the (4.2), we obtain

$$U_{n+1}(x_m) = \frac{6(\Delta t)^2}{h^2}\Big(c_{m-1}^n - 2c_m^n + c_{m+1}^n\Big) + 2U_n(x_m) - U_{n-1}(x_m).$$

According to the cubic B-spline functions (Table 1), we have

$$\begin{aligned} c_{m-1}^{n+1} + 4c_m^{n+1} + c_{m+1}^{n+1} &= \frac{6(\Delta t)^2}{h^2}\Big(c_{m-1}^n - 2c_m^n + c_{m+1}^n\Big) + 2\Big(c_{m-1}^n + 4c_m^n + c_{m+1}^n\Big) - \Big(c_{m-1}^{n-1} + 4c_m^{n-1} + c_{m+1}^{n-1}\Big) \\ &= \mathcal{X}_m^n, \end{aligned} \tag{4.4}$$

where $n = 0, 1, \ldots$ and $0 \leq m \leq N$. For when $n = 0$, the initial condition $u_t(x, 0) = g(x)$, with the central finite difference method, is written the following:

$$U_t(x_m, t_n) = \frac{U_{n+1}(x_m) - U_{n-1}(x_m)}{2\Delta t} = g(x_m).$$

Therefore,

$$U_{n-1}(x_m) = U_{n+1}(x_m) - 2\Delta t g(x_m)$$

and according to the cubic B-spline functions, we have

$$c_{m-1}^{n-1} + 4c_m^{n-1} + c_{m+1}^{n-1} = c_{m-1}^{n+1} + 4c_m^{n+1} + c_{m+1}^{n+1} - 2\Delta t g(x_m),$$

so the equation (4.4) turns to:

$$c_{m-1}^{n+1} + 4c_m^{n+1} + c_{m+1}^{n+1} = \frac{3(\Delta t)^2}{h^2}\Big(c_{m-1}^n - 2c_m^n + c_{m+1}^n\Big) + \Big(c_{m-1}^n + 4c_m^n + c_{m+1}^n\Big) + \Delta t g(x_m). \tag{4.5}$$

The system (4.4) consists of $(N+1)$ equations in $(N+3)$ unknown coefficients. Therefore, we still need two equations. To this end, by imposing the boundary condition $u(1, t) = h(t)$ and the overspecified condition (1.1d), we have

$$U_{n+1}(x_l) = c_{l-1}^{n+1} + 4c_l^{n+1} + c_{l+1}^{n+1} = k(t_{n+1}), \qquad x_l = a, \quad 1 \leq l \leq N-1,$$

$$U_{n+1}(x_N) = c_{N-1}^{n+1} + 4c_N^{n+1} + c_{N+1}^{n+1} = p(t_{n+1}).$$

Then a system of $(N+3)$ linear equations in the $(N+3)$ unknown coefficients is obtained. This system can be written in the matrix vector form as follows:

$$AX = B, \tag{4.6}$$

where

$$X = \Big[c_{-1}^{n+1}, c_0^{n+1}, c_1^{n+1}, \ldots, c_{N-1}^{n+1}, c_N^{n+1}, c_{N+1}^{n+1}\Big]^T,$$

$$B = \Big[k(t_{n+1}), \mathcal{X}_0^n, \mathcal{X}_1^n, \ldots, \mathcal{X}_{N-1}^n, \mathcal{X}_N^n, p(t_{n+1})\Big]^T,$$

where for $n = 0$, with due attention to (4.5),

$$\mathcal{X}_m^0 = \frac{3(\Delta t)^2}{h^2}\Big(c_{m-1}^0 - 2c_m^0 + c_{m+1}^0\Big) + \Big(c_{m-1}^0 + 4c_m^0 + c_{m+1}^0\Big) + \Delta t g(x_m), \qquad 0 \leq m \leq N,$$

and A is an $(N+3) \times (N+3)$-dimensional matrix given by

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 4 & 1 & \ldots & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 4 & 1 \end{pmatrix},$$

where, $A[1, l+1] = 1$, $A[1, l+2] = 4$, $A[1, l+3] = 1$.

In order to solve the system of linear equations (4.6), we need initial vector X^0. This initial vector can be obtained from the initial condition $u(x, 0) = f(x)$, the boundary condition $u(1, t) = h(t)$, and the overspecified condition (1.1d) as the following expression

$$\begin{aligned} U_0(x_m) &= c_{m-1}^0 + 4c_m^0 + c_{m+1}^0 = f(x_m), && 0 \leq m \leq N, \\ U_0(x_l) &= c_{l-1}^0 + 4c_l^0 + c_{l+1}^0 = k(0), && 1 \leq l \leq N-1, \\ U_0(x_N) &= c_{N-1}^0 + 4c_N^0 + c_{N+1}^0 = p(0). && \end{aligned}$$

Then a system of $(N+3)$ linear equations in the $(N+3)$ unknown coefficients are obtained. This system can be written in the matrix vector form as follows

$$AX^0 = B^*, \tag{4.7}$$

where

$$X^0 = \Big[c_{-1}^0, c_0^0, c_1^0, \ldots, c_{N-1}^0, c_N^0, c_{N+1}^0\Big]^T,$$

and

$$B^* = \Big[k(0), f(x_0), f(x_1), \ldots, f(x_{N-1}), f(x_N), p(0)\Big]^T.$$

The solution of the linear systems (4.7) for vector X^0 and (4.6) for vector X can be get by the TR method ([8, 14]). Finally we can obtain

$$U_n(x_0) = U(x_0, t_n) = q(t_n) = c_{-1}^n + 4c_0^n + c_1^n, \qquad n = 1, 2, \ldots,$$
$$U_n(x_m) = U(x_m, t_n) = c_{m-1}^n + 4c_m^n + c_{m+1}^n, \qquad n = 1, 2, \ldots, \qquad m = 1, 2, \ldots, N.$$

Theorem 4.1. The collocation approximation $U_n(x)$ for the solution $u_n(x)$ of the inverse wave equation (1.1) satisfy the following error estimate

$$\Big\|u_n - U_n\Big\|_\infty \leq \gamma h^2,$$

for sufficiently small h, where γ is a positive constant.

Proof . See [5]. □

Theorem 4.2. Let $u(x,t)$ be the solution of the initial boundary value problem (1.1). Also, suppose that $U_n(x)$ is the collocation approximation of the solution $u_n(x)$ after the temporal discretization stage. Then the error estimate of the totally discrete scheme is given by

$$\Big\|u_n - U_n\Big\|_\infty \leq \kappa\Big((\Delta t)^2 + h^2\Big),$$

where κ is some finite constant.

Proof . The time discretization process (4.3) that we use to discretize the system (1.1) in time variable is of the two order convergence (see, [13]). So, according to the Theorem 4.1, we have

$$\Big\|u_n - U_n\Big\|_\infty \leq \kappa\Big((\Delta t)^2 + h^2\Big),$$

where κ is some finite constant. □

5 Stability of the scheme

In this Section, we present the stability analysis for our proposed scheme. To this end, we state and prove the following theorem based on the Von-Neumann method.

Theorem 5.1. The scheme (4.4) for solving inverse wave equation (1.1) is conditionally stable, if $\frac{(\Delta t)^2}{h^2} < \frac{1}{6}$.

Proof . By putting $c_m^n = \xi^n e^{im\psi h}$ in equation (4.4), where ξ is the amplification factor for the scheme, ψ is the mode number, h is the space length and $i = \sqrt{-1}$, and simplifying it, we have

$$\xi(\cos\theta + 2) = \frac{6(\Delta t)^2}{h^2}(\cos\theta - 1) + (\cos\theta + 2), \tag{5.1}$$

where, $\theta = \psi h$. Dividing the both sides of the equation (5.1) by $(\cos\theta + 2)$, we have

$$\xi = \frac{-12(\Delta t)^2}{h^2}\frac{\sin^2\frac{\theta}{2}}{\cos\theta + 2} + 1,$$

so,

$$\Big|\xi\Big| = \Big|1 - \frac{12(\Delta t)^2}{h^2}\frac{\sin^2\frac{\theta}{2}}{\cos\theta + 2}\Big|.$$

If $|\xi| \leq 1$ then the scheme (4.4) is stable. To do this, if $\frac{(\Delta t)^2}{h^2} < \frac{1}{6}$, then $|\xi| \leq 1$, and the proof is complete. □

6 Numerical experiments

In this Section, we are going to demonstrate numerically, some of results for the unknown boundary condition in the inverse problem (1.1). The propose of this Section is to illustrate the applicability of the present method is described in Section 4 for solving this inverse problem. As expected the inverse problems are ill-posed, therefore it is necessary to investigate the stability of the present method by giving a test problem. The proposed method is written in the MATLAB 7.14 (R2012a) and is tested on a personal computer with Intel(R) Core(TM)2 Duo CPU and 4GB RAM. Numerical results are compared with the finite difference method (FDM) and the radial basis function (RBF) method (Multiquadrics-RBF) [10].

Remark 6.1. In all examples, we take $t_M = 1$, $a = 0.01$, $h = 0.01$, $\Delta t = 0.001$, and compute the unknown boundary condition $u(0, t)$ for different values of time steps. Solutions $u(x, t)$ are also computed for the different values of time $t = 0.1, 0.4, 0.7, 1$ with space step length $h = 0.01$.

Example 6.2. In this example, we consider the following one-dimensional inverse problem, for estimating unknown boundary condition $q(t)$.

$$\begin{array}{lll} u_{tt}(x,t) = u_{xx}(x,t), & & 0 < x < 1, \quad 0 < t < 1, \\ u(x,0) = x^2, & u_t(x,0) = 0, & 0 \le x \le 1, \\ u(1,t) = 1 + t^2, & u(0.01,t) = (0.01)^2 + t^2, & 0 \le t \le 1, \end{array}$$

The exact solutions of this problem are

$$u(x,t) = x^2 + t^2, \qquad q(t) = t^2, \qquad 0 \le x \le 1, \qquad 0 \le t \le 1.$$

The numerical results of the unknown boundary condition $u(0, t)$ is reported in Table 2. To clarify the accuracy of the present method, the corresponding graphical illustration is presented in Figure 2. Also, the graphical illustration corresponding to the difference between the exact and numerical solutions of $u(x, t)$ is presented in Figure 3. The obtained numerical solutions for $u(x, t)$ are given in Tables 3 and 6, and the graphical illustrations corresponding to these Tables are presented in Figure 4.

Table 2: The comparison among the exact and numerical solutions for $u(0, t) = q(t)$ in Example 6.2.

t	B-spline			FDM		RBF	
	$q(t)$	$q^*(t)$	$\lvert q(t) - q^*(t)\rvert$	$q^*(t)$	$\lvert q(t) - q^*(t)\rvert$	$q^*(t)$	$\lvert q(t) - q^*(t)\rvert$
0.1	0.010000	0.008665	$1.334744e-03$	-0.023419	$3.341870e-02$	0.020145	$1.014451e-02$
0.2	0.040000	0.039309	$6.910453e-04$	0.012355	$2.764466e-02$	0.050173	$1.017265e-02$
0.3	0.090000	0.089235	$7.645219e-04$	0.067826	$2.217416e-02$	0.100203	$1.020269e-02$
0.4	0.160000	0.159209	$7.910390e-04$	0.147471	$1.252899e-02$	0.170188	$1.018757e-02$
0.5	0.250000	0.249393	$6.065713e-04$	0.255769	$5.769081e-03$	0.260101	$1.010145e-02$
0.6	0.360000	0.359306	$6.937972e-04$	0.397198	$3.719826e-02$	0.370105	$1.010534e-02$
0.7	0.490000	0.489556	$4.438077e-04$	0.576237	$8.623678e-02$	0.500095	$1.009509e-02$
0.8	0.640000	0.638335	$1.665023e-03$	0.797363	$1.573629e-01$	0.650133	$1.013280e-02$
0.9	0.810000	0.809173	$8.269771e-04$	1.065055	$2.550547e-01$	0.820247	$1.024718e-02$
1	1.000000	0.999627	$3.734836e-04$	1.383791	$3.837905e-01$	1.010284	$1.028422e-02$
Execution time (second)	63.710959			655.363505		660.053746	

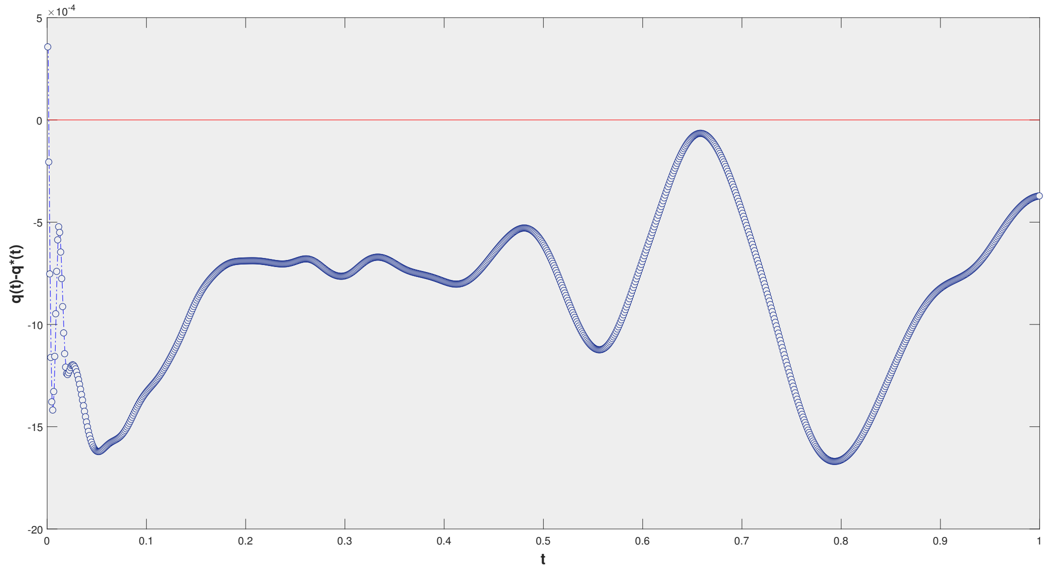


Figure 2: Difference between the exact and numerical solutions of $u(0,t)$ (using cubic B-spline method) at different time levels in Example 6.2.

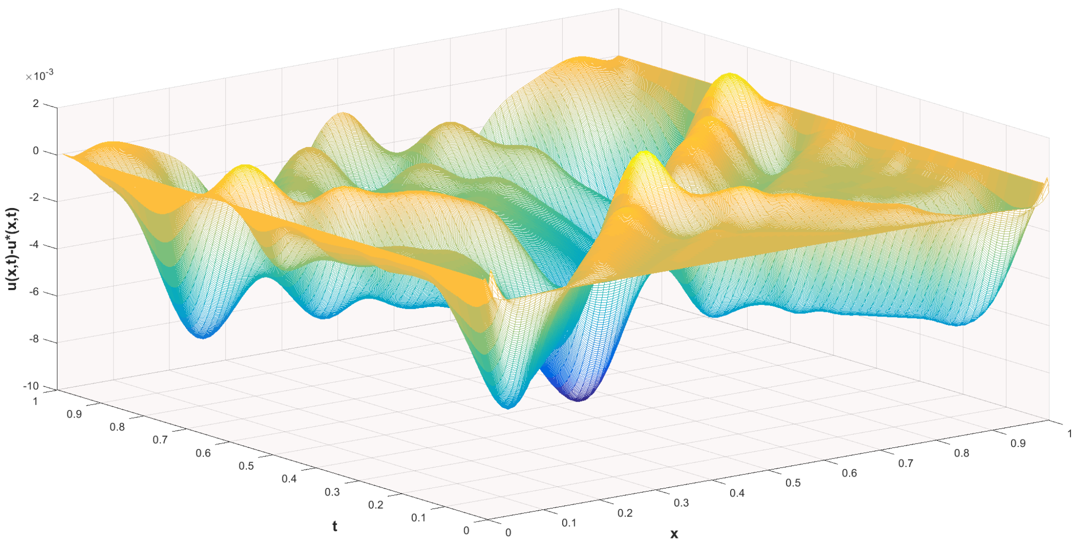


Figure 3: Difference between the exact and numerical solutions of $u(x,t)$ (using cubic B-spline method) in Example 6.2.

Table 3: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t = 0.1$ for Example 6.2.

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$
0.1	0.020000	0.025980	$5.980018e-03$	0.281876	$2.618755e-01$	0.080882	$6.088199e-02$
0.2	0.050000	0.052632	$2.632262e-03$	0.514170	$4.641700e-01$	0.115037	$6.503706e-02$
0.3	0.100000	0.100167	$1.668558e-04$	0.581775	$4.817751e-01$	0.165037	$6.503702e-02$
0.4	0.170000	0.171013	$1.012523e-03$	0.652335	$4.823352e-01$	0.235037	$6.503708e-02$
0.5	0.260000	0.260831	$8.305401e-04$	0.742344	$4.823441e-01$	0.325037	$6.503713e-02$
0.6	0.370000	0.370968	$9.675648e-04$	0.852336	$4.823359e-01$	0.435037	$6.503721e-02$
0.7	0.500000	0.500289	$2.888326e-04$	0.981821	$4.818208e-01$	0.565037	$6.503745e-02$
0.8	0.650000	0.652114	$2.113812e-03$	1.115606	$4.656059e-01$	0.715039	$6.503879e-02$
0.9	0.820000	0.825945	$5.944882e-03$	1.098631	$2.786313e-01$	0.881830	$6.182991e-02$
1	1.010000	1.009995	$5.378808e-06$	–	–	0.575730	$2.557303e-01$

Table 4: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t = 0.4$ for Example 6.2.

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$
0.1	0.170000	0.171559	$1.558871e-03$	0.182022	$1.202229e-02$	0.234580	$6.457960e-02$
0.2	0.200000	0.201761	$1.760764e-03$	0.083894	$1.161062e-01$	0.329997	$1.299973e-01$
0.3	0.250000	0.252231	$2.231220e-03$	0.248293	$1.707446e-03$	0.444678	$1.946779e-01$
0.4	0.320000	0.323350	$3.349613e-03$	0.609328	$2.893275e-01$	0.575730	$2.557303e-01$
0.5	0.410000	0.417517	$7.516586e-03$	0.813802	$4.038023e-01$	0.674546	$2.645461e-01$
0.6	0.520000	0.523644	$3.644327e-03$	0.822459	$3.024589e-01$	0.776779	$2.567790e-01$
0.7	0.650000	0.651987	$1.987092e-03$	0.680976	$3.097568e-02$	0.846297	$1.962968e-01$
0.8	0.800000	0.801716	$1.716448e-03$	0.718246	$8.175390e-02$	0.931618	$1.316177e-01$
0.9	0.970000	0.971273	$1.273001e-03$	1.000972	$3.097183e-02$	1.036196	$6.619598e-02$
1	1.160000	1.159994	$6.107287e-06$	–	–	1.159997	$3.231078e-06$

Table 5: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t = 0.7$ for Example 6.2.

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$
0.1	0.500000	0.500824	$8.238517e-04$	0.539812	$3.981189e-02$	0.566255	$6.625531e-02$
0.2	0.530000	0.535545	$5.544546e-03$	0.453964	$7.603589e-02$	0.663942	$1.339421e-01$
0.3	0.580000	0.584956	$4.955926e-03$	0.458463	$1.215369e-01$	0.769286	$1.892855e-01$
0.4	0.650000	0.653058	$3.058458e-03$	0.202756	$4.472444e-01$	0.845312	$1.953122e-01$
0.5	0.740000	0.742334	$2.333865e-03$	0.011933	$7.280665e-01$	0.936058	$1.960584e-01$
0.6	0.850000	0.853017	$3.016800e-03$	0.396319	$4.536810e-01$	1.045315	$1.953154e-01$
0.7	0.980000	0.984549	$4.548550e-03$	0.853315	$1.266847e-01$	1.169795	$1.897953e-01$
0.8	1.130000	1.135695	$5.694983e-03$	1.047237	$8.276316e-02$	1.265560	$1.355603e-01$
0.9	1.300000	1.300890	$8.900114e-04$	1.300612	$6.118885e-04$	1.367871	$6.787105e-02$
1	1.490000	1.489993	$7.360499e-06$	–	–	1.490049	$4.892595e-05$

Table 6: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t=1$ for Example 6.2.

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$
0.1	1.010000	1.009840	$1.597410e-04$	0.953249	$5.675088e-02$	1.002341	$7.659472e-03$
0.2	1.040000	1.040851	$8.506884e-04$	0.624658	$4.153418e-01$	1.037527	$2.473141e-03$
0.3	1.090000	1.093820	$3.820391e-03$	0.645609	$4.443907e-01$	1.091218	$1.218128e-03$
0.4	1.160000	1.164019	$4.019215e-03$	0.727493	$4.325075e-01$	1.163432	$3.431617e-03$
0.5	1.250000	1.250346	$3.463538e-04$	0.812071	$4.379292e-01$	1.254169	$4.169362e-03$
0.6	1.360000	1.363526	$3.525692e-03$	0.887814	$4.721858e-01$	1.363432	$3.431632e-03$
0.7	1.490000	1.494238	$4.237672e-03$	0.929387	$5.606133e-01$	1.491218	$1.218170e-03$
0.8	1.640000	1.641035	$1.035010e-03$	1.033373	$6.066271e-01$	1.637527	$2.473020e-03$
0.9	1.810000	1.809836	$1.642921e-04$	1.475618	$3.343816e-01$	1.802347	$7.653413e-03$
1	2.000000	1.999993	$7.126144e-06$	–	–	2.000056	$5.554890e-05$

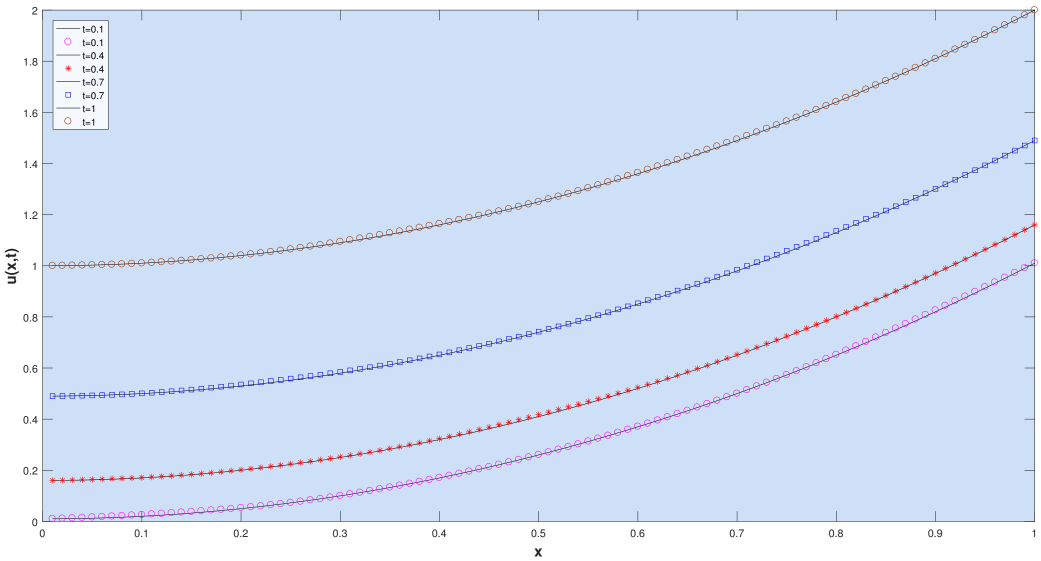


Figure 4: The comparison between the exact solution (shown by continuous lines) and numerical solution of $u(x,t)$ (using cubic B-spline method) at different time levels in Example 6.2.

Example 6.3. In this example, we consider the following one-dimensional inverse problem, for estimating unknown boundary condition $q(t)$.

$$\begin{array}{lll} u_{tt}(x,t)=u_{xx}(x,t), & & 0<x<1, \quad 0<t<1, \\ u(x,0)=e^{-x}, & u_t(x,0)=-e^{-x}, & 0\le x\le 1, \\ u(1,t)=e^{-t-1}, & u(0.01,t)=e^{-t-0.01}, & 0\le t\le 1, \end{array}$$

The exact solutions of this problem are

$$u(x,t)=e^{-t-x}, \qquad q(t)=e^{-t}, \qquad 0\le x\le 1, \qquad 0\le t\le 1.$$

The numerical results of the unknown boundary condition $u(0,t)$ is reported in Table 7. To clarify the accuracy of the present method, the corresponding graphical illustration is presented in Figure 5. Also, the graphical illustration corresponding to the difference between the exact and numerical solutions of $u(x,t)$ is presented in Figure 6. The obtained numerical solutions for $u(x,t)$ are given in Tables 8 and 11, and the graphical illustrations corresponding to these Tables are presented in Figure 7.

Table 7: The comparison among the exact and numerical solutions for $u(0,t) = q(t)$ in Example 6.3.

t	B-spline			FDM		RBF	
	$q(t)$	$q^*(t)$	$\|q(t)-q^*(t)\|$	$q^*(t)$	$\|q(t)-q^*(t)\|$	$q^*(t)$	$\|q(t)-q^*(t)\|$
0.1	0.904837	0.904024	8.130902e − 04	0.871590	3.324791e − 02	0.895971	8.866572e − 03
0.2	0.818731	0.818521	2.099746e − 04	0.792640	2.609112e − 02	0.810719	8.011885e − 03
0.3	0.740818	0.740596	2.220373e − 04	0.720016	2.080263e − 02	0.733580	7.238487e − 03
0.4	0.670320	0.670182	1.381749e − 04	0.658110	1.220978e − 02	0.663781	6.538692e − 03
0.5	0.606531	0.606474	5.657870e − 05	0.611317	4.785912e − 03	0.600625	5.905503e − 03
0.6	0.548812	0.548658	1.531983e − 04	0.584027	3.521576e − 02	0.543479	5.332581e − 03
0.7	0.496585	0.496425	1.602629e − 04	0.580636	8.405034e − 02	0.491771	4.814183e − 03
0.8	0.449329	0.449061	2.682956e − 04	0.605534	1.562052e − 01	0.444984	4.345092e − 03
0.9	0.406570	0.406447	1.231597e − 04	0.663116	2.565463e − 01	0.402649	3.920544e − 03
1	0.367879	0.367951	7.145120e − 05	0.757774	3.898944e − 01	0.364173	3.706071e − 03
Execution time (second)	69.980792			652.772187		521.060564	

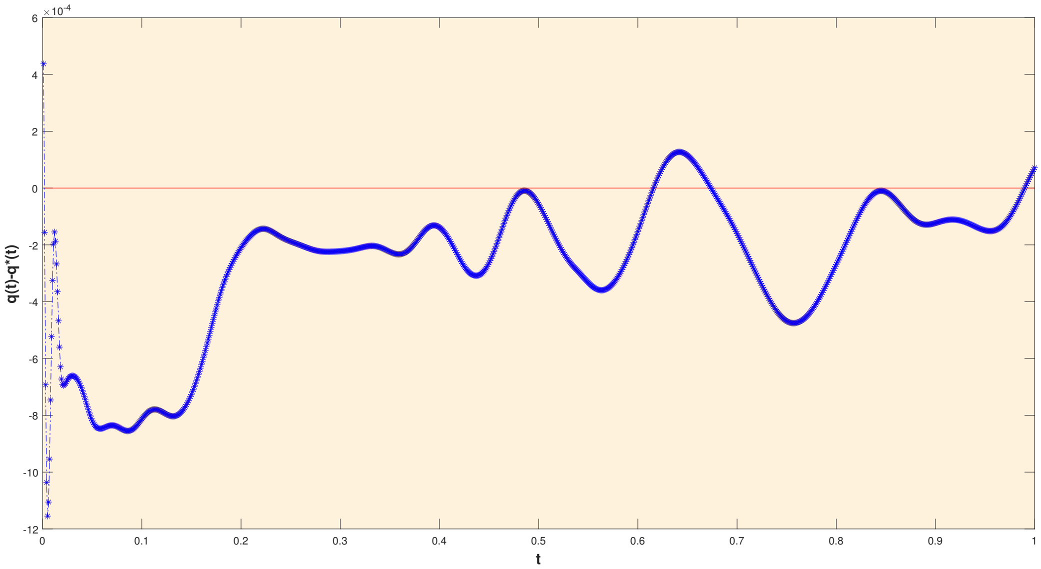


Figure 5: Difference between the exact and numerical solutions of $u(0,t)$ (using cubic B-spline method) at different time levels in Example 6.3.

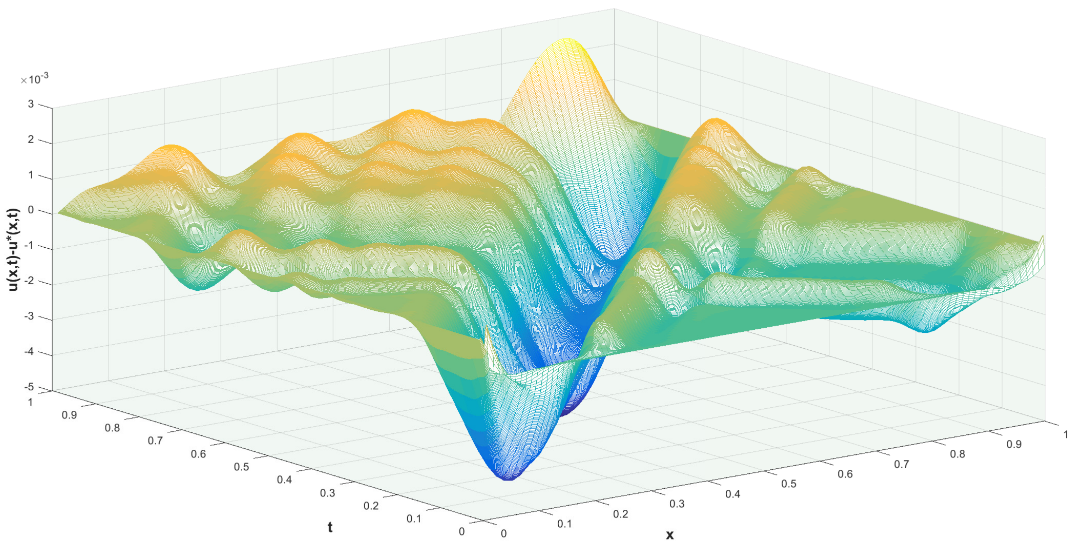


Figure 6: Difference between the exact and numerical solutions of $u(x,t)$ (using cubic B-spline method) in Example 6.3.

Table 8: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t=0.1$ for Example 6.3.

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$
0.1	0.818731	0.823112	$4.381017e-03$	1.077677	$2.589467e-01$	0.954919	$1.361880e-01$
0.2	0.740818	0.743757	$2.938683e-03$	1.201400	$4.605820e-01$	0.891976	$1.511582e-01$
0.3	0.670320	0.670515	$1.950860e-04$	1.148475	$4.781546e-01$	0.821467	$1.511473e-01$
0.4	0.606531	0.607636	$1.105114e-03$	1.085247	$4.787167e-01$	0.757669	$1.511387e-01$
0.5	0.548812	0.549713	$9.017288e-04$	1.027540	$4.787282e-01$	0.699943	$1.511309e-01$
0.6	0.496585	0.497650	$1.065000e-03$	0.975308	$4.787225e-01$	0.647709	$1.511240e-01$
0.7	0.449329	0.449918	$5.886690e-04$	0.927542	$4.782135e-01$	0.600447	$1.511176e-01$
0.8	0.406570	0.408501	$1.931524e-03$	0.868694	$4.621240e-01$	0.557682	$1.511123e-01$
0.9	0.367879	0.369052	$1.172179e-03$	0.644449	$2.765700e-01$	0.509446	$1.415668e-01$
1	0.332871	0.332870	$7.323592e-07$	–	–	0.333104	$2.330910e-04$

Table 9: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t=0.4$ for Example 6.3.

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$
0.1	0.606531	0.606883	$3.527633e-04$	0.620143	$1.361222e-02$	0.746847	$1.403166e-01$
0.2	0.548812	0.549153	$3.411326e-04$	0.620143	$1.361222e-02$	0.837249	$2.884375e-01$
0.3	0.496585	0.497542	$9.565775e-04$	0.494452	$2.133008e-03$	0.932793	$4.362074e-01$
0.4	0.449329	0.453334	$4.004753e-03$	0.736199	$2.868705e-01$	1.029026	$5.796970e-01$

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Table 9 – Continued from previous page

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\lvert u(x,t)-u^*(x,t)\rvert$	$u^*(x,t)$	$\lvert u(x,t)-u^*(x,t)\rvert$	$u^*(x,t)$	$\lvert u(x,t)-u^*(x,t)\rvert$
0.5	0.406570	0.410554	$3.984227e-03$	0.807468	$4.008980e-01$	1.012243	$6.056733e-01$
0.6	0.367879	0.368582	$7.024805e-04$	0.668337	$3.004574e-01$	0.953045	$5.851659e-01$
0.7	0.332871	0.333424	$5.532629e-04$	0.363895	$3.102422e-02$	0.776488	$4.436171e-01$
0.8	0.301194	0.301240	$4.599823e-05$	0.220256	$8.093847e-02$	0.596297	$2.951024e-01$
0.9	0.272532	0.272734	$2.019523e-04$	0.303362	$3.082995e-02$	0.596297	$2.951024e-01$
1	0.246597	0.246596	$9.247281e-07$	–	–	0.246799	$2.022716e-04$

Table 10: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t = 0.7$ for Example 6.3.

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\lvert u(x,t)-u^*(x,t)\rvert$	$u^*(x,t)$	$\lvert u(x,t)-u^*(x,t)\rvert$	$u^*(x,t)$	$\lvert u(x,t)-u^*(x,t)\rvert$
0.1	0.449329	0.449792	$4.633991e-04$	0.487433	$3.810402e-02$	0.591922	$1.425929e-01$
0.2	0.406570	0.407439	$8.697706e-04$	0.330168	$7.640202e-02$	0.697904	$2.913347e-01$
0.3	0.367879	0.368407	$5.271778e-04$	0.248213	$1.196667e-01$	0.778275	$4.103954e-01$
0.4	0.332871	0.332897	$2.563786e-05$	−0.108870	$4.417410e-01$	0.750983	$4.181121e-01$
0.5	0.301194	0.300740	$4.539631e-04$	−0.420574	$7.217680e-01$	0.718803	$4.176089e-01$
0.6	0.272532	0.272728	$1.958297e-04$	−0.177906	$4.504378e-01$	0.689172	$4.166404e-01$
0.7	0.246597	0.249164	$2.567163e-03$	0.120732	$1.258649e-01$	0.656659	$4.100616e-01$
0.8	0.223130	0.225911	$2.781171e-03$	0.141050	$8.208022e-02$	0.519528	$2.963979e-01$
0.9	0.201897	0.201331	$5.659195e-04$	0.202575	$6.781108e-04$	0.349113	$1.472168e-01$
1	0.182684	0.182683	$9.101218e-07$	–	–	0.182863	$1.796914e-04$

Table 11: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t = 1$ for Example 6.3.

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\lvert u(x,t)-u^*(x,t)\rvert$	$u^*(x,t)$	$\lvert u(x,t)-u^*(x,t)\rvert$	$u^*(x,t)$	$\lvert u(x,t)-u^*(x,t)\rvert$
0.1	0.332871	0.332248	$6.232355e-04$	0.280163	$5.270840e-02$	0.299068	$3.380291e-02$
0.2	0.301194	0.299808	$1.386122e-03$	−0.110965	$4.121595e-01$	0.268016	$3.317801e-02$
0.3	0.272532	0.272292	$2.402213e-04$	−0.169539	$4.420708e-01$	0.239650	$3.288184e-02$
0.4	0.246597	0.245878	$7.189648e-04$	−0.184273	$4.308696e-01$	0.213710	$3.288696e-02$
0.5	0.223130	0.222476	$6.544451e-04$	−0.212337	$4.354675e-01$	0.189933	$3.319691e-02$
0.6	0.201897	0.200903	$9.939322e-04$	−0.265374	$4.672701e-01$	0.168071	$3.382540e-02$
0.7	0.182684	0.182831	$1.477558e-04$	−0.372146	$5.548299e-01$	0.147890	$3.479351e-02$
0.8	0.165299	0.164808	$4.907291e-04$	−0.436396	$6.016945e-01$	0.129165	$3.613355e-02$
0.9	0.149569	0.147191	$2.377986e-03$	−0.182506	$3.320751e-01$	0.111688	$3.788021e-02$
1	0.135335	0.135334	$8.531876e-07$	–	–	0.135317	$1.841884e-05$

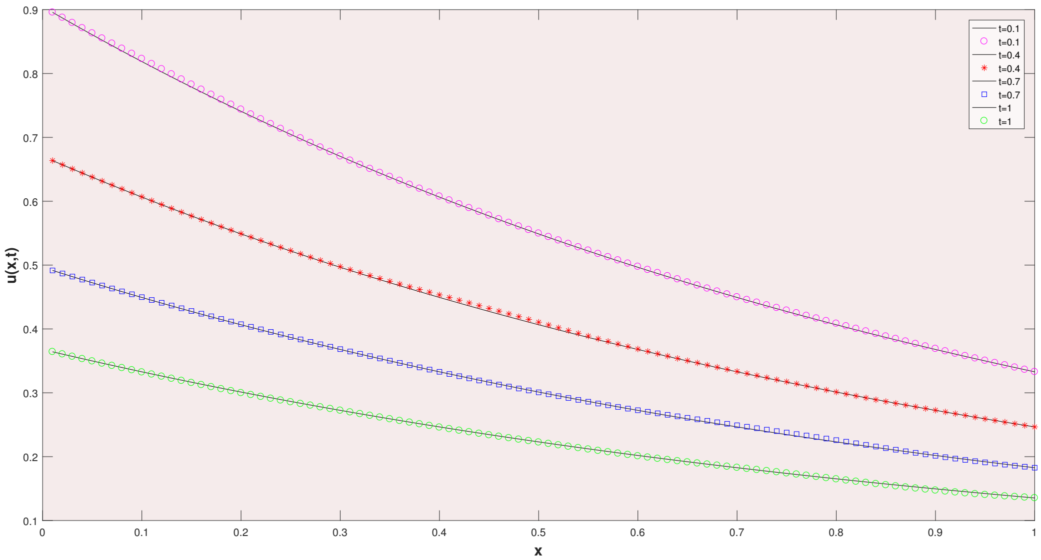

Figure 7: The comparison between the exact solution (shown by continuous lines) and numerical solution of $u(x,t)$ (using cubic B-spline method) at different time levels in Example 6.3.

Example 6.4. In this example, we consider the following one-dimensional inverse problem, for estimating unknown boundary condition $q(t)$.

$$\begin{aligned}
&u_{tt}(x,t) = u_{xx}(x,t), && && 0 < x < 1, \quad 0 < t < 1,\\
&u(x,0) = e^{-x} + x^2, && u_t(x,0) = -e^{-x} + x^3, && 0 \le x \le 1,\\
&u(1,t) = e^{-1-t} + t + t^3 + t^2 + 1, && u(0.01,t) = e^{-0.01-t} + (0.01)^3 t + (0.01)t^3 + t^2 + (0.01)^2, && 0 \le t \le 1,
\end{aligned}$$

The exact solutions of this problem are

$$u(x,t) = e^{-x-t} + x^3 t + x t^3 + t^2 + x^2, \qquad q(t) = e^{-t} + t^2, \qquad 0 \le x \le 1, \qquad 0 \le t \le 1.$$

The numerical results of the unknown boundary condition $u(0,t)$ is reported in Table 12. To clarify the accuracy of the present method, the corresponding graphical illustration is presented in Figure 8. Also, the graphical illustration corresponding to the difference between the exact and numerical solutions of $u(x,t)$ is presented in Figure 9. The obtained numerical solutions for $u(x,t)$ are given in Tables 13 and 16, and the graphical illustrations corresponding to these Tables are presented in Figure 10.

Table 12: The comparison among the exact and numerical solutions for $u(0,t) = q(t)$ in Example 6.4.

t	B-spline			FDM		RBF	
	$q(t)$	$q^*(t)$	$\lvert q(t)-q^*(t)\rvert$	$q^*(t)$	$\lvert q(t)-q^*(t)\rvert$	$q^*(t)$	$\lvert q(t)-q^*(t)\rvert$
0.1	0.914837	0.913971	$8.660856e-04$	0.905848	$8.989653e-03$	0.906065	$8.771927e-03$
0.2	0.858731	0.857784	$9.464724e-04$	0.862863	$4.132155e-03$	0.850898	$7.833081e-03$
0.3	0.830818	0.829813	$1.005385e-03$	0.837482	$6.663429e-03$	0.823968	$6.850699e-03$
0.4	0.830320	0.829341	$9.786953e-04$	0.833696	$3.375686e-03$	0.824564	$5.756086e-03$
0.5	0.856531	0.855589	$9.415711e-04$	0.855497	$1.033765e-03$	0.852049	$4.481289e-03$
0.6	0.908812	0.907811	$1.000581e-03$	0.906877	$1.934758e-03$	0.905853	$2.959027e-03$
0.7	0.986585	0.985672	$9.132446e-04$	0.991827	$5.242119e-03$	0.985463	$1.122617e-03$
0.8	1.089329	1.088401	$9.282853e-04$	1.114340	$2.501131e-02$	1.090423	$1.094077e-03$
0.9	1.216570	1.215638	$9.317507e-04$	1.278407	$6.183750e-02$	1.220326	$3.756695e-03$
1	1.367879	1.366845	$1.034832e-03$	1.488020	$1.201404e-01$	1.374791	$6.911522e-03$
Execution time (second)	65.668541			652.776792		644.512736	

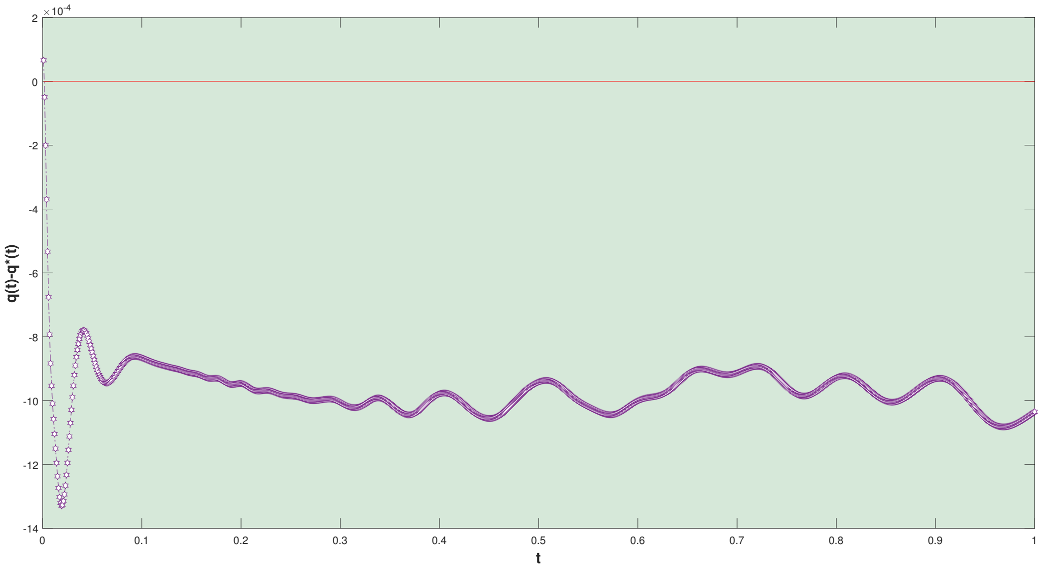

Figure 8: Difference between the exact and numerical solutions of $u(0,t)$ (using cubic B-spline method) at different time levels in Example 6.4.

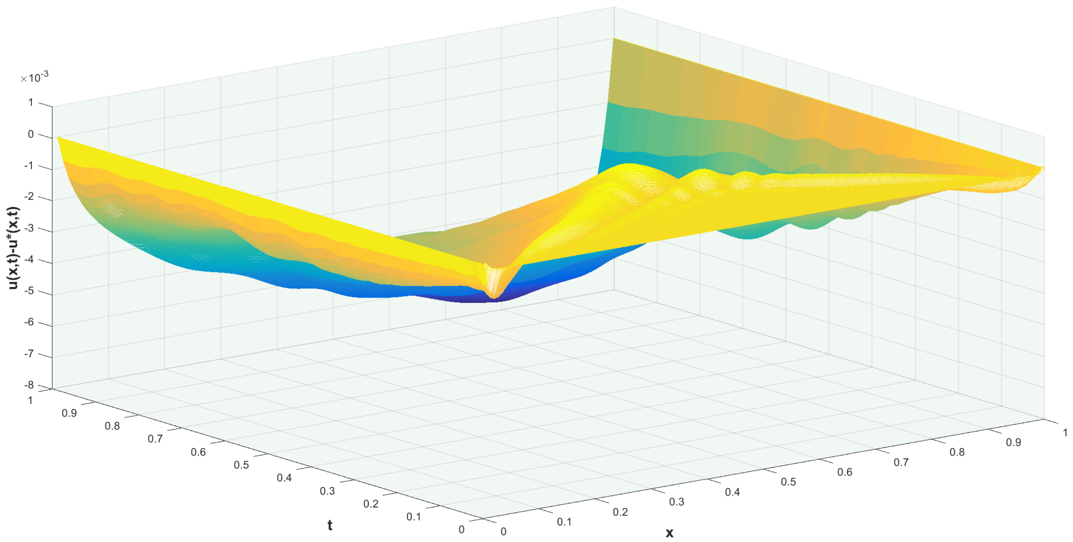

Figure 9: Difference between the exact and numerical solutions of $u(x,t)$ (using cubic B-spline method) in Example 6.4.

Table 13: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t = 0.1$ for Example 6.4.

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$
0.1	0.838931	0.840156	$1.225420e-03$	0.873342	$3.441157e-02$	0.848038	$9.107163e-03$
0.2	0.791818	0.791588	$2.302895e-04$	0.866688	$7.486999e-02$	0.808592	$1.677383e-02$
0.3	0.773320	0.773359	$3.861744e-05$	0.851866	$7.854609e-02$	0.790091	$1.677144e-02$
0.4	0.783331	0.783472	$1.410608e-04$	0.861996	$7.866501e-02$	0.800100	$1.676979e-02$
0.5	0.821812	0.821952	$1.407497e-04$	0.900478	$7.866658e-02$	0.838581	$1.676889e-02$
0.6	0.888785	0.888919	$1.334666e-04$	0.967450	$7.866465e-02$	0.905554	$1.676869e-02$
0.7	0.984329	0.984438	$1.090076e-04$	1.062909	$7.857970e-02$	1.001098	$1.676913e-02$
0.8	1.108570	1.108513	$5.635559e-05$	1.184500	$7.593073e-02$	1.125340	$1.677008e-02$
0.9	1.261679	1.262461	$7.812644e-04$	1.307083	$4.540377e-02$	1.277182	$1.550273e-02$
1	1.443871	1.443863	$7.627782e-06$	–	–	1.443911	$3.955469e-05$

Table 14: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t = 0.4$ for Example 6.4.

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$
0.1	0.783331	0.785713	$2.382181e-03$	0.798268	$1.493698e-02$	0.794257	$1.092677e-02$
0.2	0.764812	0.766876	$2.064653e-03$	0.750160	$1.465174e-02$	0.793605	$2.879333e-02$
0.3	0.776585	0.778378	$1.792780e-03$	0.768238	$8.347337e-03$	0.822263	$4.567737e-02$
0.4	0.820529	0.821439	$9.104828e-04$	0.862080	$4.155134e-02$	0.884966	$6.443663e-02$

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Table 14 – Continued from previous page

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$
0.5	0.898570	0.899019	$4.489820e-04$	0.962535	$6.396549e-02$	0.972430	$7.386084e-02$
0.6	1.012679	1.013050	$3.701450e-04$	1.061237	$4.855784e-02$	1.084234	$7.155453e-02$
0.7	1.164871	1.166479	$1.608034e-03$	1.169318	$4.446830e-03$	1.220897	$5.602584e-02$
0.8	1.357194	1.359391	$2.196918e-03$	1.343395	$1.379966e-02$	1.395993	$3.879876e-02$
0.9	1.591732	1.594924	$3.191949e-03$	1.596546	$4.814360e-03$	1.611969	$2.023716e-02$
1	1.870597	1.870582	$1.543023e-05$	–	–	1.870584	$1.270792e-05$

Table 15: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t = 0.7$ for Example 6.4.

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$
0.1	0.984329	0.986298	$1.968797e-03$	0.983863	$4.663639e-04$	1.001641	$1.731209e-02$
0.2	1.010770	1.013246	$2.476368e-03$	0.993900	$1.686954e-02$	1.049319	$3.854954e-02$
0.3	1.069679	1.072586	$2.906872e-03$	1.057700	$1.197977e-02$	1.125281	$5.560109e-02$
0.4	1.164871	1.168597	$3.725425e-03$	1.104268	$6.060310e-02$	1.224523	$5.965171e-02$
0.5	1.300194	1.304409	$4.214970e-03$	1.180883	$1.193115e-01$	1.361178	$6.098403e-02$
0.6	1.479532	1.484365	$4.833615e-03$	1.398356	$8.117605e-02$	1.539535	$6.000302e-02$
0.7	1.706797	1.711342	$4.545349e-03$	1.681135	$2.566225e-02$	1.765689	$5.889251e-02$
0.8	1.985930	1.990570	$4.639779e-03$	1.970360	$1.556982e-02$	2.034372	$4.844204e-02$
0.9	2.320897	2.324994	$4.097545e-03$	2.320349	$5.472488e-04$	2.346340	$2.544312e-02$
1	2.715684	2.715663	$2.090602e-05$	–	–	2.715655	$2.854397e-05$

Table 16: Comparison among the exact and numerical solutions for $u(x,t)$ at times $t = 1$ for Example 6.4.

	B-spline			FDM		RBF	
x	$u(x,t)$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$	$u^*(x,t)$	$\|u(x,t)-u^*(x,t)\|$
0.1	1.443871	1.447118	$3.247006e-03$	1.455208	$1.133733e-02$	1.450647	$6.776126e-03$
0.2	1.549194	1.553738	$4.543321e-03$	1.479057	$7.013733e-02$	1.563739	$1.454526e-02$
0.3	1.689532	1.694760	$5.227813e-03$	1.479057	$7.013733e-02$	1.709959	$2.042769e-02$
0.4	1.870597	1.876565	$5.968326e-03$	1.790142	$8.045527e-02$	1.894756	$2.415942e-02$
0.5	2.098130	2.104406	$6.275574e-03$	2.023378	$7.475189e-02$	2.123606	$2.547614e-02$
0.6	2.377897	2.384710	$6.813883e-03$	2.312440	$6.545664e-02$	2.402005	$2.410857e-02$
0.7	2.715684	2.723082	$7.398453e-03$	2.633767	$8.191654e-02$	2.735465	$1.978170e-02$
0.8	3.117299	3.124920	$7.620664e-03$	3.017213	$1.000861e-01$	3.129513	$1.221368e-02$
0.9	3.588569	3.595498	$6.929588e-03$	3.529901	$5.866737e-02$	3.589720	$1.151353e-03$
1	4.135335	4.135309	$2.612864e-05$	–	–	4.135291	$4.427479e-05$

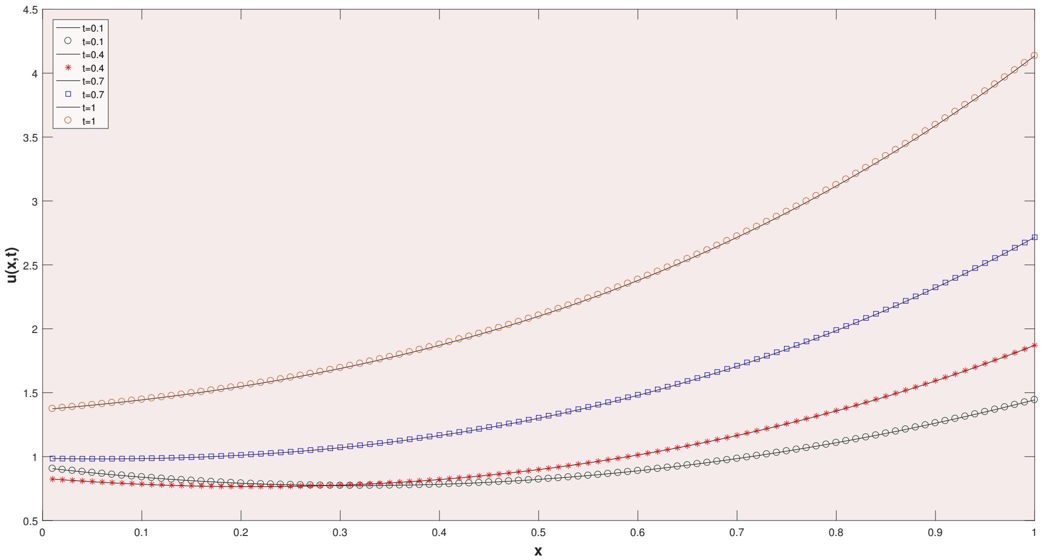

Figure 10: The comparison between the exact solution (shown by continuous lines) and numerical solution of $u(x,t)$ (using cubic B-spline method) at different time levels in Example 6.4.

7 Conclusion

The cubic B-spline method has been employed to estimate unknown boundary condition for the inverse wave problems (1.1). Since the coefficient matrix of the system obtained from interpolating is usually ill-posed, hence to regularize the resultant ill-posed linear system of equations, we have applied the TR method to obtain a stable numerical approximation to the solution. The stability and convergence analysis of the proposed method have been discussed, and shown that the convergence rate of the proposed method is $O\Big((\Delta t)^2+h^2\Big)$. Numerical comparisons have been made between the implementations of the proposed method and finite difference method (FDM) and the radial basis function (RBF) method. The results obtained are quite satisfactory and competent with the solutions available in the literature. The obtained numerical solutions by the presented method is the most accurate in comparison with FDM and RBF method and are in good agreement with the exact solutions. The strong point of the method is its easy and simple computation with low-storage space and cost. Therefore, it follows that the proposed method is efficient and powerful in solving the inverse wave equation.

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