Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 549–562
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.24900.2848

Some new Hermite-Hadamard type inequalities for $p-$convex functions with generalized fractional integral operators

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(Communicated by Choonkil Park)

Abstract

By use of definition of a generalized fractional integral operators, proposed by Raina and Agarwal et. al, we establish a fractional Hermite-Hadamard type inequalities for $p-$convex functions and an identity with a parameter. We derive several parameterized integral inequalities associated with this identity, and provide two examples to illustrate the obtained results.

Keywords: Hermite-Hadamard inequality, Fractional integral operators, $p-$convex functions
2020 MSC: Primary 26D15; Secondary 26A51

1 Introduction

A particularly important mathematical result that's receiving renewed interest is the Hermite-Hadamard inequality, which is the first fundamental result for convex functions and has many applications, with an accessible geometric interpretation. Discovered by C. Hermite and J. Hadamard, inequality for convex functions received attention in the literature, and is paraphrased as follows: [19, p.137]:

$$u\left(\frac{w+k}{2}\right) \leq \frac{1}{k-w}\int_w^k u(\zeta)d\zeta \leq \frac{u(w)+u(k)}{2}, \tag{1.1}$$

provided that $u : \mathring{\mathbf{I}} \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on an interval $\mathring{\mathbf{I}}$ of reals with $w, k \in \mathring{\mathbf{I}}$ defined by:

$$u(\varepsilon\zeta + (1-\varepsilon)\xi) \leq \varepsilon u(\zeta) + (1-\varepsilon)u(\xi) \tag{1.2}$$

for $\zeta, \xi \in \mathring{\mathbf{I}}$ and $\varepsilon \in [0, 1]$. For f, the concave function, the inequalities found in (1.1) hold in the opposite direction. The Hermite-Hadamard inequality is believed to be the most helpful inequality in mathematical analysis. It is clear that this inequality is related to the concept of convexity, and it is easy to obtain from Jensen's inequality. Mathematicians

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Received: October 2021 *Accepted:* April 2022

are attempting to broaden the scope of convex functions by proiding novel modification in (1.2). Over the last two decades, the types of equivariant innovative amendments that have been performed in (1.2) have led to numerous novel theorems, extensions, and generalizations that have in turn stimulated new inequality theorems. There are many novel Hermite-Hadamard inequalities, as well as applications in other disciplines of pure and applied mathematics [5, 6, 7, 10, 12, 14, 16, 17, 24, 25, 27, 28, 29, 30, 31, 32]. The **p**-convex function is one of the generalizations of the convex function. The inequality established by Hadamard for convex functions has been a topic of interest in recent years, and an array of improvements and generalizations has emerged. The papers [3, 4, 15, 18, 23, 26] referenced in the references discovered several notable proofs, expansions, and applications. Our goal in this research is to provide a new parameter to define an identity, based on generalized fractional integrals, and to produce new fractional integral inequalities based on Hermite-Hadamard types. In addition to supporting the findings, examples are presented to back up the validity of the results. This document is laid down in the following manner. In Section 2, some preliminary principles and basics are discussed to prepare the reader for the rest of the article. Section 3 discusses the issue with examples, and a Conclusion section which comprises all constructed results about the topic.

2 Preliminaries and Assumptions

Definition 2.1. [8] Let $\mathring{\mathbf{I}} \subseteq (0,\infty)$ be a real interval and $0 \neq \mathbf{p} \in \mathbf{R}$. A function $u : \mathring{\mathbf{I}} \to \mathbf{R}$ is said to be $\mathbf{p}-$convex function, if

$$u(\sqrt[\mathbf{p}]{\varepsilon\zeta^{\mathbf{p}} + (1-\varepsilon)\xi^{\mathbf{p}}}) \leq \varepsilon u(\zeta) + (1-\varepsilon)u(\xi),$$

provided that $\zeta, \xi \in \mathring{\mathbf{I}}$ and $\varepsilon \in [0,1]$. If the inequality is reversed, the u is said to be $\mathbf{p}-$concave function. It may be observed that for $\mathbf{p} = 1, -1$, respectively, $\mathbf{p}-$convexity reduces to the ordinary convexity and harmonically convexity of u on $\mathring{\mathbf{I}} \subset \mathbf{R}^{+}$ [9]

Definition 2.2. [21] Let $[w,k]$ be a finite interval on the real axis and $u \in [w,k]$. The right-hand side and the left-hand side Riemann-Liouville fractional integrals $\mathcal{J}^{\alpha}_{w+}u$ and $\mathcal{J}^{\alpha}_{k-}u$ of order $\alpha > 0$, respectively, are defined by:

$$\left(\mathcal{J}^{\alpha}_{w+}u\right)(\zeta) = \frac{1}{\Gamma(\alpha)}\int_{w}^{\zeta}(\zeta-\varepsilon)^{\alpha-1}u(\varepsilon)d\varepsilon,\ \zeta > w. \tag{2.1}$$

$$\left(\mathcal{J}^{\alpha}_{k-}u\right)(\zeta) = \frac{1}{\Gamma(\alpha)}\int_{\zeta}^{k}(\varepsilon-\zeta)^{\alpha-1}u(\varepsilon)d\varepsilon,\ \zeta < k. \tag{2.2}$$

Definition 2.3. [21] The gamma function, Γ, beta function, $\mathbb{B}$ and the Hypergeometric function, ${}_2F_1$, respectively, defined by:

$$\Gamma(\zeta) := \int_{0}^{\infty} e^{-\varepsilon}\varepsilon^{\zeta}d\varepsilon,\ \zeta > 0.$$

$$\mathbb{B}(\zeta,\xi) := \frac{\Gamma(\zeta)\Gamma(\xi)}{\Gamma(\zeta+\xi)} = \int_{0}^{1}\varepsilon^{\zeta-1}(1-\varepsilon)^{\xi-1}d\varepsilon,\ \zeta,\xi > 0.$$

$${}_2F_1(w,k;c,z) := \frac{1}{\mathbb{B}(k,c-k)}\int_{0}^{1}\varepsilon^{k-1}(1-\varepsilon)^{c-k-1}(1-z\varepsilon)^{-w}d\varepsilon,\ c > k > 0; |z| < 1.$$

Raina [20] introduced a class of functions defined by:

$$\mathfrak{F}^{\sigma}_{\rho,\rho}(\zeta) = \mathfrak{F}^{\sigma(0),\sigma(1),\cdots}_{\rho,\rho}(\zeta) = \sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\rho)}\zeta^{k},\quad \rho,\rho \in \mathbf{R}^{+}; |\zeta| < \mathbf{R}, \tag{2.3}$$

where the coefficients $\sigma(k) \in \mathbf{R}^{+}$, $k \in \mathbb{N}_0$ form a bounded sequence. By using (2.3) Raina and Agarwal et al. [1, 20] defined, respectively, the left-side and right-sided fractional integral operators:

$$\left(\mathfrak{J}^{\sigma}_{\rho,\rho,w+;w}\phi\right)(\zeta) = \int_{w}^{\zeta}(\zeta-\varepsilon)^{\rho-1}\mathfrak{F}^{\sigma}_{\rho,\rho}[w(\zeta-\varepsilon)^{\rho}]\phi(\varepsilon)d\varepsilon,\ \zeta > w. \tag{2.4}$$

$$\left(\mathfrak{J}^{\sigma}_{\rho,\rho,k-;w}\phi\right)(\zeta) = \int_{\zeta}^{k}(\varepsilon-\zeta)^{\rho-1}\mathfrak{F}^{\sigma}_{\rho,\rho}[w(\varepsilon-\zeta)^{\rho}]\phi(\varepsilon)d\varepsilon,\ \zeta < k, \tag{2.5}$$

where $w \in \mathbf{R}$ and ϕ is a function such that the integrals on right hand sides exit. It is easy to verify that $\mathfrak{J}^{\sigma}_{\rho,\rho,w+;w}\phi(\zeta)$ and $\mathfrak{J}^{\sigma}_{\rho,\rho,k-;w}\phi(\zeta)$ are bounded integral operators on $L(w,k)$, provided that $\mathfrak{M} := \mathfrak{F}^{\sigma}_{\rho,\rho+1}[w(k-w)^{\rho}] < \infty$. In fact, for $\phi \in L(w,k)$, we have

$$\left\|\mathfrak{J}^{\sigma}_{\rho,\rho,w+;w}\phi\right\|_1 \leq \mathfrak{M}(k-w)^{\rho}\|\phi\|_1; \quad \left\|\mathfrak{J}^{\sigma}_{\rho,\rho,k-;w}\phi\right\|_1 \leq \mathfrak{M}(k-w)^{\rho}\|\phi\|_1$$

By setting $\rho \to \alpha$; $\sigma(0) \to 1$ and $w \to 0$ in (2.4) and (2.5), respectively, (2.1) and (2.2) are recaptured. Some Hermite-Hadamard type inequalities for generalized fractional integral operators have been proved as follows:

Theorem 2.4. [22] Let $u : [w;k] \to \mathbf{R}$ be a function with $0 \leq w < k$ and $u \in L_1[w,k]$. If u is an $s-$convex function on $[w,k]$ then we have the following inequalities for generalized fractional integral operators

$$\begin{aligned} 2^s u\left(\frac{w+k}{2}\right) &\leq \frac{1}{(k-w)^{\rho}.\mathfrak{F}^{\sigma}_{\rho,\rho}[w(k-w)^{\rho}]}\left[\left(\mathfrak{J}^{\sigma}_{\rho,\rho,k-;w}u\right)(w) + \left(\mathfrak{J}^{\sigma}_{\rho,\rho,w+;w}u\right)(k)\right] \\ &\leq \frac{u(w)+u(k)}{\mathfrak{F}^{\sigma}_{\rho,\rho+1}[w(k-w)^{\rho}]}[A_1(\rho,s) + \mathfrak{F}^{\sigma_{0;s}}_{\rho,\rho}[w(k-w)^{\rho}]], \end{aligned}$$

provided that $A_1(\rho,s)$ and $\sigma_{0,s}(k)$ are defined by (2.6).

$$A_1(\rho,s) = \int_0^1 \varepsilon^{\rho-1}(1-\varepsilon)^s.\mathfrak{F}^{\sigma}_{\rho,\rho}[w(k-w)^{\rho}\varepsilon^{\rho}]d\varepsilon; \quad \sigma_{0,s}(k) = \frac{\sigma(k)}{\rho+\rho k+s}, \ k \in \mathbb{N}_0. \tag{2.6}$$

Theorem 2.5. [21] Let $\phi : \mathring{\mathbf{I}} \subseteq \mathbf{R} \to \mathbf{R}$ be a $\mathbf{p}-$convex function such that $w,k \in \mathring{\mathbf{I}}$ with $w < k$, $\rho \in \mathbf{R}^+$ and $g(\zeta) = \sqrt[\mathbf{p}]{\zeta}$, then

$$\begin{aligned} \phi\left(\sqrt[\mathbf{p}]{\frac{w^{\mathbf{p}}+k^{\mathbf{p}}}{2}}\right) &\leq \frac{\left(\mathfrak{J}^{\sigma}_{\rho,\rho,k^{\mathbf{p}}-;w}\phi\circ g\right)(w^{\mathbf{p}}) + \left(\mathfrak{J}^{\sigma}_{\rho,\rho,w^{\mathbf{p}}+;w}\phi\circ g\right)(k^{\mathbf{p}})}{2\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}.\mathfrak{F}^{\sigma}_{\rho,\rho}[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}]} \\ &\leq \frac{\phi(w)+\phi(k)}{2}. \end{aligned}$$

Before starting our main results in Section 3, we discus some assumptions.

$$\sigma_1 := \sigma(k)\frac{{}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}}, \boldsymbol{\vartheta}+\rho k+2; \boldsymbol{\vartheta}+\rho k+4, \frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right)(1-\rho)(k^{\mathbf{p}}-w^{\mathbf{p}})}{2\mathbf{p}.w^{\mathbf{p}-1}(\boldsymbol{\vartheta}+\rho k+2)(\boldsymbol{\vartheta}+\rho k+3)\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho}\right]} \tag{2.7}$$

$$\begin{aligned} \sigma_2 := \sigma(k)\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)\Bigg\{ &\frac{(1-\rho){}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}}, \boldsymbol{\vartheta}+\rho k+2; \boldsymbol{\vartheta}+\rho k+3, \frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right)}{2\mathbf{p}w^{\mathbf{p}-1}(\boldsymbol{\vartheta}+\rho k+2)\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho}\right]} \\ &+\frac{\rho\left[\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right]^{1-\mathbf{p}}{}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}}, 1; \boldsymbol{\vartheta}+\rho k+3, \frac{\rho(w^{\mathbf{p}}-k^{\mathbf{p}})}{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)}{2\mathbf{p}(\boldsymbol{\vartheta}+\rho k+2)\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho}\right]}\Bigg\} \end{aligned} \tag{2.8}$$

$$\sigma_3 := \frac{\sigma(k)\rho\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right){}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}}, 2; \boldsymbol{\vartheta}+\rho k+3, \frac{\rho(w^{\mathbf{p}}-k^{\mathbf{p}})}{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)}{2\mathbf{p}\left[\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right]^{\mathbf{p}-1}(\boldsymbol{\vartheta}+\rho k+2)(\boldsymbol{\vartheta}+\rho k+1)\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho}\right]} \tag{2.9}$$

$$\begin{aligned} \sigma_4 := &\frac{\sigma(k)(1-\rho)\sqrt[k]{\mathbb{B}(1+k\boldsymbol{\vartheta}+\rho kk,1)}\sqrt[s]{\frac{(1-\rho)\left|u'\right|^s+(1+\rho)\left|u'\right|^s}{2}}}{2\mathbf{p}.w^{k(\mathbf{p}-1)}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]} \\ &\times\sqrt[k]{{}_2F_1\left(k\frac{\mathbf{p}-1}{\mathbf{p}}, k\boldsymbol{\vartheta}+k\rho k+1; k\boldsymbol{\vartheta}+k\rho k+2, \frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right)} \end{aligned} \tag{2.10}$$

$$\sigma_5 := \frac{\rho\sigma(k)\sqrt[k]{\mathbb{B}(1,1+k\boldsymbol{\vartheta}+\rho kk)}\sqrt[s]{\frac{2|u'|^s+\rho(|u'|^s-|u'|^s)}{2}}}{2\mathbf{p}.\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]\left[\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right]^{k(\mathbf{p}-1)}} \times \sqrt[k]{{}_2F_1\left(k\frac{\mathbf{p}-1}{\mathbf{p}},1;k\boldsymbol{\vartheta}+k\rho k+2,\frac{\rho(w^{\mathbf{p}}-k^{\mathbf{p}})}{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)} \quad (2.11)$$

$$\sigma_6 := \frac{(1-\rho)\sigma(k)}{2\mathbf{p}w^{\mathbf{p}-1}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]} \times \left\{\mathbb{B}(1+\boldsymbol{\vartheta}+\rho k,1){}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}},\boldsymbol{\vartheta}+\rho k+1;\boldsymbol{\vartheta}+\rho k+2,\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right)\right\}^{\frac{k-1}{k}}$$
$$\times\left\{(1-\rho)(\left|u'\right|^k-\left|u'\right|^k)\mathbb{B}(2+\boldsymbol{\vartheta}+\rho k,1)\times{}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}},\boldsymbol{\vartheta}+\rho k+2;\boldsymbol{\vartheta}+\rho k+3,\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right)+\left|u'\right|^k\mathbb{B}(1+\boldsymbol{\vartheta}+\rho k,1)\right.$$
$$\left.{}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}},\boldsymbol{\vartheta}+\rho k+1;\boldsymbol{\vartheta}+\rho k+2,\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right)\right\}^{\frac{1}{k}} \quad (2.12)$$

$$\sigma_7 := \frac{\rho\sigma(k)}{2\mathbf{p}.\left[\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right]^{\frac{\mathbf{p}-1}{\mathbf{p}}}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]}\left\{\mathbb{B}(1,1+\boldsymbol{\vartheta}+\rho k){}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}},1;\boldsymbol{\vartheta}+\rho k+2,\frac{\rho(w^{\mathbf{p}}-k^{\mathbf{p}})}{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)\right\}^{\frac{k-1}{k}}$$
$$\times\left\{[\rho\left|u'\right|^k+(1-\rho)\left|u'\right|^k]\mathbb{B}(1,2+\boldsymbol{\vartheta}+\rho k)\times{}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}},1;\boldsymbol{\vartheta}+\rho k+3,\frac{\rho(w^{\mathbf{p}}-k^{\mathbf{p}})}{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)+\left|u'\right|^k\mathbb{B}(2,1+\boldsymbol{\vartheta}+\rho k)\right.$$
$$\left.\times{}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}},2;\boldsymbol{\vartheta}+\rho k+3,\frac{\rho(w^{\mathbf{p}}-k^{\mathbf{p}})}{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)\right\}^{\frac{1}{k}} \quad (2.13)$$

$$\Sigma_1 := \sigma(k)\frac{{}_2F_1\left(0,\boldsymbol{\vartheta}+\rho k+2;\boldsymbol{\vartheta}+\rho k+4,\frac{w-k}{2w}\right)(k-w)}{4(\boldsymbol{\vartheta}+\rho k+2)(\boldsymbol{\vartheta}+\rho k+3)\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|(k-w)^{\rho}\right]} \quad (2.14)$$

$$\Sigma_2 := \sigma(k)\,(k-w)\,\frac{{}_2F_1\left(0,\boldsymbol{\vartheta}+\rho k+2;\boldsymbol{\vartheta}+\rho k+3,\frac{w-k}{2w}\right)+{}_2F_1\left(0,1;\boldsymbol{\vartheta}+\rho k+3,\frac{w-k}{w+k}\right)}{4(\boldsymbol{\vartheta}+\rho k+2)\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|(k-w)^{\rho}\right]} \quad (2.15)$$

$$\Sigma_3 := \sigma(k)\frac{(k-w)\,{}_2F_1\left(0,2;\boldsymbol{\vartheta}+\rho k+3,\frac{w-k}{w+k}\right)}{4(\boldsymbol{\vartheta}+\rho k+2)(\boldsymbol{\vartheta}+\rho k+1)\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|(k-w)^{\rho}\right]} \quad (2.16)$$

$$\Sigma_4 := \frac{\sigma(k)\sqrt[k]{\mathbb{B}(1+k\boldsymbol{\vartheta}+\rho kk,1)}\sqrt[s]{|u'|^s+3|u'|^s}}{\sqrt[s]{2^{s+2}}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k-w\right)^{\rho}\right]}\times\sqrt[k]{{}_2F_1\left(0,k\boldsymbol{\vartheta}+k\rho k+1;k\boldsymbol{\vartheta}+k\rho k+2,\frac{w-k}{2w}\right)} \quad (2.17)$$

$$\Sigma_5 := \frac{\sigma(k)\sqrt[k]{\mathbb{B}(1,1+k\boldsymbol{\vartheta}+\rho kk)}\sqrt[s]{3|u'|^s+|u'|^s}}{\sqrt[s]{4^{s+1}}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k-w\right)^{\rho}\right]}\times\sqrt[k]{{}_2F_1\left(0,1;k\boldsymbol{\vartheta}+k\rho k+2,\frac{w-k}{w+k}\right)} \quad (2.18)$$

$$\Delta_1 := \frac{{}_2F_1\left(0,\alpha+2;\alpha+4,\frac{w-k}{2w}\right)(k-w)\Gamma(\alpha+1)}{4(\alpha+2)(\alpha+3)} \quad (2.19)$$

$$\Delta_2 := \frac{(k-w)\,\Gamma(\alpha+1)\left[{}_2F_1\left(0,\alpha+2;\alpha+3,\frac{w-k}{2w}\right)+{}_2F_1\left(0,1;\alpha+3,\frac{w-k}{w+k}\right)\right]}{4(\alpha+2)} \quad (2.20)$$

$$\Delta_3 := \frac{(k-w)\,\Gamma(\alpha+1){}_2F_1\left(0,2;\alpha+3,\frac{w-k}{w+k}\right)}{4(\alpha+2)(\alpha+1)} \quad (2.21)$$

$$\Delta_4 := \frac{\Gamma(\alpha+1)\sqrt[s]{|u'|^s+3|u'|^s}\sqrt[k]{{}_2F_1\left(0,k\alpha+1;k\alpha+2,\frac{w-k}{2w}\right)\mathbb{B}(1+k\alpha,1)}}{\sqrt[s]{2^{s+2}}} \quad (2.22)$$

$$\Delta_5 := \frac{\Gamma(\alpha+1)\sqrt[s]{3|u'|^s+|u'|^s}\sqrt[k]{{}_2F_1\left(0,1;k\alpha+2,\frac{w-k}{w+k}\right)\mathbb{B}(1,1+k\alpha)}}{\sqrt[s]{4^{s+1}}} \quad (2.23)$$

3 Main Results

The main results which we prove in this section depend on the following lemma.

Lemma 3.1. Let $u : \mathring{\mathbf{I}} \subseteq \mathbf{R}^+ \to \mathbf{R}$ a differentiable function on $\mathring{\mathbf{I}}$, interior of $\mathring{\mathbf{I}}$, $w, k \in \mathring{\mathbf{I}}$ with $w < k$; $\mathbf{p}, \rho, \boldsymbol{\vartheta} > 0$; let $g(\zeta) = \sqrt[\mathbf{p}]{\zeta}$, $\zeta > 0$; $w \in \mathbf{R}$ and $\rho \in (0,1)$, then

$$\Omega(u,w,k) := u\left((\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right) - \frac{1}{2\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}\right]} \times$$
$$\left[\frac{\left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},[\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}]-;\frac{w}{(1-\rho)^{\rho}}} u\circ g\right)(w^{\mathbf{p}})}{[(1-\rho)(k^{\mathbf{p}} - w^{\mathbf{p}})]^{\boldsymbol{\vartheta}}} + \frac{\left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},[\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}]+;\frac{w}{\rho^{\rho}}} u\circ g\right)(k^{\mathbf{p}})}{[\rho(k^{\mathbf{p}} - w^{\mathbf{p}})]^{\boldsymbol{\vartheta}}}\right]$$
$$= \frac{k^{\mathbf{p}} - w^{\mathbf{p}}}{2\mathbf{p}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}\right]}\left[(1-\rho)\left\{\int_0^1 \varepsilon^{\boldsymbol{\vartheta}}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}\varepsilon^{\rho}\right]\right.\right.$$
$$\times\, u'\left((\varepsilon\left(\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}\right) + (1-\varepsilon)w^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right)\left[\varepsilon\left(\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}\right) + (1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\Big\}$$
$$+\rho\left\{\int_0^1 (1-\varepsilon)^{\boldsymbol{\vartheta}}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}(1-\varepsilon)^{\rho}\right] u'\left(((1-\varepsilon)\left(\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}\right) + \varepsilon k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right)\right.$$
$$\left.\left.\times\, \left[(1-\varepsilon)\left(\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}\right) + \varepsilon k^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\right\}\right] \quad (3.1)$$

Proof . Integrating by parts

$$\mathring{\mathbf{I}}_1 := \int_0^1 \varepsilon^{\boldsymbol{\vartheta}}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}\varepsilon^{\rho}\right] u'\left((\varepsilon\left(\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}\right) + (1-\varepsilon)w^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right)$$
$$\times\left[\varepsilon\left(\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}\right) + (1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon$$
$$= \left|\frac{\mathbf{p}.\varepsilon^{\boldsymbol{\vartheta}}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}\varepsilon^{\rho}\right]}{(1-\rho)(k^{\mathbf{p}} - w^{\mathbf{p}})} u\left((\varepsilon\left(\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}\right) + (1-\varepsilon)w^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right)\right|_0^1$$
$$-\int_0^1 \frac{\mathbf{p}.\varepsilon^{\boldsymbol{\vartheta}-1}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}}\left[w\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}\varepsilon^{\rho}\right]}{(1-\rho)(k^{\mathbf{p}} - w^{\mathbf{p}})} u\left((\varepsilon\left(\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}\right) + (1-\varepsilon)w^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right) d\varepsilon$$
$$= \frac{\mathbf{p}.\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}\right]}{(1-\rho)(k^{\mathbf{p}} - w^{\mathbf{p}})} u\left((\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right)$$
$$-\int_0^1 \frac{\mathbf{p}.\varepsilon^{\boldsymbol{\vartheta}-1}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}}\left[w\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}\varepsilon^{\rho}\right]}{(1-\rho)(k^{\mathbf{p}} - w^{\mathbf{p}})} u\left((\varepsilon\left(\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}\right) + (1-\varepsilon)w^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right) d\varepsilon,$$

setting $\zeta \to \varepsilon\left(\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}\right) + (1-\varepsilon)w^{\mathbf{p}}$ so that, $d\zeta = (1-\rho)(k^{\mathbf{p}} - w^{\mathbf{p}})d\varepsilon$ and $0 \le \varepsilon \le 1 \Leftrightarrow w^{\mathbf{p}} \le \zeta \le \rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}$, we have

$$\mathring{\mathbf{I}}_1 = \frac{\mathbf{p}.\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}\right]}{(1-\rho)(k^{\mathbf{p}} - w^{\mathbf{p}})} u\left((\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right) - \int_{w^{\mathbf{p}}}^{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}} \frac{\mathbf{p}\left[\zeta - w^{\mathbf{p}}\right]^{\boldsymbol{\vartheta}-1}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}}\left[\frac{w(\zeta - w^{\mathbf{p}})^{\rho}}{(1-\rho)^{\rho}}\right]}{[(1-\rho)(k^{\mathbf{p}} - w^{\mathbf{p}})]^{1+\boldsymbol{\vartheta}}}(u\circ g)(\zeta)d\zeta.$$

$$\Rightarrow \frac{[(1-\rho)(k^{\mathbf{p}} - w^{\mathbf{p}})]^{1+\boldsymbol{\vartheta}}}{\mathbf{p}}\mathring{\mathbf{I}}_1 = [(1-\rho)(k^{\mathbf{p}} - w^{\mathbf{p}})]^{\boldsymbol{\vartheta}} u\left((\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right)$$
$$\times\, \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}\right] - \left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},[\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}]-;\frac{w}{(1-\rho)^{\rho}}} u\circ g\right)(w^{\mathbf{p}}). \quad (3.2)$$

Again integrating by parts

$$\begin{aligned}
\mathring{\mathbf{I}}_2 :=& \int_0^1 \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}(1-\varepsilon)^{\rho}\right] u'\left(((1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right) \\
& \times(1-\varepsilon)^{\boldsymbol{\vartheta}}\left[(1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon k^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon \\
& \left.\frac{\mathbf{p}(1-\varepsilon)^{\boldsymbol{\vartheta}}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}(1-\varepsilon)^{\rho}\right]}{\rho(k^{\mathbf{p}}-w^{\mathbf{p}})} u\left(((1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right)\right|_0^1 \\
& +\int_0^1 \frac{\mathbf{p}(1-\varepsilon)^{\boldsymbol{\vartheta}-1}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}(1-\varepsilon)^{\rho}\right]}{\rho(k^{\mathbf{p}}-w^{\mathbf{p}})} u\left(((1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right) d\varepsilon \\
=& -\frac{\mathbf{p}.\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]}{\rho(k^{\mathbf{p}}-w^{\mathbf{p}})} u\left((\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right)+\int_0^1 \frac{\mathbf{p}(1-\varepsilon)^{\boldsymbol{\vartheta}-1}}{\rho(k^{\mathbf{p}}-w^{\mathbf{p}})} \\
& \times\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}(1-\varepsilon)^{\rho}\right] u\left(((1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right) d\varepsilon,
\end{aligned}$$

setting $\xi \to \varepsilon k^{\mathbf{p}}+(1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)$ so that, $d\xi=\rho(k^{\mathbf{p}}-w^{\mathbf{p}})d\varepsilon$ and $0\le\varepsilon\le 1 \Leftrightarrow \rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\le\xi\le k^{\mathbf{p}}$, we have

$$\mathring{\mathbf{I}}_2 = -\frac{\mathbf{p}.\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]}{\rho(k^{\mathbf{p}}-w^{\mathbf{p}})} u\left((\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right)+\int_{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}^{k^{\mathbf{p}}} \frac{\mathbf{p}\left[k^{\mathbf{p}}-\xi\right]^{\boldsymbol{\vartheta}-1}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}}\left[\frac{w(k^{\mathbf{p}}-\xi)^{\rho}}{\rho^{\rho}}\right]}{[\rho(k^{\mathbf{p}}-w^{\mathbf{p}})]^{1+\boldsymbol{\vartheta}}}(u\circ g)(\xi)d\xi.$$

$$\begin{aligned}
\Rightarrow \frac{[\rho(k^{\mathbf{p}}-w^{\mathbf{p}})]^{1+\boldsymbol{\vartheta}}}{\mathbf{p}}\mathring{\mathbf{I}}_2 =& -[\rho(k^{\mathbf{p}}-w^{\mathbf{p}})]^{\boldsymbol{\vartheta}} u\left((\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right) \\
& \times\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]+\left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},[\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}]+;\frac{w}{\rho^{\rho}}} u\circ g\right)(k^{\mathbf{p}}). \quad (3.3)
\end{aligned}$$

Substraction of (3.2) and (3.3) yields the desired result (3.1). □

Remark 3.2.

- On letting $\rho\to\frac{1}{2}$; $\mathbf{p},\boldsymbol{\vartheta}\to 1$; $w=0$; $\sigma(0)=1$ Lemma 3.1 coincides with [11, Theorem 1]
- For $\mathbf{p},\boldsymbol{\vartheta},\sigma(0)\to 1$; $w,=0$ Lemma 3.1 coincides with [2, Lemma 2.1].

Theorem 3.3. Let $u:\mathring{\mathbf{I}}\subseteq\mathbf{R}^+\to\mathbf{R}$ a differentiable function on $\mathring{\mathbf{I}}$, interior of $\mathring{\mathbf{I}}$, $w,k\in\mathring{\mathbf{I}}$ with $w<k$ such that $|u'|$ is $\mathbf{p}-$convex for $\mathbf{p},\rho,\boldsymbol{\vartheta}>0$; let $g(\zeta)=\sqrt[\mathbf{p}]{\zeta}$, $\zeta>0$; $w\in\mathbf{R}$ and $\rho\in(0,1)$, then

$$\begin{aligned}
&\Bigg|u\left((\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right)-\frac{1}{2\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]}\times\Bigg[\frac{\left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},[\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}]-;\frac{w}{(1-\rho)^{\rho}}} u\circ g\right)(w^{\mathbf{p}})}{[(1-\rho)(k^{\mathbf{p}}-w^{\mathbf{p}})]^{\boldsymbol{\vartheta}}} \\
&+\frac{\left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},[\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}]+;\frac{w}{\rho^{\rho}}} u\circ g\right)(k^{\mathbf{p}})}{[\rho(k^{\mathbf{p}}-w^{\mathbf{p}})]^{\boldsymbol{\vartheta}}}\Bigg]\Bigg| \qquad (3.4) \\
&\le |u'(w)|\,\mathfrak{F}^{\sigma_1}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]+\left|u'\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)\right|\mathfrak{F}^{\sigma_2}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]+|u'(k)|\,\mathfrak{F}^{\sigma_3}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right], \\
&\le |u'(w)|\left\{\mathfrak{F}^{\sigma_1}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]+\rho\mathfrak{F}^{\sigma_2}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]\right\} \\
&+|u'(k)|\left\{\mathfrak{F}^{\sigma_3}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]+(1-\rho)\mathfrak{F}^{\sigma_2}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]\right\} \qquad (3.5)
\end{aligned}$$

provided that σ_1, σ_2 and σ_3 are defined by, respectively, (2.7), (2.8) and (2.9).

Proof . By relation (3.1), using the property of absolute, the following holds:

$$|\Omega(u,w,k)|\le\frac{(k^{\mathbf{p}}-w^{\mathbf{p}})[(1-\rho)|\mathring{\mathbf{I}}_1|+\rho|\mathring{\mathbf{I}}_2|]}{2\mathbf{p}.\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]} \qquad (3.6)$$

By $\mathbf{p}-$convexity of $|u'|$

$$
\begin{aligned}
|\mathring{\mathbf{I}}_1| =& \left|\int_0^1 \varepsilon^{\boldsymbol{\vartheta}} \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\varepsilon^{\rho}\right] u'\left(\left(\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\left[\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\right| \\
\leq& \int_0^1 \varepsilon^{\boldsymbol{\vartheta}} \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\varepsilon^{\rho}\right] \left|u'\left(\left(\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right|\left[\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon \\
=& \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\int_0^1 \varepsilon^{\boldsymbol{\vartheta}+\rho k}\left[\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} \\
& \left|u'\left(\left(\varepsilon\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)^{\mathbf{p}}+(1-\varepsilon)w^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right| d\varepsilon \\
\leq& \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{w^{\mathbf{p}-1}\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\int_0^1 \varepsilon^{\boldsymbol{\vartheta}+\rho k}\left[1-\varepsilon\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}}\left\{\varepsilon\left|u'\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)\right|+(1-\varepsilon)|u'(w)|\right\} d\varepsilon \\
=& \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{w^{\mathbf{p}-1}\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\left\{\int_0^1 \varepsilon^{\boldsymbol{\vartheta}+\rho k+1}\left[1-\varepsilon\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}}\left|u'\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)\right| d\varepsilon\right. \\
& \left.+|u'(w)|\int_0^1 \varepsilon^{\boldsymbol{\vartheta}+\rho k+1}(1-\varepsilon)\left[1-\varepsilon\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\right\} \\
=& \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{w^{\mathbf{p}-1}\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\left\{\mathbb{B}(\boldsymbol{\vartheta}+\rho k+2,1)\left|u'\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)\right|\right. \\
& \times {}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}},\boldsymbol{\vartheta}+\rho k+2;\boldsymbol{\vartheta}+\rho k+3,\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right) \\
& \left.+\mathbb{B}(\boldsymbol{\vartheta}+\rho k+2,2)\left|u'(w)\right| \times {}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}},\boldsymbol{\vartheta}+\rho k+2;\boldsymbol{\vartheta}+\rho k+4,\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right)\right\}. \qquad (3.7)
\end{aligned}
$$

Again by $\mathbf{p}-$convexity of $|u'|$

$$
\begin{aligned}
|\mathring{\mathbf{I}}_2| =& \left|\int_0^1 \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}(1-\varepsilon)^{\rho}\right] u'\left(\left((1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right. \\
& \left.\times (1-\varepsilon)^{\boldsymbol{\vartheta}}\left[(1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\right| \\
\leq& \int_0^1 \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}(1-\varepsilon)^{\rho}\right]\left|u'\left(\left((1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right| \\
& \times (1-\varepsilon)^{\boldsymbol{\vartheta}}\left[(1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon \\
=& \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\int_0^1 (1-\varepsilon)^{\boldsymbol{\vartheta}+\rho k}\left[(1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} \\
& \times\left|u'\left(\left((1-\varepsilon)\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)^{\mathbf{p}}+\varepsilon.k^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right| d\varepsilon \\
\leq& \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{\left[\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right]^{\mathbf{p}-1}\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\int_0^1\left[1-\varepsilon\rho\frac{w^{\mathbf{p}}-k^{\mathbf{p}}}{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} \\
& \times (1-\varepsilon)^{\boldsymbol{\vartheta}+\rho k}\left\{(1-\varepsilon)\left|u'\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)\right|+\varepsilon|u'(k)|\right\} d\varepsilon \\
=& \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{\left[\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right]^{\mathbf{p}-1}\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\left\{\int_0^1 (1-\varepsilon)^{\boldsymbol{\vartheta}+\rho k+1}\right. \\
& \left.\times\left[1-\varepsilon\rho\frac{w^{\mathbf{p}}-k^{\mathbf{p}}}{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}}\left|u'\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)\right| d\varepsilon+|u'(k)|\int_0^1 \varepsilon(1-\varepsilon)^{\boldsymbol{\vartheta}+\rho k}\left[1-\varepsilon\rho\frac{w^{\mathbf{p}}-k^{\mathbf{p}}}{w^{\mathbf{p}}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\right\}
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{\left[\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right]^{\mathbf{p}-1}\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\left\{\mathbb{B}(1,\boldsymbol{\vartheta}+\rho k+2)\right.\\
&\times\left|u'\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)\right|{}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}},1;\boldsymbol{\vartheta}+\rho k+3,\rho\frac{w^{\mathbf{p}}-k^{\mathbf{p}}}{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)\\
&\left.+\mathbb{B}(2,\boldsymbol{\vartheta}+\rho k+1)\left|u'(k)\right|\times{}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}},2;\boldsymbol{\vartheta}+\rho k+3,\rho\frac{w^{\mathbf{p}}-k^{\mathbf{p}}}{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)\right\}
\end{aligned}
\tag{3.8}
$$

Combining the inequalities (3.6)-(3.8) yields the desired inequality (3.4) □

Corollary 3.4. Let $u:\mathring{\mathbf{I}}\subseteq\mathbf{R}^{+}\to\mathbf{R}$ a differentiable function on $\mathring{\mathbf{I}}$, $w,k\in\mathring{\mathbf{I}}$ with $w<k$ such that $|u'|$ is convex and $\rho,\boldsymbol{\vartheta}>0$, $w\in\mathbf{R}$, then

$$
\begin{aligned}
&\left|u\left(\frac{w+k}{2}\right)-\frac{\left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},\frac{w+k}{2}-;2^{\rho}w}\right)u(w)+\left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},\frac{w+k}{2}+;2^{\rho}w}\right)u(k)}{2^{1-\boldsymbol{\vartheta}}(k-w)^{\boldsymbol{\vartheta}}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k-w\right)^{\rho}\right]}\right|\\
&\leq\left|u'(w)\right|\mathfrak{F}^{\Sigma_1}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k-w\right)^{\rho}\right]+\left|u'\left(\frac{w+k}{2}\right)\right|\mathfrak{F}^{\Sigma_2}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k-w\right)^{\rho}\right]+\left|u'(k)\right|\mathfrak{F}^{\Sigma_3}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k-w\right)^{\rho}\right]\\
&\leq\frac{2\mathfrak{F}^{\Sigma_1}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k-w\right)^{\rho}\right]+\mathfrak{F}^{\Sigma_2}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k-w\right)^{\rho}\right]}{2}\left|u'(w)\right|+\frac{2\mathfrak{F}^{\Sigma_3}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k-w\right)^{\rho}\right]+\mathfrak{F}^{\Sigma_2}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k-w\right)^{\rho}\right]}{2}\left|u'(k)\right|
\end{aligned}
\tag{3.9}
$$

provided that Σ_1, Σ_2 and Σ_3 are defined by, respectively, (2.14), (2.15) and (2.16).

Proof . The proof directly follows from Theorem 3.3 for $\mathbf{p}=1$, $\rho=\frac{1}{2}$. □

Corollary 3.5. Let $u:\mathring{\mathbf{I}}\subseteq\mathbf{R}^{+}\to\mathbf{R}$ a differentiable function on $\mathring{\mathbf{I}}$, $w,k\in\mathring{\mathbf{I}}$ with $w<k$ such that $|u'|$ is convex and $\alpha>0$, then

$$
\begin{aligned}
\left|u\left(\frac{w+k}{2}\right)-\Gamma(\alpha+1)\frac{\left(\mathcal{J}^{\alpha}_{\frac{w+k}{2}-}u\right)(w)+\left(\mathcal{J}^{\alpha}_{\frac{w+k}{2}+}u\right)(k)}{2^{1-\alpha}(k-w)^{\alpha}}\right|&\leq\left|u'(w)\right|\Delta_1+\left|u'(k)\right|\Delta_3+\left|u'\left(\frac{w+k}{2}\right)\right|\Delta_2\\
&\leq\frac{\left|u'(w)\right|\left[2\Delta_1+\Delta_2\right]+\left|u'(k)\right|\left[2\Delta_3+\Delta_2\right]}{2}
\end{aligned}
\tag{3.10}
$$

provided that Δ_1, Δ_2 and Δ_3 are defined by, respectively, (2.19), (2.20) and (2.21).

Proof . The proof directly follows from Corollary 3.4 for $w=0$, $\sigma(0)=1$, $\boldsymbol{\vartheta}=\alpha$ □

Theorem 3.6. Let $u:\mathring{\mathbf{I}}\subseteq\mathbf{R}^{+}\to\mathbf{R}$ a differentiable function on $\mathring{\mathbf{I}}$, interior of $\mathring{\mathbf{I}}$, $w,k\in\mathring{\mathbf{I}}$ with $w<k$ such that $\left|u'\right|^{s}$ is $\mathbf{p}-$convex for $\mathbf{p},\rho,\boldsymbol{\vartheta}>0$; let $g(\zeta)=\sqrt[\mathbf{p}]{\zeta}$, $\zeta>0$; $w\in\mathbf{R}$, $\rho\in(0,1)$ and $s>1$ such that $s=\frac{k}{k-1}$, then

$$
\begin{aligned}
&\left|u\left((\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right)-\frac{1}{2\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]}\times\right.\\
&\left.\left[\frac{\left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},[\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}]-;\frac{w}{(1-\rho)^{\rho}}}u\circ g\right)(w^{\mathbf{p}})}{[(1-\rho)(k^{\mathbf{p}}-w^{\mathbf{p}})]^{\boldsymbol{\vartheta}}}+\frac{\left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},[\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}]+;\frac{w}{\rho^{\rho}}}u\circ g\right)(k^{\mathbf{p}})}{[\rho(k^{\mathbf{p}}-w^{\mathbf{p}})]^{\boldsymbol{\vartheta}}}\right]\right|\\
&\leq(k^{\mathbf{p}}-w^{\mathbf{p}})\left[\mathfrak{F}^{\sigma_4}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]+\mathfrak{F}^{\sigma_5}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\right]\right]
\end{aligned}
\tag{3.11}
$$

provided that σ_4, σ_5 are defined by, respectively, (2.10), (2.11).

Proof . By $\mathbf{p}-$convexity of $\left|u^{'}\right|^{s}$ and Hölder inequality:

$$\begin{aligned}
|\mathring{\mathbf{I}}_1| &= \left|\int_0^1 \varepsilon^{\boldsymbol{\vartheta}} \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\varepsilon^{\rho}\right] u'\left(\left(\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\left[\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\right| \\
&\le \int_0^1 \varepsilon^{\boldsymbol{\vartheta}} \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\varepsilon^{\rho}\right]\left|u'\left(\left(\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right| \\
&\quad\times\left[\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon \\
&= \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\int_0^1 \varepsilon^{\boldsymbol{\vartheta}+\rho k}\left[\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} \\
&\quad\times\left|u'\left(\left(\varepsilon\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)^{\mathbf{p}}+(1-\varepsilon)w^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right| d\varepsilon \\
&\le \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\times\left\{\int_0^1 \varepsilon^{k(\boldsymbol{\vartheta}+\rho k)}\left[\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{k(1-\mathbf{p})}{\mathbf{p}}}\right\}^{\frac{1}{k}} d\varepsilon \\
&\quad\times\left\{\int_0^1\left|u'\left(\left(\varepsilon\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)^{\mathbf{p}}+(1-\varepsilon)w^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right|^{s} d\varepsilon\right\}^{\frac{1}{s}} \\
&\le \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{w^{k(\mathbf{p}-1)}\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\times\left\{\int_0^1 \varepsilon^{k(\boldsymbol{\vartheta}+\rho k)}\left[1-\varepsilon\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right]^{\frac{k(1-\mathbf{p})}{\mathbf{p}}} d\varepsilon\right\}^{\frac{1}{k}} \\
&\quad\times\left\{\int_0^1\left[\varepsilon\left|u'\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)\right|^{s}+(1-\varepsilon)\left|u^{'}\right|^{s}\right] d\varepsilon\right\}^{\frac{1}{s}} \\
&\le \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}\sqrt[k]{\mathbb{B}(1+k\boldsymbol{\vartheta}+\rho kk,1)}}{w^{k(\mathbf{p}-1)}\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\times\sqrt[s]{\frac{(1-\rho)\left|u'\right|^{s}+(1+\rho)\left|u'\right|^{s}}{2}} \\
&\quad\times\sqrt[k]{{}_2F_1\left(k\frac{\mathbf{p}-1}{\mathbf{p}},k\boldsymbol{\vartheta}+k\rho k+1;k\boldsymbol{\vartheta}+k\rho k+2,\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right)}. \qquad (3.12)
\end{aligned}$$

Again by $\mathbf{p}-$convexity of $\left|u^{'}\right|^{s}$ and Hölder inequality:

$$\begin{aligned}
|\mathring{\mathbf{I}}_2| &= \left|\int_0^1 \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}(1-\varepsilon)^{\rho}\right] u'\left(\left((1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right. \\
&\quad\left.\times\ (1-\varepsilon)^{\boldsymbol{\vartheta}}\left[(1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\right| \\
&\le \int_0^1 \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}(1-\varepsilon)^{\rho}\right]\left|u'\left(\left((1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right| \\
&\quad\times(1-\varepsilon)^{\boldsymbol{\vartheta}}\left[(1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon \\
&= \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\int_0^1 (1-\varepsilon)^{\boldsymbol{\vartheta}+\rho k}\left[(1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} \\
&\quad\times\left|u'\left(\left((1-\varepsilon)\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)^{\mathbf{p}}+\varepsilon.k^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right| d\varepsilon \\
&\le \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\times\left\{\int_0^1 (1-\varepsilon)^{k(\boldsymbol{\vartheta}+\rho k)}\left[(1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right]^{\frac{k(1-\mathbf{p})}{\mathbf{p}}} d\varepsilon\right\}^{\frac{1}{k}} \\
&\quad\times\left\{\int_0^1\left|u'\left(\left((1-\varepsilon)\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)^{\mathbf{p}}+\varepsilon.k^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right|^{s} d\varepsilon\right\}^{\frac{1}{s}}
\end{aligned}$$

$$\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (k^{\mathbf{p}} - w^{\mathbf{p}})^{\rho k}}{\left[\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}}\right]^{k(\mathbf{p}-1)} \Gamma(\rho k + \boldsymbol{\vartheta} + 1)} \times \left\{ \int_0^1 \left[1 - \varepsilon\rho \frac{w^{\mathbf{p}} - k^{\mathbf{p}}}{\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}}\right]^{\frac{k(1-\mathbf{p})}{\mathbf{p}}} (1-\varepsilon)^{k(\boldsymbol{\vartheta}+\rho k)} d\varepsilon \right\}^{\frac{1}{k}}$$

$$\times \left\{ \int_0^1 \left[(1-\varepsilon) \left|u'(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}})\right|^s + \varepsilon \left|u'(k)\right|^s\right] d\varepsilon \right\}^{\frac{1}{s}}$$

$$\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (k^{\mathbf{p}} - w^{\mathbf{p}})^{\rho k} \sqrt[k]{\mathbb{B}(1, 1 + k\boldsymbol{\vartheta} + \rho k k)}}{\left[\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}}\right]^{k(\mathbf{p}-1)} \Gamma(\rho k + \boldsymbol{\vartheta} + 1)} \times \sqrt[k]{{}_2F_1\left(k\frac{\mathbf{p}-1}{\mathbf{p}}, 1; k\boldsymbol{\vartheta} + k\rho k + 2, \frac{\rho(w^{\mathbf{p}} - k^{\mathbf{p}})}{\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}}}\right)}$$

$$\times \sqrt[s]{\frac{2\left|u'\right|^s + \rho(\left|u'\right|^s - \left|u'\right|^s)}{2}} \tag{3.13}$$

Combining the inequalities (3.12), (3.13) and (3.6) yields the desired inequality (3.11) □

Corollary 3.7. Let $u : \mathring{\mathbf{I}} \subseteq \mathbf{R}^+ \to \mathbf{R}$ a differentiable function on $\mathring{\mathbf{I}}$, $w, k \in \mathring{\mathbf{I}}$ with $w < k$ such that $|u'|$ is convex and $\rho, \boldsymbol{\vartheta} > 0$, $w \in \mathbf{R}$; let $s > 1$ be such that $s = \frac{k}{k-1}$, then

$$\left| u\left(\frac{w+k}{2}\right) - \frac{\left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},\frac{w+k}{2}-;2^{\rho}w}\right)u(w) + \left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},\frac{w+k}{2}+;2^{\rho}w}\right)u(k)}{2^{1-\boldsymbol{\vartheta}}(k-w)^{\boldsymbol{\vartheta}}\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k-w\right)^{\rho}\right]} \right|$$
$$\leq (k-w)\left[\mathfrak{F}^{\Sigma_4}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k-w\right)^{\rho}\right] + \mathfrak{F}^{\Sigma_5}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k-w\right)^{\rho}\right]\right] \tag{3.14}$$

provided that Σ_4, Σ_5 are defined by, respectively, (2.17), (2.18).

Proof . The proof directly follows from Theorem 3.6 for $\mathbf{p} = 1$, $\rho = \frac{1}{2}$. □

Corollary 3.8. Let $u : \mathring{\mathbf{I}} \subseteq \mathbf{R}^+ \to \mathbf{R}$ a differentiable function on $\mathring{\mathbf{I}}$, $w, k \in \mathring{\mathbf{I}}$ with $w < k$ such that $|u'|$ is convex and $\alpha > 0$; let $s > 1$ be such that $s = \frac{k}{k-1}$, then

$$\left| u\left(\frac{w+k}{2}\right) - \Gamma(\alpha+1)\frac{\left(\mathcal{J}^{\alpha}_{\frac{w+k}{2}-}u\right)(w) + \left(\mathcal{J}^{\alpha}_{\frac{w+k}{2}+}u\right)(k)}{2^{1-\alpha}(k-w)^{\alpha}} \right| \leq (k-w)[\Delta_4 + \Delta_5].$$

provided that Δ_4, Δ_5 are defined by, respectively, (2.22), (2.23).

Proof . The proof directly follows from Corollary 3.7 for $w = 0$, $\sigma(0) = 1$, $\boldsymbol{\vartheta} = \alpha$ □

Theorem 3.9. Let $u : \mathring{\mathbf{I}} \subseteq \mathbf{R}^+ \to \mathbf{R}$ a differentiable function on $\mathring{\mathbf{I}}$, interior of $\mathring{\mathbf{I}}$, $w, k \in \mathring{\mathbf{I}}$ with $w < k$ such that $\left|u^{'}\right|^k$ is $\mathbf{p}-$convex for $\mathbf{p}, \rho, \boldsymbol{\vartheta} > 0$; $k \geq; 1$ let $g(\zeta) = \sqrt[\mathbf{p}]{\zeta}$, $\zeta > 0$; $w \in \mathbf{R}$, $\rho \in (0, 1)$, then

$$\left| u\left((\rho w^{\mathbf{p}} + (1-\rho)k^{\mathbf{p}})^{\frac{1}{\mathbf{p}}}\right) - \frac{1}{2\mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}\right]} \times \right.$$
$$\left.\left[\frac{\left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},[\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}]-;\frac{w}{(1-\rho)^{\rho}}}u\circ g\right)(w^{\mathbf{p}})}{[(1-\rho)(k^{\mathbf{p}} - w^{\mathbf{p}})]^{\boldsymbol{\vartheta}}} + \frac{\left(\mathfrak{J}^{\sigma}_{\rho,\boldsymbol{\vartheta},[\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}]+;\frac{w}{\rho^{\rho}}}u\circ g\right)(k^{\mathbf{p}})}{[\rho(k^{\mathbf{p}} - w^{\mathbf{p}})]^{\boldsymbol{\vartheta}}}\right]\right|$$
$$\leq (k^{\mathbf{p}} - w^{\mathbf{p}})\left[\mathfrak{F}^{\sigma_6}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}\right] + \mathfrak{F}^{\sigma_7}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}} - w^{\mathbf{p}}\right)^{\rho}\right]\right] \tag{3.15}$$

provided that σ_6, σ_7 are defined by, respectively, (2.12), (2.13).

Proof . By $\mathbf{p}-$convexity of $\left|u^{'}\right|^{k}$ and Power-mean inequality:

$$\begin{aligned}
|\mathring{\mathbf{I}}_1| =& \left|\int_0^1 \varepsilon^{\boldsymbol{\vartheta}} \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\varepsilon^{\rho}\right] u'\left(\left(\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right.\\
&\left.\times\left[\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\right|\\
\leq& \int_0^1 \varepsilon^{\boldsymbol{\vartheta}} \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}\varepsilon^{\rho}\right]\left|u'\left(\left(\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right|\\
&\times\left[\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\\
=&\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\int_0^1 \varepsilon^{\boldsymbol{\vartheta}+\rho k}\left[\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}}\\
&\times\left|u'\left(\left(\varepsilon\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)^{\mathbf{p}}+(1-\varepsilon)w^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right| d\varepsilon\\
\leq&\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\times\left\{\int_0^1 \varepsilon^{\boldsymbol{\vartheta}+\rho k}\left[\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\right\}^{1-\frac{1}{k}}\\
&\times\left\{\int_0^1 \varepsilon^{\boldsymbol{\vartheta}+\rho k}\left[\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}}\times\left|u'\left(\left(\varepsilon\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)^{\mathbf{p}}+(1-\varepsilon)w^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right|^{k} d\varepsilon\right\}^{\frac{1}{k}}\\
\leq&\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{\sqrt[k]{w^{(k-1)(\mathbf{p}-1)}}\,\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\times\left\{\int_0^1 \varepsilon^{\boldsymbol{\vartheta}+\rho k}\left[1-\varepsilon\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\right\}^{1-\frac{1}{k}}\\
&\times\left\{\int_0^1 \varepsilon^{\boldsymbol{\vartheta}+\rho k}\left[\varepsilon\left|u'\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)\right|^{k}+(1-\varepsilon)\left|u^{'}\right|^{k}\right]\left[\varepsilon\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+(1-\varepsilon)w^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\right\}^{\frac{1}{k}}\\
\leq&\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{w^{\mathbf{p}-1}\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\left\{\mathbb{B}(1+\boldsymbol{\vartheta}+\rho k,1)\times{}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}},\boldsymbol{\vartheta}+\rho k+1;\boldsymbol{\vartheta}+\rho k+2,\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right)\right\}^{\frac{k-1}{k}}\\
&\times\left\{\left|u^{'}\right|^{k}{}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}},\boldsymbol{\vartheta}+\rho k+1;\boldsymbol{\vartheta}+\rho k+2,\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right)\right.\\
&\times\mathbb{B}(\boldsymbol{\vartheta}+\rho k+1,1)+(1-\rho)[\left|u^{'}\right|^{k}-\left|u^{'}\right|^{k}]\mathbb{B}(\boldsymbol{\vartheta}+\rho k+2,1)\\
&\left.\times{}_2F_1\left(\frac{\mathbf{p}-1}{\mathbf{p}},\boldsymbol{\vartheta}+\rho k+2;\boldsymbol{\vartheta}+\rho k+3,\frac{(1-\rho)(w^{\mathbf{p}}-k^{\mathbf{p}})}{w^{\mathbf{p}}}\right)\right\}^{\frac{1}{k}} \qquad (3.16)
\end{aligned}$$

Again by $\mathbf{p}-$convexity of $\left|u^{'}\right|^{k}$ and Power-mean inequality:

$$\begin{aligned}
|\mathring{\mathbf{I}}_2| =& \left|\int_0^1 \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[w\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}(1-\varepsilon)^{\rho}\right] u'\left(\left((1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right.\\
&\left.\times(1-\varepsilon)^{\boldsymbol{\vartheta}}\left[(1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\right|\\
\leq& \int_0^1 \mathfrak{F}^{\sigma}_{\rho,\boldsymbol{\vartheta}+1}\left[|w|\left(k^{\mathbf{p}}-w^{\mathbf{p}}\right)^{\rho}(1-\varepsilon)^{\rho}\right]\left|u'\left(\left((1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right|\\
&\times(1-\varepsilon)^{\boldsymbol{\vartheta}}\left[(1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon\\
=&\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(k^{\mathbf{p}}-w^{\mathbf{p}})^{\rho k}}{\Gamma(\rho k+\boldsymbol{\vartheta}+1)}\int_0^1 (1-\varepsilon)^{\boldsymbol{\vartheta}+\rho k}\left[(1-\varepsilon)\left(\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}\right)+\varepsilon.k^{\mathbf{p}}\right]^{\frac{1-\mathbf{p}}{\mathbf{p}}}\\
&\times\left|u'\left(\left((1-\varepsilon)\left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}}+(1-\rho)k^{\mathbf{p}}}\right)^{\mathbf{p}}+\varepsilon.k^{\mathbf{p}}\right)^{\frac{1}{\mathbf{p}}}\right)\right| d\varepsilon
\end{aligned}$$

$$\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (k^{\mathbf{p}} - w^{\mathbf{p}})^{\rho k}}{\Gamma(\rho k + \vartheta + 1)} \times \left\{ \int_0^1 (1-\varepsilon)^{\vartheta + \rho k} \left[(1-\varepsilon)\left(\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}}\right) + \varepsilon . k^{\mathbf{p}} \right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} d\varepsilon \right\}^{1-\frac{1}{k}}$$

$$\times \{ \int_0^1 (1-\varepsilon)^{\vartheta + \rho k} \left[(1-\varepsilon)\left(\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}}\right) + \varepsilon . k^{\mathbf{p}} \right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} \times \left| u' \left(\left((1-\varepsilon) \left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}}} \right)^{\mathbf{p}} + \varepsilon . k^{\mathbf{p}} \right)^{\frac{1}{\mathbf{p}}} \right) \right|^k d\varepsilon \}^{\frac{1}{k}}$$

$$\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (k^{\mathbf{p}} - w^{\mathbf{p}})^{\rho k}}{\left[\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}}\right]^{\frac{\mathbf{p}-1}{\mathbf{p}}} \Gamma(\rho k + \vartheta + 1)} \times \left\{ \int_0^1 \left[1 - \varepsilon \rho \frac{w^{\mathbf{p}} - k^{\mathbf{p}}}{\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}}} \right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} (1-\varepsilon)^{\vartheta + \rho k} d\varepsilon \right\}^{1-\frac{1}{k}}$$

$$\times \left\{ \int_0^1 (1-\varepsilon)^{\vartheta + \rho k} \left[(1-\varepsilon)\left(\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}}\right) + \varepsilon . k^{\mathbf{p}} \right]^{\frac{1-\mathbf{p}}{\mathbf{p}}} \times \left| u' \left(\left((1-\varepsilon) \left(\sqrt[\mathbf{p}]{\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}}} \right)^{\mathbf{p}} + \varepsilon . k^{\mathbf{p}} \right)^{\frac{1}{\mathbf{p}}} \right) \right|^k d\varepsilon \right\}^{\frac{1}{k}}$$

$$\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (k^{\mathbf{p}} - w^{\mathbf{p}})^{\rho k}}{\left[\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}}\right]^{\frac{\mathbf{p}-1}{\mathbf{p}}} \Gamma(\rho k + \vartheta + 1)} \left\{ \mathbb{B}(1, 1 + \vartheta + \rho k) {}_2F_1 \left(\frac{\mathbf{p}-1}{\mathbf{p}}, 1; \vartheta + \rho k + 2, \frac{\rho(w^{\mathbf{p}} - k^{\mathbf{p}})}{\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}}} \right) \right\}^{\frac{k-1}{k}}$$

$$\times \left\{ [\rho \left| u' \right|^k + (1-\rho) \left| u' \right|^k] \mathbb{B}(1, 2 + \vartheta + \rho k) \times {}_2F_1 \left(\frac{\mathbf{p}-1}{\mathbf{p}}, 1; \vartheta + \rho k + 3, \frac{\rho(w^{\mathbf{p}} - k^{\mathbf{p}})}{\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}}} \right) + \left| u' \right|^k \mathbb{B}(2, 1 + \vartheta + \rho k) \right.$$

$$\left. \times {}_2F_1 \left(\frac{\mathbf{p}-1}{\mathbf{p}}, 2; \vartheta + \rho k + 3, \frac{\rho(w^{\mathbf{p}} - k^{\mathbf{p}})}{\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}}} \right) \right\}^{\frac{1}{k}} \tag{3.17}$$

Combining the inequalities (3.6) and (3.16)-(3.17) yields the desired inequality (3.15) □

Example 3.10. Let $w = 1$; $k = 2$; $\alpha = 2$; $u(\zeta) = \zeta^{-2}$. Then obviously $\left| u' \right|$ is convex and all the conditions of Corollary 3.5 are satisfied.

$$\begin{aligned} \left| u\left(\frac{w+k}{2}\right) - \Gamma(\alpha+1) \frac{\left(\mathcal{J}^{\alpha}_{\frac{w+k}{2}-} u\right)(w) + \left(\mathcal{J}^{\alpha}_{\frac{w+k}{2}+} u\right)(k)}{2^{1-\alpha}(k-w)^{\alpha}} \right| &= \frac{28 - \ln 68719476736}{9} \\ &\cong 0.338524 \end{aligned}$$

$$|u'(w)|\, \Delta_1 + |u'(k)|\, \Delta_3 + \left| u'\left(\frac{w+k}{2}\right) \right| \Delta_2 = \frac{1737 + 64\sqrt{3}}{864} \cong 2.138716$$

$$\frac{|u'(w)|\,[2\Delta_1 + \Delta_2] + |u'(k)|\,[2\Delta_3 + \Delta_2]}{2} = \frac{386 + 27\sqrt{3}}{96} \cong 4.507972$$

It is clear that $\frac{28 - \ln 68719476736}{9} < \frac{1737+64\sqrt{3}}{864} < \frac{386+27\sqrt{3}}{96}$, which demonstrates the result described in Corollary 3.5.

Example 3.11. Let $w = 2 = \mathbf{p}$; $k = 4$; $\vartheta = 1 = k$; $\rho = \frac{1}{2}$, $u(\zeta) = \frac{\zeta^{1-\mathbf{p}}}{1-\mathbf{p}}$, $g(\zeta) = \sqrt[\mathbf{p}]{\zeta}$, $\zeta \in (0, \infty)$; let $w = 0$, $\sigma(0) = 1$. Then obviously $|u'|$ is $\mathbf{p}-$convex and all the conditions of Theorem 3.9 are satisfied.

$$\begin{aligned} & \left| u\left(\left(\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}} \right)^{\frac{1}{\mathbf{p}}} \right) - \frac{1}{2\mathfrak{F}^{\sigma}_{\rho, \vartheta+1} \left[w \left(k^{\mathbf{p}} - w^{\mathbf{p}} \right)^{\rho} \right]} \times \right. \\ & \left. \left[\frac{\left(\mathfrak{J}^{\sigma}_{\rho, \vartheta, [\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}}]-; \frac{w}{(1-\rho)^{\rho}}} u \circ g \right)(w^{\mathbf{p}})}{[(1-\rho)(k^{\mathbf{p}} - w^{\mathbf{p}})]^{\vartheta}} + \frac{\left(\mathfrak{J}^{\sigma}_{\rho, \vartheta, [\rho w^{\mathbf{p}} + (1-\rho) k^{\mathbf{p}}]+; \frac{w}{\rho^{\rho}}} u \circ g \right)(k^{\mathbf{p}})}{[\rho(k^{\mathbf{p}} - w^{\mathbf{p}})]^{\vartheta}} \right] \right| \\ = & \frac{\sqrt{10} - 3}{3\sqrt{10}} \cong 0.0171056 \end{aligned}$$

$$\begin{aligned} (k^{\mathbf{p}} - w^{\mathbf{p}}) \left[\mathfrak{F}^{\sigma_6}_{\rho, \vartheta+1} \left[|w| \left(k^{\mathbf{p}} - w^{\mathbf{p}} \right)^{\rho} \right] + \mathfrak{F}^{\sigma_7}_{\rho, \vartheta+1} \left[|w| \left(k^{\mathbf{p}} - w^{\mathbf{p}} \right)^{\rho} \right] \right] &= \frac{524213\sqrt{10} - 1625216}{2880} \\ &\cong 11.281614 \end{aligned}$$

It is clear that $\frac{\sqrt{10}-3}{3\sqrt{10}} < \frac{524213\sqrt{10}-1625216}{2880}$, which demonstrates the result described in Theorem 3.9.

Conclusions

In the development of the present paper we have established a new fractional integral inequality of the Hermite-Hadamard type which involves $p-$convex functions. The established results use the fractional integral operator defined by R. K. Raina and Agarwal. To achieve our objective, a fundamental lemma was proved which corresponds to a representative identity of the left side of the Hermite-Hadamard fractional integral inequality. Also, we have used the classical Hölder and power mean inequalities as tools to attain our results.

Since the fractional integral operator used is parametric, our results are valid for other types of fractional integrals, such as: Riemann-Liouville, Katugampola, Prabhakar fractional integral operators, between others, with a suitable choice of the parameters.

Acknowledgments

The authors thanks to Dirección de Investigación from Pontificia Universidad Católica del Ecuador for the technical support given for this work.

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