Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 1713–1733
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.25945.3167

Existence theory and stability analysis to the system of infinite point fractional order BVPs by multivariate best proximity point theorem

Mahammad Khuddush[a,*], Kapula Rajendra Prasad[b], Doddi Leela[b]

[a]Department of Mathematics, Dr. Lankapalli Bullayya College of Engineering, Resapuvanipalem, Visakhapatnam, 530013, India
[b]Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam, 530003, India

(Communicated by Oluwatosin Temitope Mewomo)

Abstract

This paper deals with the existence of solutions to the system of nonlinear infinite-point fractional order boundary value problems by an application of n-best proximity point theorem in a complete metric space. Further, we study Hyers-Ulam stability of the addressed system. An appropriate example is provided to check the validity of obtained results.

Keywords: Fractional derivative, boundary value problem, n-best proximity point theorem, metric space, Hyers-Ulam stability
2020 MSC: Primary 34A08, Secondary 34B15, 47H10, 54H25

1 Introduction

Fractional calculus is one of the useful fields of applied mathematics which has applications in the areas such as engineering, economics, control theory, chemistry, biology, medicine and other fields, see [21, 24, 25, 26, 39, 40]. Fractional differential equations can describe many phenomena in various fields of engineering and scientific disciplines In consequence, the subject of fractional differential equations is gaining much importance and attention. In recent years, there are a large number of papers dealing with the existence, uniqueness and multiple solutions of boundary value problems for nonlinear differential equations of fractional order. For examples and recent development of the topic, see [1, 3, 4, 6, 7, 13, 16, 17, 18, 20, 23, 27, 28, 29, 30, 34, 41] and references therein.

Fixed point theory is an indispensable tool for solving the equation $\aleph\mathtt{z} = \mathtt{z}$ for a mapping $\aleph$ defined on a subset of a metric space, a normed linear space or a topological vector space. As a non-self mapping $\aleph : \mathtt{A} \to \mathtt{B}$ does not necessarily have a fixed point, one often tries to determine an element $\mathtt{z}$ which is in some sense closest to $\aleph\mathtt{z}$. Best approximation theorems and best proximity point theorems are pertinent in this perspective. A classical best approximation theorem, due to Fan [12], asserts that if $\mathtt{A}$ is a non-empty compact convex subset of a Hausdorff locally convex topological vector space $\mathtt{Y}$ with a semi-norm $\mathtt{p}$ and $\aleph : \mathtt{A} \to \mathtt{Y}$ is a continuous mapping, then there is an element $\mathtt{z}$ in $\mathtt{A}$ satisfying the

*Corresponding author
Email addresses: khuddush89@gmail.com (Mahammad Khuddush), krajendra92@rediffmail.com (Kapula Rajendra Prasad), leelaravidadi@gmail.com (Doddi Leela)

Received: January 2022 *Accepted:* May 2022

condition that $\mathtt{d_p}(\mathtt{z},\aleph\mathtt{z})=\mathtt{d_p}(\aleph\mathtt{z},\mathtt{A})$. There have been many subsequent extensions and variants of Fan's Theorem, see [2, 8, 14, 15, 19, 22, 31, 32, 35, 36] and references therein.

Basically the proximity theory is useful tool to find proximity point when the given mapping is non self. Let $\mathtt{A}$ and $\mathtt{B}$ be two non empty subsets of $\mathtt{Y}$ such that $\aleph:\mathtt{A}\to\mathtt{B}$ then a point $\mathtt{z}\in\mathtt{A}$ for which $\mathtt{d}(\mathtt{z},\aleph\mathtt{z})=\mathtt{d}(\mathtt{A},\mathtt{B})$ is called a best proximity point of $\aleph$. It should be noted that best proximity point reduced to fixed point when the mapping $\aleph$ is self mapping that is $\mathtt{A}=\mathtt{B}$.

In [5], Afshari, Jarad and Abdeljawad discussed the admissibility of two multi-valued mappings in the category of complete b-metric spaces to obtain the existence of a common fixed point and by using the triangular admissibility, they proved the uniqueness of the common fixed point. As an application of their finding, the existence and uniqueness of the following fractional order boundary value problem studied,

$$\begin{gathered}{}^{\mathtt{RL}}\mathtt{D}^{\varrho}_{0+}\mathtt{z}_1(\mathtt{s})+\mathtt{f}(\mathtt{s},\mathtt{z}_2(\mathtt{s})),\ 0<\mathtt{s}<1,\ 1<\varrho\leq 2,\\ {}^{\mathtt{RL}}\mathtt{D}^{\xi}_{0+}\mathtt{z}_2(\mathtt{s})+\mathtt{g}(\mathtt{s},\mathtt{z}_1(\mathtt{s})),\ 0<\mathtt{s}<1,\ 1<\xi\leq 2,\\ \mathtt{z}_1(0)=\mathtt{z}_2(0)=0,\ \mathtt{z}_1(1)=\int_0^1\varphi(\mathtt{t})\mathtt{z}_1(\mathtt{t})d\mathtt{t},\ \mathtt{z}_2(1)=\int_0^1\varphi(\mathtt{t})\mathtt{z}_2(\mathtt{t})d\mathtt{t}.\end{gathered}$$

Recently, Prasad, Khuddush and Leela [30] established the existence of unique solution for a two-point fractional order boundary value problem,

$$\begin{gathered}{}^{\mathtt{RL}}\mathtt{D}^{\zeta}\mathtt{z}(\mathtt{s})+\mathtt{F}\big(\mathtt{s},\mathtt{z}(\mathtt{s}),\mathtt{z}(\mathtt{s}),\mathtt{z}(\mathtt{s})\big)+\varphi(\mathtt{s})\mathtt{z}(\mathtt{s})=0,\ 0<\mathtt{s}<\mathtt{T},\ 2<\zeta\leq 3,\\ \mathtt{z}(0)=\mathtt{z}''(0)=0,\ \mathtt{z}(\mathtt{T})=\int_0^{\mathtt{T}}\mathtt{z}(\mathtt{t})d\mathtt{t},\end{gathered}$$

by proving the existence and the uniqueness of solutions of the operator equation $\mathtt{A}(\mathtt{z},\mathtt{z},\mathtt{z})+\mathtt{B}\mathtt{z}=\mathtt{z}$ in a real Banach space. Motivated by aforementioned works, in this paper we study the concept of n-best proximity point in a complete metric space and establish the existence and uniqueness theorems. Moreover, as an application of our results we study the following system of n-nonlinear infinite-point fractional order boundary value problems

$$\begin{cases}{}^{\mathtt{RL}}\mathtt{D}^{\text{ß}_1}_{0+}\mathtt{z}_1(\mathtt{s})=\mathtt{g}_1\big(\mathtt{s},\mathtt{z}_1(\mathtt{s}),\mathtt{z}_2(\mathtt{s}),\cdots,\mathtt{z}_{n-1}(\mathtt{s}),\mathtt{z}_n(\mathtt{s})\big),\\ {}^{\mathtt{RL}}\mathtt{D}^{\text{ß}_2}_{0+}\mathtt{z}_2(\mathtt{s})=\mathtt{g}_2\big(\mathtt{s},\mathtt{z}_2(\mathtt{s}),\mathtt{z}_1(\mathtt{s}),\cdots,\mathtt{z}_{n-1}(\mathtt{s}),\mathtt{z}_n(\mathtt{s})\big),\\ \qquad\vdots\\ {}^{\mathtt{RL}}\mathtt{D}^{\text{ß}_{n-1}}_{0+}\mathtt{z}_{n-1}(\mathtt{s})=\mathtt{g}_{n-1}\big(\mathtt{s},\mathtt{z}_{n-1}(\mathtt{s}),\mathtt{z}_2(\mathtt{s}),\cdots,\mathtt{z}_1(\mathtt{s}),\mathtt{z}_n(\mathtt{s})\big),\\ {}^{\mathtt{RL}}\mathtt{D}^{\text{ß}_n}_{0+}\mathtt{z}_n(\mathtt{s})=\mathtt{g}_n\big(\mathtt{s},\mathtt{z}_n(\mathtt{s}),\mathtt{z}_2(\mathtt{s}),\cdots,\mathtt{z}_{n-1}(\mathtt{s}),\mathtt{z}_1(\mathtt{s})\big),\end{cases}\tag{1.1}$$

satisfying

$$\mathtt{z}_i(0)=0\ \text{ and }\ {}^{\mathtt{RL}}\mathtt{D}^{\alpha_i}_{0+}\mathtt{z}_i(1)=\sum_{\mathtt{j}=1}^{\infty}\delta_{i\mathtt{j}}\,{}^{\mathtt{RL}}\mathtt{D}^{\alpha_i}_{0+}\mathtt{z}_i\big(\sigma_i(\tau_{i\mathtt{j}})\big),\ i=1,2,\cdots,n,\tag{1.2}$$

where $0<\mathtt{s}<1$, $1<\text{ß}_i\leq 2$, $0<\alpha_i<\text{ß}_i-1$, $\delta_{ij}>0$, ${}^{\mathtt{RL}}\mathtt{D}^{\star}_{0+}$ denotes the standard Riemann-Liouville fractional derivative of order $\star\in\{\text{ß}_i,\alpha_i\}, i=1,2,\cdots,n$, $\mathtt{g}_i:[0,1]\times[0,+\infty)^n\to[0,+\infty)$, $\sigma_i:[0,1]\to[0,1]$ are continuous functions. In [30], authors studied the fixed point theorem of mixed monotone ternary operators on Banach spaces, whereas in this paper, we study n-best proximity poin theorem on metric spaces. Therefore, by applying our results we can study different kind differential equations whereas results in [30] can only applicable for certain type of differential equations.

2 Preliminaries

In this section, we construct kernel for the boundary value problem (1.1)–(1.2) and estimate bounds for it, which are useful for our later discussions.

Definition 2.1. [21] The Riemann-Liouville fractional integral of order $\xi>0$ of a function $\mathtt{g}:(0,+\infty)\to\mathbb{R}$ is defined by

$${}^{\mathtt{RL}}\mathtt{I}^{\xi}_{0+}\mathtt{g}(\mathtt{s})=\int_0^{\mathtt{s}}\frac{(\mathtt{s}-\mathtt{t})^{\xi-1}}{\Gamma(\xi)}\mathtt{g}(\mathtt{t})d\mathtt{t}.$$

Definition 2.2. [21] Let $\mathtt{k}$ be a positive integer. The Riemann-Liouville fractional derivative of order $\xi > 0$ of a continuous function $\mathtt{g} : (0, +\infty) \to \mathbb{R}$ is defined by

$$^{\mathtt{RL}}\mathtt{D}^{\xi}_{0+}\mathtt{g}(\mathtt{s}) = \frac{\mathrm{d}^{\mathtt{k}}}{\mathrm{d}\mathtt{s}^{\mathtt{k}}} \int_0^{\mathtt{s}} \frac{(\mathtt{s}-\mathtt{t})^{\mathtt{k}-\xi-1}}{\Gamma(\mathtt{k}-\xi)} \mathtt{g}(\mathtt{t})\mathrm{d}\mathtt{t},$$

where $\mathtt{k} - 1 < \xi \leq \mathtt{k}$, provided that the right-hand side exists.

Remark 2.3. ([21]) In this work we need the following composition relations:

(*a*) $^{\mathtt{RL}}\mathtt{D}^{\xi}_{0+}\,{}^{\mathtt{RL}}\mathtt{I}^{\xi}_{0+}\mathtt{g}(\mathtt{s}) = \mathtt{g}(\mathtt{s}),\ \xi > 0, \mathtt{g}(\mathtt{s}) \in L^1(0, +\infty)$;

(*b*) $^{\mathtt{RL}}\mathtt{D}^{\zeta}_{0+}\,{}^{\mathtt{RL}}\mathtt{I}^{\xi}_{0+}\mathtt{g}(\mathtt{s}) = {}^{\mathtt{RL}}\mathtt{I}^{\xi-\zeta}_{0+}\mathtt{g}(\mathtt{s}),\ \xi > \zeta > 0, \mathtt{g}(\mathtt{s}) \in L^1(0, +\infty)$.

Remark 2.4. ([10]) For $\zeta > -1$, we have

$$^{\mathtt{RL}}\mathtt{D}^{\xi}_{0+}\mathtt{s}^{\zeta} = \frac{\Gamma(\zeta+1)}{\Gamma(\zeta-\xi+1)}\mathtt{s}^{\zeta-\xi},$$

giving inparticular $^{\mathtt{RL}}\mathtt{D}^{\xi}_{0+}\mathtt{s}^{\xi-m} = 0$, $m = 1, 2, \cdots, N$, where N is the smallest integer greater than or equal to ξ.

Lemma 2.5. [21] Let $\mathtt{k}$ be a positive integer. The general solution to $^{\mathtt{RL}}\mathtt{D}^{\xi}_{0+}\mathtt{g}(\mathtt{s}) = 0$ with $\mathtt{k}-1 < \xi \leq \mathtt{k}$ is the function

$$\mathtt{g}(\mathtt{s}) = \sum_{\mathtt{j}=1}^{\mathtt{k}} a_{\mathtt{j}}\mathtt{s}^{\xi-\mathtt{j}},$$

where $a_{\mathtt{j}}$ is a real number.

Lemma 2.6. [21] Let $\zeta > 0$. Then for any function $\mathtt{g} : (0, +\infty) \to \mathbb{R}$, we have

$$^{\mathtt{RL}}\mathtt{I}^{\zeta}_{0+}\,{}^{\mathtt{RL}}\mathtt{D}^{\zeta}_{0+}\mathtt{g}(\mathtt{s}) = \mathtt{g}(\mathtt{s}) + \sum_{\mathtt{j}=1}^{\mathtt{k}} a_{\mathtt{j}}\mathtt{s}^{\zeta-\mathtt{j}},$$

where $a_{\mathtt{j}}$ is a real number and $\mathtt{k}$ is the smallest integer greater that or equal to ζ, and

$$^{\mathtt{RL}}\mathtt{D}^{\zeta}_{0+}\,{}^{\mathtt{RL}}\mathtt{I}^{\zeta}_{0+}\mathtt{g}(\mathtt{s}) = \mathtt{g}(\mathtt{s}).$$

In order to study the system of boundary value problems (1.1)–(1.2), we first consider the corresponding linear boundary value problem,

$$^{\mathtt{RL}}\mathtt{D}^{\beta_i}_{0+}\mathtt{z}_i(\mathtt{s}) = \mathtt{h}_i(\mathtt{s}),\ 0 < \mathtt{s} < 1, \tag{2.1}$$

$$\mathtt{z}_i(0) = 0 \ \text{ and } \ ^{\mathtt{RL}}\mathtt{D}^{\alpha_i}_{0+}\mathtt{z}_i(1) = \sum_{\mathtt{j}=1}^{\infty} \delta_{i\mathtt{j}}\mathtt{z}_i\big(\sigma_i(\tau_{i\mathtt{j}})\big),\ i = 1, 2, \cdots, n, \tag{2.2}$$

where $\mathtt{h}_i \in \mathrm{C}[0, 1]$ is a given function.

Lemma 2.7. Suppose $\sum_{\mathtt{j}=1}^{m} \delta_{i\mathtt{j}}\big(\sigma_i(\tau_{i\mathtt{j}})\big)^{\beta_i-1}$ converges $\frac{2\Gamma(\beta_i)}{\Gamma(\beta_i-\alpha_i)}$. The boundary value problem (2.1)–(2.2) has a unique solution

$$\mathtt{z}_i(\mathtt{s}) = \int_0^1 \mathtt{G}_i(\mathtt{s}, \mathtt{t})\mathtt{h}_i(\mathtt{t})\mathrm{d}\mathtt{t} - \mathtt{s}^{\beta_i-1}\sum_{\mathtt{j}=1}^{\infty}\delta_{i\mathtt{j}}\int_0^{\sigma_i(\tau_{i\mathtt{j}})} \frac{(\sigma_i(\tau_{i\mathtt{j}})-\mathtt{t})^{\beta_i-1}}{\Gamma(\beta_i)}\mathtt{h}_i(\mathtt{t})\mathrm{d}\mathtt{t}, \tag{2.3}$$

where

$$\mathtt{G}_i(\mathtt{s}, \mathtt{t}) = \frac{1}{\Gamma(\beta_i)}\begin{cases} \mathtt{s}^{\beta_i-1}(1-\mathtt{t})^{\beta_i-\alpha_i-1} + (\mathtt{s}-\mathtt{t})^{\beta_i-1}, & \mathtt{t} \leq \mathtt{s}, \\ \mathtt{s}^{\beta_i-1}(1-\mathtt{t})^{\beta_i-\alpha_i-1}, & \mathtt{s} \leq \mathtt{t}. \end{cases}$$

Proof . Let $z_i(s)$ be a solution of (2.1). Then, by Lemma 2.6, we have

$$z_i(s) = a_1 s^{ß_i-1} + a_2 s^{ß_i-2} + \int_0^s \frac{(s-t)^{ß_i-1}}{\Gamma(ß_i)} h_i(t)dt, \tag{2.4}$$

where a_1 and a_2 are constants. Using condition $z_i(0) = 0$, we get $a_2 = 0$. So, (2.4) reduces to

$$z_i(s) = a_1 s^{ß_i-1} + \int_0^s \frac{(s-t)^{ß_i-1}}{\Gamma(ß_i)} h_i(t)dt. \tag{2.5}$$

Before applying (2.2), we first take

$$^{RL}D_{0+}^{\alpha_i} z_i(1) = \sum_{j=1}^{m} \delta_{ij} z_i\big(\sigma_i(\tau_{ij})\big),$$

and using Remark 2.4 to get

$$\begin{aligned} a_1 =& \frac{1}{\sum_{j=1}^{m} \delta_{ij}\big(\sigma_i(\tau_{ij})\big)^{ß_i-1} - \frac{\Gamma(ß_i)}{\Gamma(ß_i-\alpha_i)}} \Bigg[\int_0^1 \frac{(1-t)^{ß_i-\alpha_i-1}}{\Gamma(ß_i-\alpha_i)} h_i(t)dt \\ & - \sum_{j=1}^{m} \delta_{ij} \int_0^{\sigma_i(\tau_{ij})} \frac{(\sigma_i(\tau_{ij})-t)^{ß_i-1}}{\Gamma(ß_i)} h_i(t)dt\Bigg]. \end{aligned}$$

Plugging a_2 value into (2.5), we obtain

$$\left.\begin{aligned} z_i(s) =& \frac{1}{\sum_{j=1}^{m} \delta_{ij}\big(\sigma_i(\tau_{ij})\big)^{ß_i-1} - \frac{\Gamma(ß_i)}{\Gamma(ß_i-\alpha_i)}} \Bigg[\int_0^1 \frac{s^{ß_i-1}(1-t)^{ß_i-\alpha_i-1}}{\Gamma(ß_i-\alpha_i)} h_i(t)dt \\ & - s^{ß_i-1}\sum_{j=1}^{m} \delta_{ij} \int_0^{\sigma_i(\tau_{ij})} \frac{(\sigma_i(\tau_{ij})-t)^{ß_i-1}}{\Gamma(ß_i)} h_i(t)dt\Bigg] + \int_0^s \frac{(s-t)^{ß_i-1}}{\Gamma(ß_i)} h_i(t)dt \end{aligned}\right\} \tag{2.6}$$

Since

$$\big|\delta_{ij} z_i\big(\sigma_i(\tau_{ij})\big)\big| \le \delta_{ij}\|z_i\|,$$

and

$$\begin{aligned} \left|\delta_{ij}\int_0^{\sigma_i(\tau_{ij})} \frac{(\sigma_i(\tau_{ij})-t)^{ß_i-1}}{\Gamma(ß_i)} h_i(t)dt\right| &\le \frac{\delta_{ij}}{\Gamma(ß_i)}\left|\int_0^{\sigma_i(\tau_{ij})} v(t)h_i(t)dt\right| \\ &\le \frac{\delta_{ij}}{\Gamma(ß_i)}\left|\int_0^{\tau_{ij}} v(t)h_i(t)dt\right| \\ &\le \frac{\delta_{ij}}{\Gamma(ß_i)}\|v\|_{L_1}\|h_i\|_{L_1}, \end{aligned}$$

where $v(t) = (\sigma_i(\tau_{ij}) - t)^{ß_i-1}$. Then, by comparison test, the series in (2.2) and

$$\sum_{j=1}^{\infty} \delta_{ij} \int_0^{\sigma_i(\tau_{ij})} \frac{(\sigma_i(\tau_{ij})-t)^{ß_i-1}}{\Gamma(ß_i)} h_i(t)dt$$

are convergent. So, by taking the limit as $m \to \infty$ in (2.6), we obtain

$$\begin{aligned} z_i(s) &= \int_0^1 \frac{s^{ß_i-1}(1-t)^{ß_i-\alpha_i-1}}{\Gamma(ß_i)} h_i(t)dt - s^{ß_i-1}\sum_{j=1}^{\infty} \delta_{ij} \int_0^{\sigma_i(\tau_{ij})} \frac{\Gamma(ß_i-\alpha_i)(\sigma_i(\tau_{ij})-t)^{ß_i-1}}{\Gamma(ß_i)^2} h_i(t)dt \\ &\quad + \int_0^s \frac{(s-t)^{ß_i-1}}{\Gamma(ß_i)} h_i(t)dt \\ &= \int_0^1 G_i(s,t)h_i(t)dt - s^{ß_i-1}\sum_{j=1}^{\infty} \delta_{ij} \int_0^{\sigma_i(\tau_{ij})} \frac{\Gamma(ß_i-\alpha_i)(\sigma_i(\tau_{ij})-t)^{ß_i-1}}{\Gamma(ß_i)^2} h_i(t)dt. \end{aligned} \tag{2.7}$$

Conversely, it is clear that (2.7) satisfies $z_i(0)=0$. Next applying the operator $D_{0^+}^{ß_i}$ and $D_{0^+}^{\alpha_i}$ to the two sides of (2.7) respectively and using Remark 2.3 and Remark 2.4, we obtain $D_{0^+}^{ß_i} z_i(s) = h_i(s)$ and

$$\begin{aligned} D_{0^+}^{\alpha_i} z_i(s) = &\int_0^1 \frac{s^{ß_i-\alpha_i-1}(1-t)^{ß_i-\alpha_i-1}}{\Gamma(ß_i-\alpha_i)} h_i(t)dt - s^{ß_i-\alpha_i-1}\sum_{j=1}^{\infty}\delta_{ij}\int_0^{\sigma_i(\tau_{ij})}\frac{(\sigma_i(\tau_{ij})-t)^{ß_i-1}}{\Gamma(ß_i)}h_i(t)dt \\ &+\int_0^s \frac{(s-t)^{ß_i-\alpha_i-1}}{\Gamma(ß_i-\alpha_i)}h_i(t)dt. \end{aligned}$$

Substituting $s=1$, we get

$$D_{0^+}^{\alpha_i} z_i(1) = 2\int_0^1 \frac{(1-t)^{ß_i-\alpha_i-1}}{\Gamma(ß_i-\alpha_i)}h_i(t)dt - \sum_{j=1}^{\infty}\delta_{ij}\int_0^{\sigma_i(\tau_{ij})}\frac{(\sigma_i(\tau_{ij})-t)^{ß_i-1}}{\Gamma(ß_i)}h_i(t)dt. \tag{2.8}$$

But,

$$\begin{aligned} \sum_{j=1}^{m}\delta_{ij} z_i\big(\sigma_i(\tau_{ij})\big) =& \sum_{j=1}^{m}\delta_{ij}\left(\sigma_i(\tau_{ij})\right)^{ß_i-1}\int_0^1 \frac{(1-t)^{ß_i-\alpha_i-1}}{\Gamma(ß_i)}h_i(t)dt \\ &- \sum_{j=1}^{m}\delta_{ij}\left(\sigma_i(\tau_{ij})\right)^{ß_i-1}\sum_{j=1}^{\infty}\delta_{ij}\int_0^{\sigma_i(\tau_{ij})}\frac{\Gamma(ß_i-\alpha_i)(\sigma_i(\tau_{ij})-t)^{ß_i-1}}{\Gamma(ß_i)^2}h_i(t)dt \\ &+\sum_{j=1}^{m}\delta_{ij}\int_0^{\sigma_i(\tau_{ij})}\frac{(\sigma_i(\tau_{ij})-t)^{ß_i-1}}{\Gamma(ß_i)}h_i(t)dt. \end{aligned}$$

Since $\sum_{j=1}^{m}\delta_{ij}\left(\sigma_i(\tau_{ij})\right)^{ß_i-1}$ converges to $\frac{2\Gamma(ß_i)}{\Gamma(ß_i-\alpha_i)}$, it follows by taking limit $m\to+\infty$ that

$$\begin{aligned} \sum_{j=1}^{\infty}\delta_{ij} z_i\big(\sigma_i(\tau_{ij})\big) =& 2\int_0^1 \frac{(1-t)^{ß_i-\alpha_i-1}}{\Gamma(ß_i-\alpha_i)}h_i(t)dt \\ &- 2\sum_{j=1}^{\infty}\delta_{ij}\int_0^{\sigma_i(\tau_{ij})}\frac{(\sigma_i(\tau_{ij})-t)^{ß_i-1}}{\Gamma(ß_i)}h_i(t)dt \\ &+\sum_{j=1}^{m}\delta_{ij}\int_0^{\sigma_i(\tau_{ij})}\frac{(\sigma_i(\tau_{ij})-t)^{ß_i-1}}{\Gamma(ß_i)}h_i(t)dt \\ =& 2\int_0^1 \frac{(1-t)^{ß_i-\alpha_i-1}}{\Gamma(ß_i-\alpha_i)}h_i(t)dt - \sum_{j=1}^{\infty}\delta_{ij}\int_0^{\sigma_i(\tau_{ij})}\frac{(\sigma_i(\tau_{ij})-t)^{ß_i-1}}{\Gamma(ß_i)}h_i(t)dt. \end{aligned} \tag{2.9}$$

By (2.8) and (2.9), we get (2.2). Which shows that the solution of the integral equation (2.3) satisfies the differential equation (2.1) under infinite-point boundary conditions (2.2). □

Remark 2.8. The supposition $\sum_{j=1}^{m}\delta_j\big(\sigma(\tau_j)\big)^{ß-1}$ converges to $\frac{2\Gamma(ß)}{\Gamma(ß-\alpha)}$ in the Lemma 2.7 is valid. For example: Let $ß=\frac{3}{2}$, $\alpha=\frac{1}{3}$, $\delta_j=\frac{18\Gamma(5/6)}{\pi^{5/2}j}$, $\tau_j=\frac{1}{j^3}$ and $\sigma(s)=s^{2/3}$. Then

$$\sum_{j=1}^{m}\delta_j\big(\sigma(\tau_j)\big)^{ß-1} = \sum_{j=1}^{m}\frac{18\Gamma(5/6)}{\pi^{5/2}j^2} \to \frac{3\Gamma(5/6)}{\sqrt{\pi}} = \frac{2\Gamma(ß)}{\Gamma(ß-\alpha)} \quad \text{as } m\to+\infty.$$

The following properties are evident from the definition of $G_i(s,t)$.

Lemma 2.9. The kernel $G_i(s,t)$ has the following properties:

(*i*) $G_i(s,t)$ is nonnegative and continuous on $[0,1]\times[0,1]$.

(*ii*) $G_i(s,t)\le G_i(1,t)$ for $s,t\in[0,1]\times[0,1]$.

3 Main Results

To establish our main results, it is fundamental to recall some notions, definitions amd lemmas which will be useful in the sequel.

Let $\mathtt{A}$ and $\mathtt{B}$ be any two nonempty subset of a metric space $(\mathtt{Y},\mathtt{d})$. Define

$$\mathcal{P}_{\mathtt{A}} = \{\mathtt{z}_2 \in \mathtt{Y} : \mathtt{d}(\mathtt{z}_1,\mathtt{z}_2) = \mathtt{d}(\mathtt{z}_1,\mathtt{A})\},$$
$$\mathtt{d}(\mathtt{A},\mathtt{B}) := \inf\{\mathtt{d}(\mathtt{z}_1,\mathtt{z}_2) : \mathtt{z}_1 \in \mathtt{A}, \mathtt{z}_2 \in \mathtt{B}\},$$
$$\mathtt{A}_0 = \{\mathtt{z}_1 \in \mathtt{A} : \mathtt{d}(\mathtt{z}_1,\mathtt{z}_2) = \mathtt{d}(\mathtt{A},\mathtt{B}), \text{ for some } \mathtt{z}_2 \in \mathtt{B}\},$$

and

$$\mathtt{B}_0 = \{\mathtt{z}_2 \in \mathtt{B} : \mathtt{d}(\mathtt{z}_1,\mathtt{z}_2) = \mathtt{d}(\mathtt{A},\mathtt{B}), \text{ for some } \mathtt{z}_1 \in \mathtt{A}\}.$$

Definition 3.1. [37] Let $\mathtt{A}$ and $\mathtt{B}$ be two nonempty subsets of a metric space $(\mathtt{Y},\mathtt{d})$. An element $\mathtt{z} \in \mathtt{A}$ is said to be a best proximity point of the nonself mapping $\aleph : \mathtt{A} \to \mathtt{B}$ iff it satisfies the condition

$$\mathtt{d}(\mathtt{z},\aleph\mathtt{z}) = \mathtt{d}(\mathtt{A},\mathtt{B}).$$

Definition 3.2. [33] Let $(\mathtt{A},\mathtt{B})$ be a pair of nonempty subsets of a metric space $(\mathtt{Y},\mathtt{d})$ with $\mathtt{A}_0 \neq \emptyset$. Then the pair $(\mathtt{A},\mathtt{B})$ has $\mathcal{P}$-property if and only if

$$\begin{pmatrix} \mathtt{d}(\mathtt{z}_1,\mathtt{y}_1) & = \mathtt{d}(\mathtt{A},\mathtt{B}) \\ \mathtt{d}(\mathtt{z}_2,\mathtt{y}_2) & = \mathtt{d}(\mathtt{A},\mathtt{B}) \end{pmatrix} \implies \mathtt{d}(\mathtt{z}_1,\mathtt{z}_2) = \mathtt{d}(\mathtt{y}_1,\mathtt{y}_2),$$

where $\mathtt{z}_1,\mathtt{z}_2 \in \mathtt{A}$ and $\mathtt{y}_1,\mathtt{y}_2 \in \mathtt{B}$.

Definition 3.3. [11] A map $\varphi : [0,+\infty) \to [0,+\infty)$ is called a $\mathtt{c}$-comparison function if it satisfies:

(*i*) φ is a monotone increasing,

(*ii*) $\sum_{n=0}^{\infty} \varphi^n(t)$ converges for all $t \in [0,+\infty)$.

If we replace the second condition by $\lim_{n\to\infty} \varphi^n(\mathtt{t}) = 0$ for all $\mathtt{t} \in [0,+\infty)$, we obtain the notion of comparison function, which is more general than the one of $\mathtt{c}$-comparison function. It is known that if φ is a comparison function, then $\varphi(\mathtt{t}) < \mathtt{t}$ for all $t > 0$ and $\varphi(0) = 0$.

Let Θ be the set of all continuous functions $\theta : [0,+\infty)^{n+3} \to [0,+\infty)$ such that for every $\mathtt{t},\mathtt{s},\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_n \in [0,+\infty)$,

$$\begin{aligned}
&\theta(0,\mathtt{t},\mathtt{s},\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_{n-3},\mathtt{z}_{n-2},\mathtt{z}_{n-1},\mathtt{z}_n) = 0,\\
&\theta(\mathtt{t},\mathtt{s},0,\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_{n-3},\mathtt{z}_{n-2},\mathtt{z}_{n-1},\mathtt{z}_n) = 0,\\
&\theta(\mathtt{t},\mathtt{s},\mathtt{z}_1,\mathtt{z}_2,0,\cdots,\mathtt{z}_{n-3},\mathtt{z}_{n-2},\mathtt{z}_{n-1},\mathtt{z}_n) = 0,\\
&\qquad\vdots\\
&\theta(\mathtt{t},\mathtt{s},\mathtt{z}_1,\mathtt{z}_2,\mathtt{z}_3,\cdots,0,\mathtt{z}_{n-2},\mathtt{z}_{n-1},\mathtt{z}_n) = 0,\\
&\theta(\mathtt{t},\mathtt{s},\mathtt{z}_1,\mathtt{z}_2,\mathtt{z}_3,\cdots,\mathtt{z}_{n-2},\mathtt{z}_{n-1},0,\mathtt{z}_n) = 0,
\end{aligned}$$

if n is odd and

$$\begin{aligned}
&\theta(0,\mathtt{t},\mathtt{s},\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_{n-3},\mathtt{z}_{n-2},\mathtt{z}_{n-1},\mathtt{z}_n) = 0,\\
&\theta(\mathtt{t},\mathtt{s},0,\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_{n-3},\mathtt{z}_{n-2},\mathtt{z}_{n-1},\mathtt{z}_n) = 0,\\
&\theta(\mathtt{t},\mathtt{s},\mathtt{z}_1,\mathtt{z}_2,0,\cdots,\mathtt{z}_{n-3},\mathtt{z}_{n-2},\mathtt{z}_{n-1},\mathtt{z}_n) = 0,\\
&\qquad\vdots\\
&\theta(\mathtt{t},\mathtt{s},\mathtt{z}_1,\mathtt{z}_2,\mathtt{z}_3,\cdots,\mathtt{z}_{n-2},0,\mathtt{z}_{n-1},\mathtt{z}_n) = 0,\\
&\theta(\mathtt{t},\mathtt{s},\mathtt{z}_1,\mathtt{z}_2,\mathtt{z}_3,\cdots,\mathtt{z}_{n-2},\mathtt{z}_{n-1},\mathtt{z}_n,0) = 0,
\end{aligned}$$

if n is even.

Definition 3.4. [37] Let θ be a continuous function in Θ and φ be a comparison function. A mapping $\aleph : \mathtt{A} \to \mathtt{B}$ is said to be a generalized almost (φ, θ)-contraction if

$$\mathtt{d}(\aleph z_1, \aleph z_2) = \varphi\big(\mathtt{d}(z_1, z_2)\big) + \theta\big(\mathtt{d}(z_2, \aleph z_1) - \mathtt{d}(\mathtt{A}, \mathtt{B}), \mathtt{d}(z_1, \aleph z_2) - \mathtt{d}(\mathtt{A}, \mathtt{B}),$$
$$\mathtt{d}(z_1, \aleph z_1) - \mathtt{d}(\mathtt{A}, \mathtt{B}), \mathtt{d}(z_2, \aleph z_2) - \mathtt{d}(\mathtt{A}, \mathtt{B})\big)$$

for all $z_1, z_2 \in \mathtt{A}$.

Definition 3.5. [38] Let $(\mathtt{Y}, \mathtt{d})$ be a metric space with $\mathtt{A} \neq \emptyset$ and $\mathtt{B} \neq \emptyset$ are closed subsets. Let $\aleph : \mathtt{Y}^2 \to \mathtt{Y}$ be a mapping such that $\mathtt{d}\big(z_1, \aleph(z_1, z_2)\big) = \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big)$ and $\mathtt{d}\big(z_2, \aleph(z_2, z_1)\big) = \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big)$. Then $\aleph$ has a coupled best proximity point (z_1, z_2).

Definition 3.6. [27] An element $(z_1, z_2, z_3, \cdots, z_{n-1}, z_n) \in \mathtt{Y}^n$ is called an n-fixed point of a mapping $\aleph : \mathtt{Y}^n \to \mathtt{Y}$ if

$$\aleph(z_1, z_2, z_3, z_4, \cdots, z_{n-1}, z_n) = z_1,\ \aleph(z_2, z_1, z_3, z_4, \cdots, z_{n-1}, z_n) = z_2,$$
$$\aleph(z_3, z_2, z_1, z_4, \cdots, z_{n-1}, z_n) = z_3,\ \aleph(z_4, z_1, z_3, z_1, \cdots, z_{n-1}, z_n) = z_4,$$
$$\vdots$$
$$\aleph(z_{n-1}, z_2, z_3, z_4, \cdots, z_1, z_n) = z_{n-1},\ \aleph(z_n, z_2, z_3, z_4, \cdots, z_{n-1}, z_1) = z_n.$$

Now we define n-best proximity point as follows:

Definition 3.7. Let $(\mathtt{Y}, \mathtt{d})$ be a metric space with $\mathtt{A} \neq \emptyset$ and $\mathtt{B} \neq \emptyset$ are closed subsets. Let $\aleph : \mathtt{Y}^n \to \mathtt{Y}$ be a mapping such that

$$\mathtt{d}\big(z_1, \aleph(z_1, z_2, z_3, \cdots, z_{n-1}, z_n)\big) = \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),\ \mathtt{d}\big(z_2, \aleph(z_2, z_1, z_3, \cdots, z_{n-1}, z_n)\big) = \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),$$
$$\mathtt{d}\big(z_3, \aleph(z_3, z_2, z_1, \cdots, z_{n-1}, z_n)\big) = \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),\ \mathtt{d}\big(z_4, \aleph(z_4, z_2, z_3, z_1, \cdots, z_{n-1}, z_n)\big) = \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),$$
$$\vdots$$
$$\mathtt{d}\big(z_{n-1}, \aleph(z_{n-1}, z_2, z_3, \cdots, z_1, z_n)\big) = \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),\ \mathtt{d}\big(z_n, \aleph(z_n, z_2, z_3, \cdots, z_{n-1}, z_1)\big) = \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big).$$

Then $\aleph$ has an n-best proximity point $(z_1, z_2, z_3, \cdots, z_{n-1}, z_n)$.

If $\mathtt{A} = \mathtt{B}$, then n-best proximity point $(z_1, z_2, z_3, \cdots, z_{n-1}, z_n)$ of $\aleph$ is an n-fixed pooint of $\aleph$.

Theorem 3.8. Let $(\mathtt{Y}, \mathtt{d})$ be a complete metric space. Let $\mathtt{A} \neq \emptyset$, $\mathtt{B} \neq \emptyset$ are closed subsets with $\mathtt{A}_0$ and $\mathtt{B}_0$ are nonempty. Let $\aleph : \mathtt{Y}^n \to \mathtt{Y}$ be a continuous mapping which satisfies

(*i*) $\aleph(\mathtt{A}_0 \times \mathtt{B}_0 \times \mathtt{A}_0 \times \cdots \times \mathtt{B}_0 \times \mathtt{A}_0(n \text{ products})) \subseteq \mathtt{B}_0$,

(*ii*) $\aleph(\mathtt{B}_0 \times \mathtt{A}_0 \times \mathtt{B}_0 \times \cdots \times \mathtt{A}_0 \times \mathtt{B}_0(n \text{ products})) \subseteq \mathtt{A}_0$,

(*iii*) Pair $(\mathtt{A}, \mathtt{B})$ has the $(\mathcal{P})$-property.

Let θ be a continuous function in Θ and φ be a comparison function satisfying

$$\mathtt{d}\big(\aleph(z_1, z_2, z_3, \cdots, z_{n-1}, z_n), \aleph(y_1, y_2, y_3, \cdots, y_{n-1}, y_n)\big) \leq \varphi(\eta_1) + \theta(\eta_2), \tag{3.1}$$

where $\eta_1 = \max_{1 \leq i \leq n} \mathtt{d}(z_i, y_i)$ and

$$\eta_2 = \Big(\mathtt{d}\big(y_1, \aleph(z_1, z_2, z_3, \cdots, z_{n-1}, z_n)\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big), \mathtt{d}\big(y_2, \aleph(z_2, z_1, z_3, \cdots, z_{n-1}, z_n)\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),$$
$$\mathtt{d}\big(y_3, \aleph(z_3, z_2, z_1, \cdots, z_{n-1}, z_n)\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big), \cdots, \mathtt{d}\big(y_n, \aleph(z_n, z_2, z_3, \cdots, z_{n-1}, z_1)\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),$$
$$\mathtt{d}\big(z_1, \aleph(z_1, z_2, z_3, \cdots, z_{n-1}, z_n)\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big), \mathtt{d}\big(z_2, \aleph(z_2, z_1, z_3, \cdots, z_{n-1}, z_n)\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),$$
$$\mathtt{d}\big(z_3, \aleph(z_3, z_2, z_1, \cdots, z_{n-1}, z_n)\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big), \cdots, \mathtt{d}\big(z_n, \aleph(z_n, z_2, z_3, \cdots, z_{n-1}, z_1)\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big)\Big)$$

for all $z_i, y_i \in \mathtt{Y}$. Then $(y_1, y_1, y_1, \cdots, y_1, y_1)$ is the unique n-best proximity point of $\aleph$.

Proof . Choose $z_{0,2i} \in A_0$ and $z_{0,2i-1} \in B_0$. Then, $\aleph(z_{0,2}, z_{0,1}, z_{0,3}, \cdots, z_{0,n-1}, z_{0,n})$, $\aleph(z_{0,4}, z_{0,1}, z_{0,2}, \cdots, z_{0,n-1}, z_{0,n})$, $\aleph(z_{0,6}, z_{0,1}, z$ $\cdot, z_{0,n-1}, z_{0,n}), \cdot \quad \cdot \quad \cdot \in B_0$ and $\aleph(z_{0,1}, z_{0,2}, z_{0,3}, \cdot \cdot \cdot, z_{0,n-1}, z_{0,n})$, $\aleph(z_{0,3}, z_{0,2}, z_{0,3}, \cdots, z_{0,n-1}, z_{0,n})$, $\aleph(z_{0,5}, z_{0,2}, z_{0,3}, \cdots, z_{0,n-1}, z_{0,n}), \cdots \in A_0$. So, there exists $z_{1,2i} \in A_0$ and $z_{1,2i-1} \in B_0$ such that

$$\begin{aligned}
&d\big(z_{1,1}, \aleph(z_{0,1}, z_{0,2}, z_{0,3}, \cdots, z_{0,n-1}, z_{0,n})\big) = d\big(z_{1,2}, \aleph(z_{0,2}, z_{0,1}, z_{0,3}, \cdots, z_{0,n-1}, z_{0,n})\big)\\
&d\big(z_{1,3}, \aleph(z_{0,3}, z_{0,2}, z_{0,1}, \cdots, z_{0,n-1}, z_{0,n})\big) = \cdots = d\big(z_{1,n}, \aleph(z_{0,n}, z_{0,2}, z_{0,3}, \cdots, z_{0,n-1}, z_{0,1})\big)\\
&= d(A, B).
\end{aligned}$$

Continuing this process, we construct n-sequences $(z_{k,2i})$ in A and $(z_{k,2i-1})$ in B for $i = 1, 2 \cdots, n$ such that

$$\begin{aligned}
&d\big(z_{k+1,1}, \aleph(z_{k,1}, z_{k,2}, z_{k,3}, \cdots, z_{k,n-1}, z_{k,n})\big) = d(A, B),\\
&d\big(z_{k+1,2}, \aleph(z_{k,2}, z_{k,1}, z_{k,3}, \cdots, z_{k,n-1}, z_{k,n})\big) = d(A, B),\\
&d\big(z_{k+1,3}, \aleph(z_{k,3}, z_{k,2}, z_{k,1}, \cdots, z_{k,n-1}, z_{k,n})\big) = d(A, B),\\
&\qquad\vdots\\
&d\big(z_{k+1,n-1}, \aleph(z_{k,n-1}, z_{k,2}, z_{k,3}, \cdots, z_{k,1}, z_{k,n})\big) = d(A, B),\\
&d\big(z_{k+1,n}, \aleph(z_{k,n}, z_{k,2}, z_{k,3}, \cdots, z_{k,n-1}, z_{k,1})\big) = d(A, B)
\end{aligned}$$

for $k = 0, 1, 2, \cdots$.

Case (i) Suppose there exists $k \in \mathbb{N}$ such that $d(z_{k,1}, z_{k+1,1}) = d(z_{k,2}, z_{k+1,2}) = \cdots = d(z_{k,n}, z_{k+1,n}) = 0$. Thus,

$$\begin{aligned}
d(A, B) &\le d\big(z_{k,1}, \aleph(z_{k,1}, z_{k,2}, z_{k,3}, \cdots, z_{k,n-1}, z_{k,n})\big)\\
&\le d(z_{k,1}, z_{k+1,1}) + d\big(z_{k+1,1}, \aleph(z_{k,1}, z_{k,2}, z_{k,3}, \cdots, z_{k,n-1}, z_{k,n})\big)\\
&= d(A, B).
\end{aligned}$$

Thus we have $d(A, B) = d\big(z_{k,1}, \aleph(z_{k,1}, z_{k,2}, z_{k,3}, \cdots, z_{k,n-1}, z_{k,n})\big)$. Similarly, we obtain

$$\begin{aligned}
&d(A, B) = d\big(z_{k,2}, \aleph(z_{k,2}, z_{k,1}, z_{k,3}, \cdots, z_{k,n-1}, z_{k,n})\big),\\
&d(A, B) = d\big(z_{k,3}, \aleph(z_{k,3}, z_{k,2}, z_{k,1}, \cdots, z_{k,n-1}, z_{k,n})\big),\\
&\qquad\vdots\\
&d(A, B) = d\big(z_{k,n}, \aleph(z_{k,n}, z_{k,2}, z_{k,3}, \cdots, z_{k,n-1}, z_{k,1})\big).
\end{aligned}$$

Therefore, $(z_{k,1}, z_{k,2}, z_{k,3}, \cdots, z_{k,n-1}, z_{k,n})$ is an n-best proximity point of $\aleph$.

Case (ii) Suppose that $d(z_{k,i}, z_{k+1,i}) > 0$ for some i. Since pair (A, B) has the $\mathcal{P}$-property, $d\big(z_{k,1}, \aleph(z_{k-1,1}, z_{k-1,2}, z_{k-1,3}, \cdots, z_{k-1,n-1}, z_{k-1,n})\big) = d(A, B)$ and $d\big(z_{k+1,1}, \aleph(z_{k,1}, z_{k,2}, z_{k,3}, \cdots, z_{k,n-1}, z_{k,n})\big) = d(A, B)$ and using (3.1), we have

$$\begin{aligned}
d(z_{k,1}, z_{k+1,1}) &= d\big(\aleph(z_{k-1,1}, z_{k-1,2}, z_{k-1,3}, \cdots, z_{k-1,n-1}, z_{k-1,n}), \aleph(z_{k,1}, z_{k,2}, z_{k,3}, \cdots, z_{k,n-1}, z_{k,n})\big)\\
&\le \varphi\big(\max_{1\le i\le n} d(z_{k-1,i}, z_{k,i})\big) + \theta(\eta_3),\\
&= \varphi\big(\max_{1\le i\le n} d(z_{k-1,i}, z_{k,i})\big),
\end{aligned}$$

where

$$\begin{aligned}
\eta_3 = \Big(&d\big(z_{k,1}, \aleph(z_{k-1,1}, z_{k-1,2}, z_{k-1,3}, \cdots, z_{k-1,n-1}, z_{k-1,n})\big) - d\big(A, B\big),\\
&d\big(2_{k,2}, \aleph(z_{k-1,2}, z_{k-1,1}, z_{k-1,3}, \cdots, z_{k-1,n-1}, z_{k-1,n})\big) - d\big(A, B\big),\\
&d\big(z_{k,3}, \aleph(z_{k-1,3}, z_2, z_{k-1,1}, \cdots, z_{k-1,n-1}, z_{k-1,n})\big) - d\big(A, B\big),\\
&\qquad\vdots\\
&d\big(z_{k,n}, \aleph(z_{k-1,n}, z_{k-1,2}, z_{k-1,3}, \cdots, z_{k-1,n-1}, z_{k-1,1})\big) - d\big(A, B\big),\\
&d\big(z_{k-1,1}, \aleph(z_{k-1,1}, z_{k-1,2}, z_{k-1,3}, \cdots, z_{k-1,n-1}, z_{k-1,n})\big) - d\big(A, B\big),\\
&d\big(2_{k-1,2}, \aleph(z_{k-1,2}, z_{k-1,1}, z_{k-1,3}, \cdots, z_{k-1,n-1}, z_{k-1,n})\big) - d\big(A, B\big),\\
&d\big(z_{k-1,3}, \aleph(z_{k-1,3}, z_2, z_{k-1,1}, \cdots, z_{k-1,n-1}, z_{k-1,n})\big) - d\big(A, B\big),\\
&\qquad\vdots\\
&d\big(z_{k-1,n}, \aleph(z_{k-1,n}, z_{k-1,2}, z_{k-1,3}, \cdots, z_{k-1,n-1}, z_{k-1,1})\big) - d\big(A, B\big)\Big).
\end{aligned}$$

Since $\mathtt{d}\big(\mathtt{z}_{k,1}, \aleph(\mathtt{z}_{k-1,1}, \mathtt{z}_{k-1,2}, \mathtt{z}_{k-1,3}, \cdots, \mathtt{z}_{k-1,n-1}, \mathtt{z}_{k-1,n})\big) = \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big)$, it follows from the property of θ that $\theta(\eta_3) = 0$. Similarly, we can prove that

$$\mathtt{d}(\mathtt{z}_{k,\mathtt{j}}, \mathtt{z}_{k+1,\mathtt{j}}) \leq \varphi\big(\max_{1\leq i\leq n} \mathtt{d}(\mathtt{z}_{k-1,i}, \mathtt{z}_{k,i})\big),\ \mathtt{j} = 2, 3, \cdots, n.$$

Combining above all, we get

$$\max_{1\leq \mathtt{j}\leq n} \mathtt{d}(\mathtt{z}_{k,\mathtt{j}}, \mathtt{z}_{k+1,\mathtt{j}}) \leq \varphi\big(\max_{1\leq i\leq n} \mathtt{d}(\mathtt{z}_{k-1,i}, \mathtt{z}_{k,i})\big). \tag{3.2}$$

Repeating (3.2) k-times, we obtain

$$\begin{aligned}\max_{1\leq \mathtt{j}\leq n} \mathtt{d}(\mathtt{z}_{k,\mathtt{j}}, \mathtt{z}_{k+1,\mathtt{j}}) &\leq \varphi\big(\max_{1\leq i\leq n} \mathtt{d}(\mathtt{z}_{k-1,i}, \mathtt{z}_{k,i})\big)\\ &\leq \varphi^2\big(\max_{1\leq i\leq n} \mathtt{d}(\mathtt{z}_{k-2,i}, \mathtt{z}_{k-1,i})\big)\\ &\quad\vdots\\ &\leq \varphi^k\big(\max_{1\leq i\leq n} \mathtt{d}(\mathtt{z}_{0,i}, \mathtt{z}_{1,i})\big).\end{aligned}$$

Thus,

$$\lim_{k\to+\infty} \mathtt{d}(\mathtt{z}_{k,\mathtt{j}}, \mathtt{z}_{k+1,\mathtt{j}}) = 0,\ \mathtt{j} = 1, 2, \cdots, n.$$

On other hand,

$$\begin{aligned}\mathtt{d}(\mathtt{A}, \mathtt{B}) &\leq \mathtt{d}\big(\mathtt{z}_{k,1}, \aleph(\mathtt{z}_{k,1}, \mathtt{z}_{k,2}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,n})\big)\\ &\leq \mathtt{d}(\mathtt{z}_{k,1}, \mathtt{z}_{k+1,1}) + \mathtt{d}\big(\mathtt{z}_{k+1,1}, \aleph(\mathtt{z}_{k,1}, \mathtt{z}_{k,2}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,n})\big)\\ &\leq \mathtt{d}(\mathtt{z}_{k,1}, \mathtt{z}_{k+1,1}) + \mathtt{d}(\mathtt{A}, \mathtt{B}).\end{aligned}$$

Letting $k \to +\infty$ in the above inequalities, we get

$$\mathtt{d}\big(\mathtt{z}_{k,1}, \aleph(\mathtt{z}_{k,1}, \mathtt{z}_{k,2}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,n})\big) = \mathtt{d}(\mathtt{A}, \mathtt{B}).$$

Similary, we can prove

$$\mathtt{d}\big(\mathtt{z}_{k,\mathtt{j}}, \aleph(\mathtt{z}_{k,1}, \mathtt{z}_{k,2}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,n})\big) = \mathtt{d}(\mathtt{A}, \mathtt{B}),\ \mathtt{j} = 2, 3, \cdots, n.$$

Consider $\varepsilon > 0$. Since $\varphi^k\big(\max_{1\leq i\leq n} \mathtt{d}(\mathtt{z}_{0,i}, \mathtt{z}_{1,i})\big)$ as $k \to +\infty$, there exists $k_0 \in \mathbb{N}$ such that

$$\mathtt{d}(\mathtt{z}_{k,i}, \mathtt{z}_{k+1,i}) < \frac{1}{2}(\varepsilon - \varphi(\varepsilon))$$

hold for all $k \geq k_0$ and $i = 1, 2, \cdots, n$.

Claim: $\max_{1\leq i\leq n} \mathtt{d}(\mathtt{z}_{k,i}, \mathtt{z}_{m,i}) < \varepsilon$ for all $m > k \geq k_0$. (3.3)

We use the induction on m to prove above claim. Assume inequality (3.3) holds for $m = p$. Now, we prove relation (3.3) for $m = p + 1$. By using the triangular inequality, we have

$$\mathtt{d}(\mathtt{z}_{k,1}, \mathtt{z}_{p+1,1}) \leq \mathtt{d}(\mathtt{z}_{k,1}, \mathtt{z}_{k+1,1}) + \mathtt{d}(\mathtt{z}_{k+1,1}, \mathtt{z}_{p+1,1}). \tag{3.4}$$

Since pair $(\mathtt{A}, \mathtt{B})$ has the $\mathcal{P}$-property, $\mathtt{d}\big(\mathtt{z}_{k+1,1}, \aleph(\mathtt{z}_{k,1}, \mathtt{z}_{k,2}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,n})\big) = \mathtt{d}\big(\mathtt{z}_{p+1,1}, \aleph(\mathtt{z}_{p,1}, \mathtt{z}_{p,2}, \mathtt{z}_{p,3}, \cdots, \mathtt{z}_{p,n-1}, \mathtt{z}_{p,n})\big) = \mathtt{d}(\mathtt{A}, \mathtt{B})$ and using (3.1), we have

$$\begin{aligned}\mathtt{d}(\mathtt{z}_{k+1,1}, \mathtt{z}_{p+1,1}) &= \mathtt{d}\big(\aleph(\mathtt{z}_{k,1}, \mathtt{z}_{k,2}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,n}), \aleph(\mathtt{z}_{p,1}, \mathtt{z}_{p,2}, \mathtt{z}_{p,3}, \cdots, \mathtt{z}_{p,n-1}, \mathtt{z}_{p,n})\big)\\ &\leq \varphi\big(\max_{1\leq i\leq n} \mathtt{d}(\mathtt{z}_{k,i}, \mathtt{z}_{p,i})\big) + \theta(\ell_1),\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}\ell_1 = \Big(& \mathtt{d}\big(\mathtt{z}_{p,1}, \aleph(\mathtt{z}_{k,1}, \mathtt{z}_{k,2}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,n})\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),\\ & \mathtt{d}\big(\mathtt{z}_{p,2}, \aleph(\mathtt{z}_{k,2}, \mathtt{z}_{k,1}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,n})\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),\\ & \vdots \\ & \mathtt{d}\big(\mathtt{z}_{p,n}, \aleph(\mathtt{z}_{k,n}, \mathtt{z}_{k,2}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,1})\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),\\ & \mathtt{d}\big(\mathtt{z}_{k,1}, \aleph(\mathtt{z}_{k,1}, \mathtt{z}_{k,2}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,n})\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),\\ & \mathtt{d}\big(\mathtt{z}_{k,2}, \aleph(\mathtt{z}_{k,2}, \mathtt{z}_{k,1}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,n})\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),\\ & \vdots \\ & \mathtt{d}\big(\mathtt{z}_{k,n}, \aleph(\mathtt{z}_{k,n}, \mathtt{z}_{k,2}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,1})\big) - \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big)\Big).\end{aligned}$$

By using the properties of θ and $\lim\limits_{k\to+\infty} \mathtt{d}\big(\mathtt{z}_{k,1}, \aleph(\mathtt{z}_{k,1}, \mathtt{z}_{k,2}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,n})\big) = \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big)$, we get

$$\limsup_{k\to+\infty} \theta(\ell_1) = 0.$$

Similarly, we can have

$$\begin{aligned}\mathtt{d}(\mathtt{z}_{k+1,i}, \mathtt{z}_{p+1,i}) &= \mathtt{d}\big(\aleph(\mathtt{z}_{k,1}, \mathtt{z}_{k,2}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,n}), \aleph(\mathtt{z}_{p,1}, \mathtt{z}_{p,2}, \mathtt{z}_{p,3}, \cdots, \mathtt{z}_{p,n-1}, \mathtt{z}_{p,n})\big)\\ &\le \varphi\big(\max_{1\le i\le n} \mathtt{d}(\mathtt{z}_{k,i}, \mathtt{z}_{p,i})\big) + \theta(\ell_i),\ i = 2, 3, \cdots, n\end{aligned} \tag{3.6}$$

and

$$\limsup_{k\to+\infty} \theta(\ell_i) = 0, \quad i = 2, 3, \cdots, n.$$

Thus for k_0 large enough, we have

$$\theta(\ell_i) < \frac{1}{2}(\varepsilon - \varphi(\varepsilon)),\ i = 1, 2, \cdots, n.$$

Combining all the relations, from $i = 1$ to n, we obtain

$$\max_{1\le i\le n} \mathtt{d}(\mathtt{z}_{k,i}, \mathtt{z}_{p+1,i}) < \varepsilon.$$

Hence, the claim. Thus $(\mathtt{z}_{k,2i})$ and $(\mathtt{z}_{k,2i-1})$ are Cauchy sequences in $\mathtt{A}$ and $\mathtt{B}$ respectively. Since $(\mathtt{Y}, \mathtt{d})$ is complete, there exist $\mathtt{y}_i \in \mathtt{Y}$, $i = 1, 2, \cdots, n$ such that

$$\lim_{k\to+\infty} \mathtt{z}_{k,i} = \mathtt{y}_i,\ i = 1, 2, \cdots, n.$$

Since $\mathtt{A}$ and $\mathtt{B}$ are closed, we get $\mathtt{y}_{2i} \in \mathtt{A}$ and $\mathtt{y}_{2i-1} \in \mathtt{B}$. Since $\aleph$ is continuous,

$$\lim_{k\to+\infty} \mathtt{d}\big(\mathtt{z}_{k,1}, \aleph(\mathtt{z}_{k,1}, \mathtt{z}_{k,2}, \mathtt{z}_{k,3}, \cdots, \mathtt{z}_{k,n-1}, \mathtt{z}_{k,n})\big) = \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),$$

implies

$$\mathtt{d}\big(\mathtt{y}_1, \aleph(\mathtt{y}_1, \mathtt{y}_2, \mathtt{y}_3, \cdots, \mathtt{y}_{n-1}, \mathtt{y}_n)\big) = \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big).$$

Similarly, we can prove

$$\begin{aligned}\mathtt{d}\big(\mathtt{y}_2, \aleph(\mathtt{y}_2, \mathtt{y}_1, \mathtt{y}_3, \cdots, \mathtt{y}_{n-1}, \mathtt{y}_n)\big) &= \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),\\ \mathtt{d}\big(\mathtt{y}_3, \aleph(\mathtt{y}_3, \mathtt{y}_2, \mathtt{y}_1, \cdots, \mathtt{y}_{n-1}, \mathtt{y}_n)\big) &= \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big),\\ &\vdots\\ \mathtt{d}\big(\mathtt{y}_n, \aleph(\mathtt{y}_n, \mathtt{y}_2, \mathtt{y}_3, \cdots, \mathtt{y}_{n-1}, \mathtt{y}_1)\big) &= \mathtt{d}\big(\mathtt{A}, \mathtt{B}\big).\end{aligned}$$

Thus, $(\mathtt{y}_1, \mathtt{y}_2, \mathtt{y}_3, \cdots, \mathtt{y}_{n-1}, \mathtt{y}_n)$ is an n-best proximity point of $\aleph$. Now, we show that $\mathtt{y}_1 = \mathtt{y}_2 = \mathtt{y}_3 = \cdots = \mathtt{y}_{n-1} = \mathtt{y}_n$. Using the $\mathcal{P}$-property of pair $(\mathtt{A}, \mathtt{B})$, we get

$$\mathtt{d}(\mathtt{y}_1, \mathtt{y}_2) = \mathtt{d}\big(\aleph(\mathtt{y}_1, \mathtt{y}_2, \mathtt{y}_3, \cdots, \mathtt{y}_{n-1}, \mathtt{y}_n), \aleph(\mathtt{y}_2, \mathtt{y}_1, \mathtt{y}_3, \cdots, \mathtt{y}_{n-1}, \mathtt{y}_n)\big).$$

Using inequality (3.1), we get

$$\begin{aligned} \mathtt{d}(\mathtt{y}_1,\mathtt{y}_2) &= \mathtt{d}\big(\aleph(\mathtt{y}_1,\mathtt{y}_2,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n),\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)\\ &\le \varphi\big(\max\{\mathtt{d}(\mathtt{y}_1,\mathtt{y}_2),\mathtt{d}(\mathtt{y}_2,\mathtt{y}_1)\}\big)+\theta(\ell)\\ &\le \varphi\big(\mathtt{d}(\mathtt{y}_1,\mathtt{y}_2)\big), \end{aligned} \tag{3.7}$$

since

$$\begin{aligned} \ell = \Big(&\mathtt{d}\big(\mathtt{y}_2,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big),\mathtt{d}\big(\mathtt{y}_1,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big),\\ &\mathtt{d}\big(\mathtt{y}_3,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big),\cdots,\mathtt{d}\big(\mathtt{y}_n,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big),\\ &\mathtt{d}\big(\mathtt{y}_1,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big),\mathtt{d}\big(\mathtt{y}_2,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big),\\ &\mathtt{d}\big(\mathtt{y}_3,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big),\cdots,\mathtt{d}\big(\mathtt{y}_n,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big)\Big)\\ = \Big(&0,\mathtt{d}\big(\mathtt{y}_1,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big),\mathtt{d}\big(\mathtt{y}_3,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big),\\ &\cdots,\mathtt{d}\big(\mathtt{y}_n,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big),\mathtt{d}\big(\mathtt{y}_1,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big),\\ &0,\mathtt{d}\big(\mathtt{y}_3,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big),\cdots,\mathtt{d}\big(\mathtt{y}_n,\aleph(\mathtt{y}_2,\mathtt{y}_1,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)-\mathtt{d}\big(\mathtt{A},\mathtt{B}\big)\Big). \end{aligned}$$

So, $\theta(\ell)=0$. Since $\varphi(\mathtt{t})<\mathtt{t}$ for all $\mathtt{t}>0$, we have from (3.7) that

$$\mathtt{d}(\mathtt{y}_1,\mathtt{y}_2)\le\varphi\big(\mathtt{d}(\mathtt{y}_1,\mathtt{y}_2)\big)<\mathtt{d}(\mathtt{y}_1,\mathtt{y}_2)\implies \mathtt{y}_1=\mathtt{y}_2.$$

Similarly, we can prove $\mathtt{y}_1=\mathtt{y}_2=\mathtt{y}_3=\cdots=\mathtt{y}_n$. To prove the uniqueness, let ζ be another n-best proximity point. Then,

$$\begin{aligned} \mathtt{d}(\mathtt{y}_1,\zeta) &= \mathtt{d}\big(\aleph(\mathtt{y}_1,\mathtt{y}_1,\mathtt{y}_1,\cdots,\mathtt{y}_1,\mathtt{y}_1),\aleph(\zeta,\zeta,\zeta,\cdots,\zeta,\zeta)\big)\\ &\le\varphi\big(\mathtt{d}(\mathtt{y}_1,\zeta)\big). \end{aligned}$$

Again, since $\varphi(\mathtt{t})<\mathtt{t}$ for all $\mathtt{t}>0$, we conclude that $\mathtt{d}(\mathtt{y}_1,\zeta)=0$. Thus, $\mathtt{y}_1=\zeta$. The proof is completed. □

Corollary 3.9. Let $(\mathtt{Y},\mathtt{d})$ be a complete metric space. Let $\mathtt{A}\ne\emptyset$, $\mathtt{B}\ne\emptyset$ are closed subsets with $\mathtt{A}_0$ and $\mathtt{B}_0$ are nonempty. Let $\aleph:\mathtt{Y}^n\to\mathtt{Y}$ be a continuous mapping which satisfies

(*i*) $\aleph(\mathtt{A}_0\times\mathtt{A}_0\times\cdots\times\mathtt{A}_0(n \text{ products}))\subseteq\mathtt{B}_0$ (or) $\aleph(\mathtt{B}_0\times\mathtt{B}_0\times\cdots\times\mathtt{B}_0(n \text{ products}))\subseteq\mathtt{A}_0$,

(*ii*) Pair $(\mathtt{A},\mathtt{B})$ has the $\mathcal{P}$-property.

Let θ be a continuous function in Θ and φ be a comparison function satisfying

$$\mathtt{d}\big(\aleph(\mathtt{z}_1,\mathtt{z}_2,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n),\aleph(\mathtt{y}_1,\mathtt{y}_2,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)\le\varphi(\eta_1)+\theta(\eta_2),\quad \text{for all}\quad \mathtt{z}_i,\mathtt{y}_i\in\mathtt{Y}.$$

where η_1 and η_2 are defined in Theorem 3.8. Then $(\mathtt{y}_1,\mathtt{y}_1,\mathtt{y}_1,\cdots,\mathtt{y}_1,\mathtt{y}_1)$ is the unique n-best proximity point of $\aleph$.

Proof . Choose $\mathtt{z}_{0,i}\in\mathtt{A}_0$. Since $\aleph(\mathtt{A}_0\times\mathtt{A}_0\times\cdots\times\mathtt{A}_0)\subseteq\mathtt{B}_0$, we get, $\aleph(\mathtt{z}_{0,1},\mathtt{z}_{0,2},\mathtt{z}_{0,3},\cdots,\mathtt{z}_{0,n-1},\mathtt{z}_{0,n})$, $\aleph(\mathtt{z}_{0,2},\mathtt{z}_{0,1},\mathtt{z}_{0,3},\cdots,\mathtt{z}_{0,n-1},\mathtt{z}_{0,n})$, $\cdots$ $\aleph(\mathtt{z}_{0,n},\mathtt{z}_{0,2},\mathtt{z}_{0,3},\cdots,\mathtt{z}_{0,n-1},\mathtt{z}_{0,1})\in\mathtt{B}_0$. Then by following Theorem 3.8, we get that $(\mathtt{y}_1,\mathtt{y}_2,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)$ is the unique n-best proximity point of $\aleph$. □

Take $\mathtt{B}=\mathtt{A}$ in Theorem 3.8, we have the following result.

Corollary 3.10. Let $\mathtt{A}$ a closed subsets of a complete metric space $(\mathtt{Y},\mathtt{d})$. Let $\aleph:\mathtt{Y}^n\to\mathtt{Y}$ be a continuous mapping with $\aleph(\mathtt{A}\times\mathtt{A}\times\cdots\times\mathtt{A}(n\,\text{times}))\subseteq\mathtt{A}$. Suppose there exists a comparison function φ and $\theta\in\Theta$ such that

$$\mathtt{d}\big(\aleph(\mathtt{z}_1,\mathtt{z}_2,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n),\aleph(\mathtt{y}_1,\mathtt{y}_2,\mathtt{y}_3,\cdots,\mathtt{y}_{n-1},\mathtt{y}_n)\big)\le\varphi(\eta_1)+\theta(\eta_3),\quad \text{for all}\quad \mathtt{z}_i,\mathtt{y}_i\in\mathtt{Y}.$$

where η_1 is defined in Theorem 3.8 and

$$\begin{aligned} \eta_3 = \Big(&\mathtt{d}\big(\mathtt{y}_1,\aleph(\mathtt{z}_1,\mathtt{z}_2,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n)\big),\mathtt{d}\big(\mathtt{y}_2,\aleph(\mathtt{z}_2,\mathtt{z}_1,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n)\big),\\ &\mathtt{d}\big(\mathtt{y}_3,\aleph(\mathtt{z}_3,\mathtt{z}_2,\mathtt{z}_1,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n)\big),\cdots,\mathtt{d}\big(\mathtt{y}_n,\aleph(\mathtt{z}_n,\mathtt{z}_2,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_1)\big),\\ &\mathtt{d}\big(\mathtt{z}_1,\aleph(\mathtt{z}_1,\mathtt{z}_2,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n)\big),\mathtt{d}\big(\mathtt{z}_2,\aleph(\mathtt{z}_2,\mathtt{z}_1,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n)\big),\\ &\mathtt{d}\big(\mathtt{z}_3,\aleph(\mathtt{z}_3,\mathtt{z}_2,\mathtt{z}_1,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n)\big),\cdots,\mathtt{d}\big(\mathtt{z}_n,\aleph(\mathtt{z}_n,\mathtt{z}_2,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_1)\big)\Big). \end{aligned}$$

Then there exists a unique n-fixed point of $\aleph$.

4 Existence of Solutions for System of Fractional Order BVPs (1.1)–(1.2)

In this section, we derive necessary conditions for the existence of solutions for the problem (1.1)–(1.2). On the set $\mathtt{Y} = C([0,1],\mathbb{R})$ of all continuous real-valued functions defined on $[0,1]$, we take the metric $\mathtt{d}$ on $\mathtt{Y}$ defined by

$$\mathtt{d}(\mathtt{z},\mathtt{y}) = \sup_{\mathtt{s}\in[0,1]} |\mathtt{z}(\mathtt{s}) - \mathtt{y}(\mathtt{s})|$$

for all $\mathtt{z},\mathtt{y} \in \mathtt{Y}$.

Define an operator $\aleph : \mathtt{Y}^n \to \mathtt{Y}$ by

$$\aleph(\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n) = \sum_{i=1}^{n} \aleph_i \mathtt{z}_i,$$

where

$$\begin{aligned}
\aleph_1\mathtt{z}_1(\mathtt{s}) = &\int_0^1 \mathtt{G}_1(\mathtt{s},\mathtt{t})\mathtt{g}_1(\mathtt{t},\mathtt{z}_1(\mathtt{t}),\mathtt{z}_2(\mathtt{t}),\cdots,\mathtt{z}_n(\mathtt{t}))\mathtt{dt}\\
&- \mathtt{s}^{ß_1-1}\sum_{\mathtt{j}=1}^{\infty}\delta_{1\mathtt{j}}\int_0^{\sigma_1(\tau_{1\mathtt{j}})}\frac{(\sigma_1(\tau_{1\mathtt{j}})-\mathtt{t})^{ß_1-1}}{\Gamma(ß_1)}\mathtt{g}_1(\mathtt{t},\mathtt{z}_1(\mathtt{t}),\mathtt{z}_2(\mathtt{t}),\cdots,\mathtt{z}_n(\mathtt{t}))\mathtt{dt},\\
\aleph_2\mathtt{z}_2(\mathtt{s}) = &\int_0^1 \mathtt{G}_2(\mathtt{s},\mathtt{t})\mathtt{g}_2(\mathtt{t},\mathtt{z}_2(\mathtt{t}),\mathtt{z}_1(\mathtt{t}),\cdots,\mathtt{z}_n(\mathtt{t}))\mathtt{dt}\\
&- \mathtt{s}^{ß_2-1}\sum_{\mathtt{j}=1}^{\infty}\delta_{2\mathtt{j}}\int_0^{\sigma_2(\tau_{2\mathtt{j}})}\frac{(\sigma_2(\tau_{2\mathtt{j}})-\mathtt{t})^{ß_2-1}}{\Gamma(ß_2)}\mathtt{g}_2(\mathtt{t},\mathtt{z}_2(\mathtt{t}),\mathtt{z}_1(\mathtt{t}),\cdots,\mathtt{z}_n(\mathtt{t}))\mathtt{dt},\\
&\vdots\\
\aleph_n\mathtt{z}_n(\mathtt{s}) = &\int_0^1 \mathtt{G}_n(\mathtt{s},\mathtt{t})\mathtt{g}_n(\mathtt{t},\mathtt{z}_n(\mathtt{t}),\mathtt{z}_2(\mathtt{t}),\cdots,\mathtt{z}_1(\mathtt{t}))\mathtt{dt}\\
&- \mathtt{s}^{ß_n-1}\sum_{\mathtt{j}=1}^{\infty}\delta_{n\mathtt{j}}\int_0^{\sigma_n(\tau_{n\mathtt{j}})}\frac{(\sigma_n(\tau_{n\mathtt{j}})-\mathtt{t})^{ß_n-1}}{\Gamma(ß_n)}\mathtt{g}_n(\mathtt{t},\mathtt{z}_n(\mathtt{t}),\mathtt{z}_2(\mathtt{t}),\cdots,\mathtt{z}_1(\mathtt{t}))\mathtt{dt}.
\end{aligned}$$

Then from Lemma 2.7, it is clear that $(\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_n)$ is a solution of the system (1.1)–(1.2) if and only if it is a fixed point of $\aleph$.

Theorem 4.1. Suppose there exists $\mathtt{M} > 0$ such that

$$|\mathtt{g}_i(\mathtt{s},\mathtt{z}_i,\mathtt{z}_2,\cdots,\mathtt{z}_1,\cdots,\mathtt{z}_n) - \mathtt{g}_i(\mathtt{s},\mathtt{y}_i,\mathtt{y}_2,\cdots,\mathtt{y}_1,\cdots,\mathtt{y}_n)| \le \mathtt{M}\sum_{i=1}^{n}|\mathtt{z}_i - \mathtt{y}_i| \quad \text{for all} \quad \mathtt{z}_i,\mathtt{y}_i \in \mathtt{Y} \tag{4.1}$$

and $i = 1,2,\cdots,n$. Further, there exist $\wp_1 > 0$ and $0 < \wp_2 < 1$ such that

$$|\mathtt{g}_i(\mathtt{t},0,0,\cdots,0,0)| \le \wp_1$$

and

$$\sum_{i=1}^{n}\left[\mathtt{G}_i^\star + \frac{2\Gamma(ß_i)}{\Gamma(ß_i+1)\Gamma(ß_i-\alpha_i)}\right] \le \wp_2,$$

where $\mathtt{G}_i^\star = \max_{\mathtt{s}\in[0,1]}\int_0^1 \mathtt{G}_i(\mathtt{s},\mathtt{t})\mathtt{dt}$. Then there exists a unique solution to the boundary value problem (1.1)–(1.2).

Proof . Define a set $\mathtt{A}$ by $\mathtt{A} = \{\mathtt{z}\in\mathtt{Y} : |\mathtt{z}(s)| \le \wp_3 \text{ for } s\in[0,1]\}$, where $\wp_3 > \dfrac{\wp_1\wp_2}{1-n\wp_2}$. Then it is clear that $\mathtt{A}$ is a closed subset of $\mathtt{Y}$. Now, we prove $\aleph\big(\mathtt{A}\times\mathtt{A}\times\cdots\times\mathtt{A}(n\,\text{times})\big)\subseteq\mathtt{A}$. Let $(\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_n)\in\mathtt{A}^n$. Then,

$$
\begin{aligned}
&|\aleph(\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_n)| \le \sum_{i=1}^{n}|\aleph_i\mathtt{z}_i(\mathtt{s})| = \sum_{i=1}^{n}\bigg|\int_0^1 \mathtt{G}_i(\mathtt{s},\mathtt{t})\mathtt{g}_i(\mathtt{t},\mathtt{z}_i(\mathtt{t}),\mathtt{z}_2(\mathtt{t}),\cdots,\mathtt{z}_n(\mathtt{t}))d\mathtt{t}\\
&\quad - \mathtt{s}^{\text{ß}_i-1}\sum_{\mathtt{j}=1}^{\infty}\delta_{i\mathtt{j}}\int_0^{\sigma_i(\tau_{i\mathtt{j}})}\frac{(\sigma_i(\tau_{i\mathtt{j}})-\mathtt{t})^{\text{ß}_i-1}}{\Gamma(\text{ß}_i)}\mathtt{g}_i(\mathtt{t},\mathtt{z}_i(\mathtt{t}),\mathtt{z}_2(\mathtt{t}),\cdots,\mathtt{z}_n(\mathtt{t}))d\mathtt{t}\bigg|\\
&\le \sum_{i=1}^{n}\bigg[\int_0^1|\mathtt{G}_i(\mathtt{s},\mathtt{t})|\big[|\mathtt{g}_i(\mathtt{t},\mathtt{z}_i(\mathtt{t}),\mathtt{z}_2(\mathtt{t}),\cdots,\mathtt{z}_n(\mathtt{t}))-\mathtt{g}_i(\mathtt{t},0,0,\cdots,0)|+|\mathtt{g}_i(\mathtt{t},0,0,\cdots,0)|\big]d\mathtt{t}\\
&\quad + |\mathtt{s}|^{\text{ß}_i-1}\sum_{\mathtt{j}=1}^{\infty}\delta_{i\mathtt{j}}\int_0^{\sigma_i(\tau_{i\mathtt{j}})}\frac{(\sigma_i(\tau_{i\mathtt{j}})-\mathtt{t})^{\text{ß}_i-1}}{\Gamma(\text{ß}_i)}\big[|\mathtt{g}_i(\mathtt{t},\mathtt{z}_i(\mathtt{t}),\mathtt{z}_2(\mathtt{t}),\cdots,\mathtt{z}_n(\mathtt{t}))-\mathtt{g}_i(\mathtt{t},0,0,\cdots,0)|\\
&\qquad + |\mathtt{g}_i(\mathtt{t},0,0,\cdots,0)|\big]d\mathtt{t}\bigg]\\
&\le \sum_{i=1}^{n}\bigg[\int_0^1|\mathtt{G}_i(\mathtt{s},\mathtt{t})|\big[\sum_{\mathtt{j}=1}^{n}|\mathtt{z}_\mathtt{j}(\mathtt{t})|+|\mathtt{g}_i(\mathtt{t},0,0,\cdots,0)|\big]d\mathtt{t} + |\mathtt{s}|^{\text{ß}_i-1}\sum_{\mathtt{j}=1}^{\infty}\delta_{i\mathtt{j}}\int_0^{\sigma_i(\tau_{i\mathtt{j}})}\frac{(\sigma_i(\tau_{i\mathtt{j}})-\mathtt{t})^{\text{ß}_i-1}}{\Gamma(\text{ß}_i)}\\
&\qquad\times\big[\sum_{\mathtt{j}=1}^{n}|\mathtt{z}_\mathtt{j}(\mathtt{t})|+|\mathtt{g}_i(\mathtt{t},0,0,\cdots,0)|\big]d\mathtt{t}\bigg]\\
&\le \sum_{i=1}^{n}\bigg[\int_0^1|\mathtt{G}_i(\mathtt{s},\mathtt{t})|(n\wp_3+\wp_1)d\mathtt{t} + \sum_{\mathtt{j}=1}^{\infty}\delta_{i\mathtt{j}}\int_0^{\sigma_i(\tau_{i\mathtt{j}})}\frac{(\sigma_i(\tau_{i\mathtt{j}})-\mathtt{t})^{\text{ß}_i-1}}{\Gamma(\text{ß}_i)}(n\wp_3+\wp_1)d\mathtt{t}\bigg]\\
&\le (n\wp_3+\wp_1)\sum_{i=1}^{n}\left[\mathtt{G}_i^\star + \frac{1}{\Gamma(\text{ß}_i+1)}\sum_{\mathtt{j}=1}^{\infty}\delta_{i\mathtt{j}}(\sigma_i(\tau_{i\mathtt{j}}))^{\text{ß}_i-1}\right]\\
&= (n\wp_3+\wp_1)\sum_{i=1}^{n}\left[\mathtt{G}_i^\star + \frac{2\Gamma(\text{ß}_i)}{\Gamma(\text{ß}_i+1)\Gamma(\text{ß}_i-\alpha_i)}\right] = (n\wp_3+\wp_1)\wp_2 \le \wp_3.
\end{aligned}
$$

That is, $\aleph(\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_n)\in\mathtt{A}$. Therefore, $\aleph(\mathtt{A}^n)\subseteq\mathtt{A}$. Next, let $(\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_n),(\mathtt{y}_1,\mathtt{y}_2,\cdots,\mathtt{y}_n)\in\mathtt{Y}^n$. Then

$$
\begin{aligned}
&\mathtt{d}\big(\aleph(\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_n),\aleph(\mathtt{y}_1,\mathtt{y}_2,\cdots,\mathtt{y}_n)\big) = \sup_{\mathtt{s}\in[0,1]}|\aleph(\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_n)-\aleph(\mathtt{y}_1,\mathtt{y}_2,\cdots,\mathtt{y}_n)|\\
&\le \sup_{\mathtt{s}\in[0,1]}\sum_{i=1}^{n}|\aleph_i(\mathtt{z}_i(\mathtt{s})-\mathtt{y}_i(\mathtt{s}))|\\
&\le \sup_{\mathtt{s}\in[0,1]}\sum_{i=1}^{n}\bigg[\int_0^1|\mathtt{G}_i(\mathtt{s},\mathtt{t})|\big[|\mathtt{g}_i(\mathtt{t},\mathtt{z}_i(\mathtt{t}),\mathtt{z}_2(\mathtt{t}),\cdots,\mathtt{z}_1(\mathtt{t}),\cdots,\mathtt{z}_n(\mathtt{t}))\\
&\qquad - \mathtt{g}_i(\mathtt{t},\mathtt{y}_i(\mathtt{t}),\mathtt{y}_2(\mathtt{t}),\cdots,\mathtt{y}_1(\mathtt{t}),\cdots,\mathtt{y}_n(\mathtt{t}))|\big]d\mathtt{t}\\
&\quad + |\mathtt{s}|^{\text{ß}_i-1}\sum_{\mathtt{j}=1}^{\infty}\delta_{i\mathtt{j}}\int_0^{\sigma_i(\tau_{i\mathtt{j}})}\frac{(\sigma_i(\tau_{i\mathtt{j}})-\mathtt{t})^{\text{ß}_i-1}}{\Gamma(\text{ß}_i)}\big[|\mathtt{g}_i(\mathtt{t},\mathtt{z}_i(\mathtt{t}),\mathtt{z}_2(\mathtt{t}),\cdots,\mathtt{z}_1(\mathtt{t}),\cdots,\mathtt{z}_n(\mathtt{t}))\\
&\qquad - \mathtt{g}_i(\mathtt{t},\mathtt{y}_i(\mathtt{t}),\mathtt{y}_2(\mathtt{t}),\cdots,\mathtt{y}_1(\mathtt{t}),\cdots,\mathtt{y}_n(\mathtt{t}))|\big]d\mathtt{t}\bigg]
\end{aligned}
$$

$$\begin{aligned}
&\leq \sum_{i=1}^{n}\Bigg[\int_0^1 |\mathtt{G}_i(\mathtt{s},\mathtt{t})|\sum_{\mathtt{k}=1}^{n}|\mathtt{z}_\mathtt{k}(\mathtt{t})-\mathtt{y}_\mathtt{k}(\mathtt{t})|\mathrm{d}\mathtt{t}\\
&\qquad+\sum_{\mathtt{j}=1}^{\infty}\delta_{i\mathtt{j}}\int_0^{\sigma_i(\tau_{i\mathtt{j}})}\frac{(\sigma_i(\tau_{i\mathtt{j}})-\mathtt{t})^{\text{ß}_i-1}}{\Gamma(\text{ß}_i)}\sum_{\mathtt{k}=1}^{n}|\mathtt{z}_\mathtt{k}(\mathtt{t})-\mathtt{y}_\mathtt{k}(\mathtt{t})|\mathrm{d}\mathtt{t}\Bigg]\\
&\leq \sum_{i=1}^{n}\Bigg[\int_0^1 |\mathtt{G}_i(\mathtt{s},\mathtt{t})|\, n\max_{1\leq i\leq n}|\mathtt{z}_i(\mathtt{s})-\mathtt{y}_i(\mathtt{s})|\mathrm{d}\mathtt{t}\\
&\qquad+\sum_{\mathtt{j}=1}^{\infty}\delta_{i\mathtt{j}}\int_0^{\sigma_i(\tau_{i\mathtt{j}})}\frac{(\sigma_i(\tau_{i\mathtt{j}})-\mathtt{t})^{\text{ß}_i-1}}{\Gamma(\text{ß}_i)}\, n\max_{1\leq i\leq n}|\mathtt{z}_i(\mathtt{s})-\mathtt{y}_i(\mathtt{s})|\mathrm{d}\mathtt{t}\Bigg]\\
&\leq n\sum_{i=1}^{n}\Bigg[\int_0^1 |\mathtt{G}_i(\mathtt{s},\mathtt{t})|\max_{1\leq i\leq n}\left\{\sup_{\mathtt{s}\in[0,1]}|\mathtt{z}_i(\mathtt{s})-\mathtt{y}_i(\mathtt{s})|\right\}\mathrm{d}\mathtt{t}\\
&\qquad+n\sum_{\mathtt{j}=1}^{\infty}\delta_{i\mathtt{j}}\int_0^{\sigma_i(\tau_{i\mathtt{j}})}\frac{(\sigma_i(\tau_{i\mathtt{j}})-\mathtt{t})^{\text{ß}_i-1}}{\Gamma(\text{ß}_i)}\max_{1\leq i\leq n}\left\{\sup_{\mathtt{s}\in[0,1]}|\mathtt{z}_i(\mathtt{s})-\mathtt{y}_i(\mathtt{s})|\right\}\mathrm{d}\mathtt{t}\Bigg]\\
&\leq n\sum_{i=1}^{n}\Bigg[\int_0^1 |\mathtt{G}_i(\mathtt{s},\mathtt{t})|\mathrm{d}\mathtt{t}+\sum_{\mathtt{j}=1}^{\infty}\delta_{i\mathtt{j}}\int_0^{\sigma_i(\tau_{i\mathtt{j}})}\frac{(\sigma_i(\tau_{i\mathtt{j}})-\mathtt{t})^{\text{ß}_i-1}}{\Gamma(\text{ß}_i)}\mathrm{d}\mathtt{t}\Bigg]\max_{1\leq i\leq n}\mathtt{d}(\mathtt{z}_i,\mathtt{y}_i)\\
&\leq n\sum_{i=1}^{n}\left[\mathtt{G}_i^\star+\frac{2\Gamma(\text{ß}_i)}{\Gamma(\text{ß}_i+1)\Gamma(\text{ß}_i-\alpha_i)}\right]\max_{1\leq i\leq n}\mathtt{d}(\mathtt{z}_i,\mathtt{y}_i)\\
&\leq n\wp_2\max_{1\leq i\leq n}\mathtt{d}(\mathtt{z}_i,\mathtt{y}_i).
\end{aligned}$$

Define $\varphi:[0,+\infty)\to[0,+\infty)$ by $\varphi(\mathtt{s})=n\wp_2\mathtt{s}$ and $\theta:[0,+\infty)^{2n}\to[0,+\infty)$ by $\theta(\mathtt{t}_1,\mathtt{t}_2,\cdots,\mathtt{t}_n,\mathtt{s}_1,\mathtt{s}_2,\cdots,\mathtt{s}_n)=\inf\{\mathtt{t}_1,\mathtt{t}_2,\cdots,\mathtt{t}_n,\mathtt{s}_1,\mathtt{s}_2,\cdots,\mathtt{s}_n\}$. Then

$$\begin{aligned}
\mathtt{d}\big(\aleph(\mathtt{z}_1,\mathtt{z}_2,\cdots,\mathtt{z}_n),\aleph(\mathtt{y}_1,\mathtt{y}_2,\cdots,\mathtt{y}_n)\big)&\leq\wp_2\max_{1\leq i\leq n}\mathtt{d}(\mathtt{z}_i,\mathtt{y}_i)\\
&\leq\varphi(\max_{1\leq i\leq n}\mathtt{d}(\mathtt{z}_i,\mathtt{y}_i))\\
&\leq\varphi(\max_{1\leq i\leq n}\mathtt{d}(\mathtt{z}_i,\mathtt{y}_i))+\theta(\ell_4),
\end{aligned}$$

where

$$\begin{aligned}
\ell_4=\Big(&\mathtt{d}\big(\mathtt{y}_1,\aleph(\mathtt{z}_1,\mathtt{z}_2,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n)\big),\mathtt{d}\big(\mathtt{y}_2,\aleph(\mathtt{z}_2,\mathtt{z}_1,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n)\big),\\
&\mathtt{d}\big(\mathtt{y}_3,\aleph(\mathtt{z}_3,\mathtt{z}_2,\mathtt{z}_1,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n)\big),\cdots,\mathtt{d}\big(\mathtt{y}_n,\aleph(\mathtt{z}_n,\mathtt{z}_2,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_1)\big),\\
&\mathtt{d}\big(\mathtt{z}_1,\aleph(\mathtt{z}_1,\mathtt{z}_2,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n)\big),\mathtt{d}\big(\mathtt{z}_2,\aleph(\mathtt{z}_2,\mathtt{z}_1,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n)\big),\\
&\mathtt{d}\big(\mathtt{z}_3,\aleph(\mathtt{z}_3,\mathtt{z}_2,\mathtt{z}_1,\cdots,\mathtt{z}_{n-1},\mathtt{z}_n)\big),\cdots,\mathtt{d}\big(\mathtt{z}_n,\aleph(\mathtt{z}_n,\mathtt{z}_2,\mathtt{z}_3,\cdots,\mathtt{z}_{n-1},\mathtt{z}_1)\big)\Big)
\end{aligned}$$

□

5 Hyers-Ulam Stability Analysis

In this section, we derive necessary conditions for the stability analysis of Hyers-Ulam's type. For some positive $\varepsilon_i>0$, $i=1,2,\cdots,n$, consider the system of inequalities given by

$$\begin{cases}
\left|{}^{\mathtt{RL}}\mathtt{D}_{0^+}^{\text{ß}_1}\mathtt{z}_1(\mathtt{s})-\mathtt{g}_1\big(\mathtt{s},\mathtt{z}_1(\mathtt{s}),\mathtt{z}_2(\mathtt{s}),\cdots,\mathtt{z}_{n-1}(\mathtt{s}),\mathtt{z}_n(\mathtt{s})\big)\right|\leq\varepsilon_1,\\
\left|{}^{\mathtt{RL}}\mathtt{D}_{0^+}^{\text{ß}_2}\mathtt{z}_2(\mathtt{s})-\mathtt{g}_2\big(\mathtt{s},\mathtt{z}_2(\mathtt{s}),\mathtt{z}_1(\mathtt{s}),\cdots,\mathtt{z}_{n-1}(\mathtt{s}),\mathtt{z}_n(\mathtt{s})\big)\right|\leq\varepsilon_2,\\
\qquad\vdots\\
\left|{}^{\mathtt{RL}}\mathtt{D}_{0^+}^{\text{ß}_{n-1}}\mathtt{z}_{n-1}(\mathtt{s})-\mathtt{g}_{n-1}\big(\mathtt{s},\mathtt{z}_{n-1}(\mathtt{s}),\mathtt{z}_2(\mathtt{s}),\cdots,\mathtt{z}_1(\mathtt{s}),\mathtt{z}_n(\mathtt{s})\big)\right|\leq\varepsilon_{n-1},\\
\left|{}^{\mathtt{RL}}\mathtt{D}_{0^+}^{\text{ß}_n}\mathtt{z}_n(\mathtt{s})-\mathtt{g}_n\big(\mathtt{s},\mathtt{z}_n(\mathtt{s}),\mathtt{z}_2(\mathtt{s}),\cdots,\mathtt{z}_{n-1}(\mathtt{s}),\mathtt{z}_1(\mathtt{s})\big)\right|\leq\varepsilon_n,
\end{cases}\tag{5.1}$$

for $s \in [0,1] := I$. Then the system of boundary value problems (1.1)-(1.2) is Hyers–Ulam stable if there exist constants $L_i > 0$, $i = 1, 2, \cdots, n$ such that for each solution $(z_1, z_2, \cdots, z_n) \in C^n(I, \mathbb{R})$ of the system of inequalities (5.1), there exists a unique solution $(y_1, y_2, \cdots, y_n) \in C^n(I, \mathbb{R})$ with

$$|(z_1, z_2, \cdots, z_n)(s) - (y_1, y_2, \cdots, y_n)(s)| \leq \left(\sum_{i=1}^{n} L_i + M\wp_2\right)\varepsilon. \tag{5.2}$$

Remark 5.1. We say that $(z_1, z_2, \cdots, z_n) \in C^n(I, \mathbb{R})$ is a solution of the system of inequalities (5.1) if there exist functions $\psi_i \in C^n(I, \mathbb{R})$ which depend upon z_i respectively such that

(*i*) $|\psi_i| \leq \varepsilon_i$, $s \in I$, and

(*ii*) for $s \in I$,

$$\begin{cases} {}^{RL}D_{0^+}^{ß_1} z_1(s) = g_1\big(s, z_1(s), z_2(s), \cdots, z_{n-1}(s), z_n(s)\big) + \psi_1(s), \\ {}^{RL}D_{0^+}^{ß_2} z_2(s) = g_2\big(s, z_2(s), z_1(s), \cdots, z_{n-1}(s), z_n(s)\big) + \psi_2(s), \\ \vdots \\ {}^{RL}D_{0^+}^{ß_{n-1}} z_{n-1}(s) = g_{n-1}\big(s, z_{n-1}(s), z_2(s), \cdots, z_1(s), z_n(s)\big) + \psi_{n-1}(s), \\ {}^{RL}D_{0^+}^{ß_n} z_n(s) = g_n\big(s, z_n(s), z_2(s), \cdots, z_{n-1}(s), z_1(s)\big) + \psi_n(s), \end{cases} \tag{5.3}$$

Lemma 5.2. Let $(z_1, z_2, \cdots, z_n) \in C^n(I, \mathbb{R})$ be the solution of the system of inequalities (5.1). Then the following inequalities hold:

$$\begin{cases} \Bigg| z_1(s) - \int_0^1 G_1(s,t) g_1(t, z_1(t), z_2(t), \cdots, z_n(t)) dt \\ \quad + s^{ß_1 - 1} \sum_{j=1}^{\infty} \delta_{1j} \int_0^{\sigma_1(\tau_{1j})} \frac{(\sigma_1(\tau_{1j}) - t)^{ß_1 - 1}}{\Gamma(ß_1)} g_1(t, z_1(t), z_2(t), \cdots, z_n(t)) dt \Bigg| \leq L_1 \varepsilon_1, \\ \Bigg| z_2(s) - \int_0^1 G_2(s,t) g_2(t, z_2(t), z_1(t), \cdots, z_n(t)) dt \\ \quad + s^{ß_2 - 1} \sum_{j=1}^{\infty} \delta_{2j} \int_0^{\sigma_2(\tau_{2j})} \frac{(\sigma_2(\tau_{2j}) - t)^{ß_2 - 1}}{\Gamma(ß_2)} g_2(t, z_2(t), z_1(t), \cdots, z_n(t)) dt \Bigg| \leq L_2 \varepsilon_2, \\ \vdots \\ \Bigg| z_n(s) - \int_0^1 G_n(s,t) g_n(t, z_n(t), z_2(t), \cdots, z_1(t)) dt \\ \quad + s^{ß_n - 1} \sum_{j=1}^{\infty} \delta_{nj} \int_0^{\sigma_n(\tau_{nj})} \frac{(\sigma_n(\tau_{nj}) - t)^{ß_n - 1}}{\Gamma(ß_n)} g_n(t, z_n(t), z_2(t), \cdots, z_1(t)) dt \Bigg| \leq L_n \varepsilon_n. \end{cases}$$

Proof . From the Remark 5.1 and Lemma 2.7, the solution of (5.3) and (1.2) is given by

$$\begin{cases} z_1(s) = \int_0^1 G_1(s,t)[g_1(t,z_1(t),z_2(t),\cdots,z_n(t))+\psi_1(t)]dt \\ -s^{\beta_1-1}\sum_{j=1}^{\infty}\delta_{1j}\int_0^{\sigma_1(\tau_{1j})}\frac{(\sigma_1(\tau_{1j})-t)^{\beta_1-1}}{\Gamma(\beta_1)}[g_1(t,z_1(t),z_2(t),\cdots,z_n(t))+\psi_1(t)]dt \\ z_2(s) = \int_0^1 G_2(s,t)[g_2(t,z_2(t),z_1(t),\cdots,z_n(t))+\psi_2(t)]dt \\ -s^{\beta_2-1}\sum_{j=1}^{\infty}\delta_{2j}\int_0^{\sigma_2(\tau_{2j})}\frac{(\sigma_2(\tau_{2j})-t)^{\beta_2-1}}{\Gamma(\beta_2)}[g_2(t,z_2(t),z_2(t),\cdots,z_n(t))+\psi_2(t)]dt \\ \qquad\vdots \\ z_n(s) = \int_0^1 G_n(s,t)[g_n(t,z_n(t),z_2(t),\cdots,z_1(t))+\psi_n(t)]dt \\ -s^{\beta_n-1}\sum_{j=1}^{\infty}\delta_{nj}\int_0^{\sigma_n(\tau_{nj})}\frac{(\sigma_n(\tau_{nj})-t)^{\beta_n-1}}{\Gamma(\beta_n)}[g_n(t,z_n(t),z_2(t),\cdots,z_1(t))+\psi_n(t)]dt \end{cases} \tag{5.4}$$

From first equation of the system (5.4), we have

$$\begin{aligned} &\Bigg| z_1(s) - \int_0^1 G_1(s,t)g_1(t,z_1(t),z_2(t),\cdots,z_n(t))dt \\ &\quad + s^{\beta_1-1}\sum_{j=1}^{\infty}\delta_{1j}\int_0^{\sigma_1(\tau_{1j})}\frac{(\sigma_1(\tau_{1j})-t)^{\beta_1-1}}{\Gamma(\beta_1)}g_1(t,z_1(t),z_2(t),\cdots,z_n(t))dt\Bigg| \\ &\quad \le \int_0^1 |G_1(s,t)||\psi_1(t)|dt + |s|^{\beta_1-1}\sum_{j=1}^{\infty}\delta_{1j}\int_0^{\sigma_1(\tau_{1j})}\frac{(\sigma_1(\tau_{1j})-t)^{\beta_1-1}}{\Gamma(\beta_1)}|\psi_1(t)|dt \\ &\quad \le \left[G_1^{\star} + \frac{2\Gamma(\beta_1)}{\Gamma(\beta_1+1)\Gamma(\beta_1-\alpha_1)}\right]\varepsilon_1 := L_1\varepsilon_1. \end{aligned}$$

Similarly we can prove other inequalities. □

Theorem 5.3. Suppose (4.1) holds. Then the system of boundary value problems (1.1)–(1.2) is Hyers-Ulam stabile, if

$$\sum_{i=1}^{n} L_i + M\wp_2 < 1,$$

where $\wp_2$ is defined in Theorem 4.1.

Proof . Let $(z_1, z_1, \cdots, z_n) \in C^n(I, \mathbb{R})$ be the solution to the system of inequalities (5.1) and let $(y_1, y_1, \cdots, y_n) \in$

$C^n(I,\mathbb{R})$ be the unique solution to the system of boundary value problems (1.1)–(1.2). Then by Lemma 5.2, we have

$$\begin{aligned}
|z_1(s)-y_1(s)| \le & \Big| z_1(s) - \int_0^1 G_1(s,t) g_1(t,y_1(t),y_2(t),\cdots,y_n(t))dt \\
& + s^{ß_1-1}\sum_{j=1}^{\infty}\delta_{1j}\int_0^{\sigma_1(\tau_{1j})}\frac{(\sigma_1(\tau_{1j})-t)^{ß_1-1}}{\Gamma(ß_1)} g_1(t,y_1(t),y_2(t),\cdots,y_n(t))dt\Big| \\
\le & \Big| z_1(s) - \int_0^1 G_1(s,t) g_1(t,z_1(t),z_2(t),\cdots,z_n(t))dt \\
& + s^{ß_1-1}\sum_{j=1}^{\infty}\delta_{1j}\int_0^{\sigma_1(\tau_{1j})}\frac{(\sigma_1(\tau_{1j})-t)^{ß_1-1}}{\Gamma(ß_1)} g_1(t,z_1(t),z_2(t),\cdots,z_n(t))dt\Big| \\
& + \int_0^1 |G_1(s,t)| |g_1(t,z_1(t),z_2(t),\cdots,z_n(t)) - g_1(t,y_1(t),y_2(t),\cdots,y_n(t))|dt \\
& + |s|^{ß_1-1}\sum_{j=1}^{\infty}\delta_{1j}\int_0^{\sigma_1(\tau_{1j})}\frac{(\sigma_1(\tau_{1j})-t)^{ß_1-1}}{\Gamma(ß_1)} \\
& \quad \times |g_1(t,z_1(t),z_2(t),\cdots,z_n(t)) - g_1(t,y_1(t),y_2(t),\cdots,y_n(t))|dt \\
\le & L_1\varepsilon_1 + M\int_0^1 |G_1(s,t)\sum_{j=1}^{n}|z_j(t)-y_j(t)|dt \\
& + M\sum_{j=1}^{\infty}\delta_{1j}\int_0^{\sigma_1(\tau_{1j})}\frac{(\sigma_1(\tau_{1j})-t)^{ß_1-1}}{\Gamma(ß_1)}\sum_{j=1}^{n}|z_j(t)-y_j(t)|dt \\
\le & L_1\varepsilon_1 + M\int_0^1 |G_1(s,t)|dt\sum_{j=1}^{n}\|z_j-y_j\| \\
& + M\sum_{j=1}^{\infty}\delta_{1j}\int_0^{\sigma_1(\tau_{1j})}\frac{(\sigma_1(\tau_{1j})-t)^{ß_1-1}}{\Gamma(ß_1)}dt\sum_{j=1}^{n}\|z_j-y_j\|.
\end{aligned}$$

Taking supremum on both sides over s, we obtain

$$\|z_1-y_1\| \le L_1\varepsilon_1 + M\left[G_i^\star + \frac{2\Gamma(ß_1)}{\Gamma(ß_1+1)\Gamma(ß_1-\alpha_1)}\right]\sum_{j=1}^{n}\|z_j-y_j\|.$$

Similarly, we can have

$$\|z_i-y_i\| \le L_i\varepsilon_i + M\left[G_i^\star + \frac{2\Gamma(ß_i)}{\Gamma(ß_i+1)\Gamma(ß_i-\alpha_i)}\right]\sum_{j=1}^{n}\|z_j-y_j\|,\ i=2,3,\cdots,n.$$

Adding all the above inequalities from $i=1$ to n, we obtain

$$\begin{aligned}
\sum_{i=1}^{n}\|z_i-y_i\| &\le \sum_{i=1}^{n}L_i\varepsilon_i + M\sum_{i=1}^{n}\left[G_i^\star + \frac{2\Gamma(ß_i)}{\Gamma(ß_i+1)\Gamma(ß_i-\alpha_i)}\right]\sum_{j=1}^{n}\|z_j-y_j\| \\
&\le \sum_{i=1}^{n}L_i\varepsilon_i + M\wp_2\sum_{j=1}^{n}\|z_j-y_j\|.
\end{aligned}$$

That is

$$\sum_{i=1}^{n}\|z_i-y_i\| \le \frac{1}{1-M\wp_2}\sum_{i=1}^{n}L_i\varepsilon_i.$$

Let $\varepsilon=\max\{\varepsilon_i : i=1,2,\cdots,n\}$. Then, we have

$$\begin{aligned}
\|(z_1,z_2,\cdots,z_n)-(y_1,y_2,\cdots,y_n)\| &\le \sum_{j=1}^{n}\|z_j-y_j\| \\
&\le \frac{\sum_{i=1}^{n}L_i}{1-M\wp_2}\varepsilon.
\end{aligned}$$

□

Example 5.4.

$$\begin{cases} {}^{\mathtt{RL}}\mathtt{D}_{0^+}^{3/2}\mathtt{z}_1(\mathtt{s}) = 1 + \mathtt{s} + \dfrac{1}{3}\sin\big(\mathtt{z}_1(\mathtt{s}) - \mathtt{z}_2(\mathtt{s})\big), \\ {}^{\mathtt{RL}}\mathtt{D}_{0^+}^{5/4}\mathtt{z}_2(\mathtt{s}) = 2\sin(\mathtt{s}) + \dfrac{1}{3}\cos\big(\mathtt{z}_2(\mathtt{s}) - \mathtt{z}_1(\mathtt{s})\big), \end{cases} \tag{5.5}$$

for $0 < \mathtt{s} < 1$, satisfying

$$\mathtt{z}_i(0) = 0 \quad \text{and} \quad {}^{\mathtt{RL}}\mathtt{D}_{0^+}^{\alpha_i}\mathtt{z}_i(1) = \sum_{\mathtt{j}=1}^{\infty} \delta_{i\mathtt{j}}\, {}^{\mathtt{RL}}\mathtt{D}_{0^+}^{\alpha_i}\mathtt{z}_i\big(\sigma_i(\tau_{i\mathtt{j}})\big),\ i = 1, 2. \tag{5.6}$$

So, we have $n = 2$, $ß_1 = \frac{3}{2}$, $ß_2 = \frac{5}{4}$. Let $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{5}$, $\delta_{1\mathtt{j}} = \frac{18\Gamma(5/6)}{\pi^{5/2}\mathtt{j}}$, $\delta_{2\mathtt{j}} = \frac{60\sqrt{2}\Gamma(19/20)\sin(\pi/20)}{\pi^2\Gamma(3/4)\mathtt{j}}$, $\tau_{1\mathtt{j}} = \frac{1}{\mathtt{j}^3}$, $\tau_{2\mathtt{j}} = \frac{1}{\mathtt{j}}$, $\sigma_1(\mathtt{s}) = \mathtt{s}^{2/3}$ and $\sigma_2(\mathtt{s}) = \mathtt{s}^4$. Then

$$\sum_{\mathtt{j}=1}^{m} \delta_{1\mathtt{j}}\big(\sigma_1(\tau_{1\mathtt{j}})\big)^{ß_1-1} = \sum_{\mathtt{j}=1}^{m} \frac{18\Gamma(5/6)}{\pi^{5/2}\mathtt{j}^2} \to \frac{3\Gamma(5/6)}{\sqrt{\pi}} = \frac{2\Gamma(ß_1)}{\Gamma(ß_1 - \alpha_1)} \quad \text{as} \quad m \to +\infty$$

and

$$\begin{aligned} \sum_{\mathtt{j}=1}^{m} \delta_{2\mathtt{j}}\big(\sigma_2(\tau_{2\mathtt{j}})\big)^{ß_2-1} &= \sum_{\mathtt{j}=1}^{m} \frac{60\sqrt{2}\Gamma(19/20)\sin(\pi/20)}{\pi^2\Gamma(3/4)\mathtt{j}^2} \\ &\to \frac{10\sqrt{2}\Gamma(19/20)\sin(\pi/20)}{\Gamma(3/4)} = \frac{2\Gamma(ß_2)}{\Gamma(ß_2 - \alpha_2)} \quad \text{as} \quad m \to +\infty. \end{aligned}$$

Let $\mathtt{z}_i, \mathtt{y}_i \in \mathtt{Y}$. Then

$$\begin{aligned} |\mathtt{g}_1(\mathtt{s}, \mathtt{z}_1, \mathtt{z}_2) - \mathtt{g}_1(\mathtt{s}, \mathtt{y}_1, \mathtt{y}_2)| &\le \frac{1}{3}\big|\sin\big(\mathtt{z}_1(\mathtt{s}) - \mathtt{z}_2(\mathtt{s})\big) - \sin\big(\mathtt{y}_1(\mathtt{s}) - \mathtt{y}_2(\mathtt{s})\big)\big| \\ &\le \frac{1}{3}\big|\big(\mathtt{z}_1(\mathtt{s}) - \mathtt{z}_2(\mathtt{s})\big) - \big(\mathtt{y}_1(\mathtt{s}) - \mathtt{y}_2(\mathtt{s})\big)\big| \\ &\le \frac{1}{3}\sum_{i=1}^{2}|\mathtt{z}_i - \mathtt{y}_i|, \end{aligned}$$

$$\begin{aligned} |\mathtt{g}_2(\mathtt{s}, \mathtt{z}_1, \mathtt{z}_2) - \mathtt{g}_2(\mathtt{s}, \mathtt{y}_1, \mathtt{y}_2)| &\le \frac{1}{3}\big|\cos\big(\mathtt{z}_1(\mathtt{s}) - \mathtt{z}_2(\mathtt{s})\big) - \cos\big(\mathtt{y}_1(\mathtt{s}) - \mathtt{y}_2(\mathtt{s})\big)\big| \\ &\le \frac{1}{3}\big|\big(\mathtt{z}_1(\mathtt{s}) - \mathtt{z}_2(\mathtt{s})\big) - \big(\mathtt{y}_1(\mathtt{s}) - \mathtt{y}_2(\mathtt{s})\big)\big| \\ &\le \frac{1}{3}\sum_{i=1}^{2}|\mathtt{z}_i - \mathtt{y}_i|, \end{aligned}$$

$|\mathtt{g}_2(\mathtt{s}, \mathtt{z}_1, \mathtt{z}_2)| = |1 + \mathtt{s}| \le 2$, $|\mathtt{g}_2(\mathtt{s}, \mathtt{z}_1, \mathtt{z}_2)| = |2\sin(\mathtt{s})| \le 2$. So, $\mathtt{M} = \frac{1}{3}$, $\wp_1 = 2$ and

$$\sum_{i=1}^{2}\mathtt{L}_i = \sum_{i=1}^{2}\left[\mathtt{G}_i^\star + \frac{2\Gamma(ß_i)}{\Gamma(ß_i + 1)\Gamma(ß_i - \alpha_i)}\right] \approx 0.5571212542 = \wp_2.$$

$$\sum_{i=1}^{n}\mathtt{L}_i + \mathtt{M}\wp_2 \approx 0..7428283389 < 1,$$

Therefore, all the conditions of of Theorem 4.1 are satisfied, so there exists a solution for the system (5.5)–(5.6). Also, in view of Theorem 5.3, the condition of Hyers–Ulam stabilities are also satisfied. Therefore, the solution of system (5.5)–(5.6) is Hyers–Ulam stable.

6 Conclusion

The proximity theory is useful tool to find proximity point when the given mapping is non-self. We established the existence of solutions to the system of nonlinear infinite-point fractional order boundary value problems by an application of n-best proximity point theorem in a complete metric space. By applying our results we can study different kind differential equations whereas results in [28, 30, 33] can only applicable for certain type of differential equations. Therefore, our findings are extended and more general. In the future, the following aspects can be explored further: (1) we study best proximity points of cyclic mappings. (2) we try to apply our obtained results for fractional difference equations, dynamical equations on time scales, fractional differential equations on time scales, etc.

Acknowledgements

The authors would like to thank the referees for their valuable suggestions and comments for the improvement of the paper.

References

[1] T. Abdeljawad, J. Alzabut, A. Mukheimer and Y. Zaidan, *Banach contraction principle for cyclical mappings on partial metric spaces*, Fixed Point Theory Appl. **2012** (2012), no. 1, 1–7.

[2] T. Abdeljawad, J. Alzabut, A. Mukheimer and Y. Zaidan, *Best proximity points for cyclical contraction mappings with θ-boundedly compact decompositions*, J. Comput. Anal. Appl. **15** (2013), no. 4, 678–85.

[3] M. Abu-Shady and M.K.A. Kaabar, *A generalized definition of the fractional derivative with applications*, Math. Prob. Engin. **2021** (2021).

[4] S. J. Achar, C. Baishya and M.K.A. Kaabar, *Dynamics of the worm transmission in wireless sensor network in the framework of fractional derivatives*, Math. Meth. Appl. Sci. **45** (2022), no. 8, 4278–4294.

[5] H. Afshari, F. Jarad and T. Abdeljawad, *On a new fixed point theorem with an application on a coupled system of fractional differential equations*, Adv. Differ. Equ., **461** (2020).

[6] J. Alzabut, R.P. Agarwal, S.R. Grace, J.M. Jonnalagadda, A.G. Selvam and C. Wang, *A survey on the oscillation of solutions for fractional difference equations*, Math. **10** (2022), no. 6.

[7] J. Alzabut, A. G. Selvam, D. Vignesh and Y. Gholami, *Solvability and stability of nonlinear hybrid Δ-difference equations of fractional order*, Int. J. Nonlinear Sci. Num. Simul. **2021** (2021).

[8] M. Aslantas H. Sahin and I. Altun, *Best proximity point theorems for cyclic p-contractions with some consequences and applications*, Nonlinear Anal. Model. Control **26** (2021), no. 1, 113–129.

[9] M.I. Ayari, *A best proximity point theorem for α-proximal Geraghty non-self mappings*, *Fixed Point Theory Appl.* **2019**(2019).

[10] Z. Bai and H. Lu, *Positive solutions for a boundary value problem of nonlinear fractional differential equations*, J. Math. Anal. Appl. **311** (2005), 495–505.

[11] V. Berinde, *Approximating fixed points of weak φ-contractions using the Picard iteration*, *Fixed Point Theory Appl.* **4**(2) (2003), 131–142.

[12] K. Fan, *Extensions of two fixed point theorems of F.E. Browder*, Math. Z. **112** (1969), 234–240.

[13] M.Q. Feng, X.M. Zhang and W.G. Ge, *New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions*, Bound. Value Prob. **2011** (2011), Art. ID 720702, 1–20.

[14] N. Goswami, R. Roy, V.N. Mishra and L.M.S. Ruiz, *Common best proximity point results for T-GKT cyclic φ-contraction mappings in partial metric spaces with some applications*, Symmetry, **13** (2021), no. 6, 1098.

[15] M. Hristov, A. Ilchev and B. Zlatanov, *On the best proximity points for p-cyclic summing contractions*, Math. **8** (2020), no. 7, 1060.

[16] S. Jabeen, Z. Zheng, M. U. Rehman, W. Wei and J. Alzabut, *Some fixed point results of weak-fuzzy graphical contraction mappings with application to integral equations*, Math. **9** (2021), no. 5.

[17] M.K.A. Kaabar, A. Refice, M.S. Souid, F. Martínez, S. Etemad, Z. Siri and S. Rezapour, *Existence and UHR stability of solutions to the implicit nonlinear FBVP in the variable order settings*, Math. **9** (2021), no. 14, 1693.

[18] M. K. A. Kaabar, M. Shabibi, J. Alzabut, S. Etemad, W. Sudsutad, F. Martinez, S. Rezapour, *Investigation of the fractional strongly singular thermostat model via fixed point techniques*, Math. 9 (2021), no. 18, 2298.

[19] A. Khemphet, *Best proximity coincidence point theorem for G-proximal generalized geraghty mapping in a metric space with graph G*, Thai J. Math. **18** (2020), no. 3, 1161–1171.

[20] M. Khuddush and K. R. Prasad, *Infinitely many positive solutions for an iterative system of conformable fractional order dynamic boundary value problems on time scales*, Turk. J. Math. **46** (2022), no. 2, 338–359.

[21] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier, North Holland, 2006.

[22] S. Komal, P. Kumam, K. Khammahawong and K. Sitthithakerngkiet, *Best proximity coincidence point theorems for generalized non-linear contraction mappings*, Filomat **32** (2018), no. 19, 6753–6768.

[23] M. M. Matar, M. I. Abbas, J. Alzabut, M. K. Kaabar, S. Etemad, S. Rezapour, *Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives*, Adv. Differ. Equ. **2021** (2021), no. 1, 1–8.

[24] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equation*, John Wiley, New York, (1993).

[25] H. Mohammadi, M.K.A. Kaabar, J. Alzabut, A.G.M. Selvam and S. Rezapour, *Complete model of Crimean-Congo hemorrhagic fever (CCHF) transmission cycle with nonlocal fractional derivative*, J. Funct. Spaces **2021** (2021), 1–12.

[26] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.

[27] K. R. Prasad, M. Khuddush, D. Leela, *Existence of solutions for n-dimensional fractional order hybrid BVPs with integral boundary conditions by an application of n-fixed point theorem*, J. Anal. **29** (2021), no. 3, 963–985.

[28] K. R. Prasad, M. Khuddush and D. Leela, *Existence, uniqueness and Hyers–Ulam stability of a fractional order iterative two-point boundary value problems*, Afr. Mat. **32** (2021)no. 7, 1227–1237.

[29] K. R. Prasad, D. Leela and M. Khuddush, *Existence and uniqueness of positive solutions for system of (p, q, r)-Laplacian fractional order boundary value problems*, Adv. Theory Nonlinear Anal. Appl. **5** (2021), 138–157.

[30] K. R. Prasad, M. Khuddush, D. Leela, *Existence of solutions for fractional order BVPs by mixed monotone ternary operator with perturbation on Banach spaces*, J. Adv. Math. Stud. **14** (2021), no. 1, 109–125.

[31] J.B. Prolla, *Fixed point theorems for set valued mappings and existence of best approximations*, Numer. Funct. Anal. Optim. **5** (1982-1983), 449–455.

[32] S. Reich, *Approximate selections, best approximations, fixed points and invariant sets*, J. Math. Anal. Appl. **62** (1978), 104–113.

[33] V.S. Raj, *A best proximity point theorem for weakly contractive non-self mappings*, Nonlinear Anal. **74** (2011), no. 14, 4804–4808.

[34] S. Rezapour, A. Imran, A. Hussain, F. Martínez, S. Etemad and M.K.A. Kaabar, *Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs*, Symmetry **13** (2021), no. 3, 469.

[35] Y. Rohen and N. Mlaiki, *Tripled best proximity point in complete metric spaces*, Open Math., **18** (2020), 204–210.

[36] H. Sahin, *Best proximity point theory on vector metric spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **70** (2021), no. 1, 130–142.

[37] B. Samet, *Some results on best proximity points*, J. Optim. Theory Appl. **159** (2013), no. 1, 281–291.

[38] W. Shatanawi and A. Pitea, *Best proximity point and best proximity coupled point in a complete metric space with (P)-property*, Filomat **29** (2015), no. 1, 63–74.

[39] H.M. Srivastava, *Diabetes and its resulting complications: Mathematical modeling via fractional calculus*, Public

Health Open Access **4** (2020), no. 3, 1–5.

[40] H.M. Srivastava, K.M. Saad and M.M. Khader, *An efficient spectral collocation method for the dynamic simulation of the fractional epidemiological model of the Ebola virus*, Chaos, Solitons and Fractals **140** (2020), 1–7.

[41] S. Muthaiah, J. Alzabut, D. Baleanu, M.E. Samei and A. Zada, *Existence, uniqueness and stability analysis of a coupled fractional-order differential systems involving Hadamard derivatives and associated with multi-point boundary conditions*, Adv. Diff. Equ. **2021** (2021), no. 1, 1–46.