Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 877–885
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.21652.2285

Diffeological gyrogroups

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(Communicated by Mohammad Bagher Ghaemi)

Abstract

This work is intended to probe the connections between the diffeology and gyrogroup by introducing the notion of diffeological gyrogroup and proceeding with the study of some basic facts about it. The theory is developed by the study of smooth action of a diffeological gyrogroup.

Keywords: diffeology, gyrogroups, diffeological gyrogroups, action of diffeological gyrogroups
2020 MSC: 58B25, 58B99, 22A99

1 Introduction

Smooth manifolds are nice and distinguished spaces in mathematics, however, the category of these objects does not behave well under taking some constructions like subspaces, quotients, and function spaces. Treatment to this difficulty can be generalizing the category of smooth manifolds to diffeologies. The theory of diffeology goes back to J.-M. Souriau in the early 1980s, where diffeological groups were suggested at first [12], and then the general concept of diffeological spaces was introduced [13]. Diffeological spaces include smooth manifolds as a full subcategory and make a complete, co-complete, and cartesian closed category. Subspaces, quotients, function spaces of diffeological spaces inherit diffeological structures. It is worth mentioning the irrational torus $\mathbb{T}_\alpha = \mathbb{R}/(\mathbb{Z} + \alpha\mathbb{Z})$ as a typical example of a diffeological space (actually a diffeological group), which is neither a smooth manifold nor an orbifold. For the convenience of the reader, we bring the relevant material from [8] in Section 2.

On the other hand, diffeology provides a unified framework to study mathematical objects with algebraic and geometric information, such as diffeological groups, diffeological polygroups, and diffeological hypergroups (see [1]). This paper is aimed to investigate gyrogroups in diffeology. Gyrogroups are an extension of groups, initiated by A. A. Ungar [17]. The main motivation behind it is to enrich the structure of Einstein's relativistic velocity addition by inventing an additional, however natural, operator so-called gyrator as a measure of failure to associativity. The binary operation of a gyrogroup induces a map from that gyrogroup to the group of its symmetries by taking any element to its corresponding left translation, which we call the gyrotranslator. The gyrotranslator is a homomorphism if and only if the gyrogroup is a group ([14]). This gives rise to another description of gyrogroups in terms of failure to be homomorphism for gyro-translators.

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Received: October 2020 *Accepted:* July 2021

In order to study gyrogroups in the realm of differential geometry, we impose diffeologies over gyrogroups respecting key features of gyrogroups. Indeed, by a diffeological gyrogroup, we mean a gyrogroup endowed with a diffeology compatible with the binary operation and inversion operation. Thanks to the functional diffeology, we provide a characterization of diffeological gyrogroup concerning the smoothness of gyro-translators. We also observe that the gyrator of a diffeological gyrogroup is a smooth map as well. In other words, the properties of gyrogroups remain stable in diffeology. This is another reason why we work with diffeologies instead of smooth manifolds.

The structure of the paper is as follows. In Section 2 we review gyrogroups and diffeological spaces. In Section 3, we define diffeological gyrogroups and study some properties. We finally discuss the smooth action of gyrogroups in Section 4.

2 Preliminaries

2.1 Background on diffeology

In this section we review the basic definitions and constructions from diffeology theory that will be used. For a deeper discussion of diffeology we refer the reader to [8].

Definition 2.1. Let X be a non empty set. Every map $P : U \to X$ is called a **parametrization** in X, where U is a real domain, that is, an open subset of $\mathbb{R}^n$ for some non-negative integer n. A **diffeology** of X is a set $\mathcal{D}$ of parametrizations of X, called **plots**, satisfying the following axioms:

D1. (*Covering*) Every constant parametrization $\mathbf{x} : r \mapsto x$ defined on $\mathbb{R}^n$ is a plot, for all $x \in X$ and all non-negative integers n.
D2. (*Locality*) If $P : U \to X$ is a parametrization and for every point r of U there exists an open neighborhood V of r such that $P|V$ is a plot, then P is a plot.
D3. (*Smooth compatibility*) For every plot $P : U \to X$, for every real domain V and for every smooth map $F : V \to U$, $P \circ F$ is a plot.

A **diffeological space** is a pair $(X, \mathcal{D})$, where $\mathcal{D}$ is a diffeology on the underlying set X.

Let X and Y be two diffeological spaces. A map $f : X \to Y$ is called **smooth** if for every plot P of X, $f \circ P$ is a plot of the space Y. A smooth bijective map with smooth inverse is called a **diffeomorphism**. The set of all smooth maps from X to Y is denoted by $\mathrm{C}^\infty(X, Y)$, and the set of all diffeomorphisms from X to Y by $\mathrm{Diff}(X, Y)$. Once $X = Y$, smooth maps and diffeomorphisms of X are denoted by $\mathrm{C}^\infty(X)$ and $\mathrm{Diff}(X)$, respectively.

Example 2.1. Let U be any open subset of Euclidean spaces. The set of all smooth parametrizations $f\colon V \to U$ as a usual sense of smoothness in Euclidean spaces from any open subset $V \subseteq \mathbb{R}^n$ for some integer n makes a diffeology on U what is known as **standard diffeology** on U.

Definition 2.2. For any set X, all of the locally constant parametrizations in X make finest diffeology on X called **fine (or discrete) diffeology** and the set of all parametrizations in X is the coarsest diffeology on X called **coarse (or indiscrete) diffeology**.

Let $\{X_i\}_{i\in I}$ be a collection of diffeological spaces. The **product diffeology** on the $X = \prod_{i\in I} X_i$ is the coarsest diffeology such that the natural projection $P_i : \prod_{i\in I} X_i \to X_i$ for any index $i \in I$ is smooth.

For a diffeological space X, a **diffeological subspace** is a subset $Y \subseteq X$ equipped with the **subspace diffeology**: the set of all plots of X with values in Y. In this situation canonical inclusion $Y \hookrightarrow X$ is smooth.

Thus, intersection of two diffeological spaces is a diffeological subspace of each of them, even though it is empty.

Every diffeological space X admits an intrinsic topology called the D-**topology** for which the plots are continuous, by means of a subset of X is D-**open** if and only if its preimage is open by every plot.

Definition 2.3. Let X be a diffeological space and $\sim$ be an equivalence relation on X. Let $Y = X/{\sim}$ be the quotient set, and let $\pi\colon X \to Y$ be the quotient map. We define the **quotient diffeology** on Y to be the diffeology for which the plots are those maps $p\colon U \to Y$ such that for every point in U there exist an open neighborhood $V \subseteq U$ and a plot $q\colon V \to X$ such that $p|_V = \pi \circ q$.

Definition 2.4. Let Y and Z be diffeological spaces. The **standard functional diffeology** on $C^\infty(Y, Z)$ is defined as follows. A parametrization $p\colon U \to C^\infty(Y, Z)$ is a plot if the evaluation map

$$U \times Y \to Z \qquad \text{given by} \quad (u, y) \mapsto p(u)(y)$$

is smooth.

Definition 2.5. A **diffeological group** is a group G equipped with a diffeology such that the multiplication and inverse maps are smooth. Let X be a diffeological space and let G be a diffeological group. A **smooth action** of a diffeological group G on X is any smooth homomorphism $\rho : G \to \mathrm{Diff}(X)$, where $\mathrm{Diff}(X)$ together with composition of maps is a group of diffeomorphisms on X which is equipped with functional diffeology (see [8, 1.61]).

2.2 Gyrogroups

In this section, we proceed regarding the concept of gyrogroup which is a kind of generalization of a group. In a gyrogroup, the associativity law is replaced by a weaker one. Throughout this paper, we will consider that a groupoid G is a nonempty set with a binary operation. Also if $(G_1, \oplus_1)$, $(G_2, \oplus_2)$ are two groupoids then the map $f\colon G_1 \to G_2$ is said to be homomorphism if $f(a \oplus b) = f(a) \oplus f(b)$ for all $a, b \in G_1$ and $Aut(G, \oplus)$ is the the group of automorphisms of the groupoid $(G, \oplus)$.

Definition 2.6. [17] A groupoid $(G, \oplus)$ is called a **gyrogroup** if its binary operation satisfies the following axioms:

1) In G there is at least one element, 0, called a left identity, satisfying $0 \oplus a = a$;
2) For each $a \in G$ there is an element $\ominus a \in G$, called a left inverse of a satisfying
$$\ominus a \oplus a = 0;$$
3) For any $a, b \in G$ there exists $gyr[a, b] \in Aut(G, \oplus)$ such that for all $c \in G$ we have:
$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c.$$
The automorphism $gyr[a, b]$ of G is called the gyroautomorphism of G generated by $a, b \in G$ and the operator $gyr\colon G \times G \to Aut(G, \oplus)$ is called the gyrator of G.
4) The gyroautomorphism $gyr[a, b]$ generated by any $a, b \in G$ possesses the left loop property
$$gyr[a, b] = gyr[a \oplus b, b].$$

The notation $a \ominus b = a \oplus (\ominus b)$ is used in gyrogroup theory as it is common in group theory.

A gyrogroup $(G, \oplus)$ is **gyrocommutative** if its binary operation obeys the gyrocommutative law

$$a \oplus b = gyr[a, b](b \oplus a) \quad \forall a, b \in G.$$

The existence of right identity and right inverse of each element and uniqueness of them and some other facts about gyrogroups is a consequence of the above gyrogroup axioms which is explained by Ungar in the following theorem:

Theorem 2.2. [17] Let $(G, \oplus)$ be a gyrogroup. For any elements $a, b, c, x \in G$ we have:

(1) If $a \oplus b = a \oplus c$, then $b = c$ (general left cancellation law).
(2) $gyr[0, a] = I$ for any left identity 0 in G.
(3) $gyr[x, a] = I$ for any left inverse x of a in G.
(4) $gyr[a, a] = I$
(5) There is a left identity which is a right identity.
(6) There is only one left identity.
(7) Every left inverse is a right inverse.
(8) There is only one left inverse, $\ominus a$, of a, and $\ominus(\ominus a) = a$.
(9) $\ominus a \oplus (a \oplus b) = b$ (Left Cancellation Law).
(10) $gyr[a, b]x = \ominus(a \oplus b) \oplus \{a \oplus (b \oplus x)\}$ (The Gyrator Identity).
(11) $gyr[a, b]0 = 0$

(12) $gyr[a,b](\ominus x) = \ominus gyr[a,b]x$
(13) $gyr[a,0] = gyr[0,b] = I$.

Theorem 2.3. [5] Let $(V_2, \oplus_2)$ be a gyrogroup, V_1 be an arbitrary set, and $\phi\colon V_1 \to V_2$ be a bijection between V_1 and V_2. Then V_1 endowed with the induced operation:

$$a \oplus_1 b\colon = \phi^{-1}(\phi(a) \oplus_2 \phi(b)), \qquad a, b \in V_1$$

becomes a gyrogroup.

3 Diffeological Gyrogroups

In this section, we will dedicate a geometrical sense to gyrogroups by introducing the notion of diffeological gyrogroups. Diffeological gyrogroups are indeed generalizations of the diffeological groups which includes Lie groups. A Lie group is in fact a diffeological group by means of its manifold diffeology. The simplest examples of diffeological gyrogroups are obviously obtained by putting the coarse diffeology (definition 2.2) on any gyrogroup so we are probing over the examples with a finer diffeology.

Definition 3.1. *Let G be a gyrogroup. A diffeology $\mathcal{D}$ on G is called a quasi-gyrogroup diffeology if the left gyrotranslations on G are all smooth. We call G along with such a diffeology $\mathcal{D}$ on it a quasi-diffeological gyrogroup.*

Proposition 3.1. Let G be diffeological gyrogroup and $g \in G$. The left and right gyrotranslations on G are diffeomorphisms.

Proof . The left gyrotranslation $L_g\colon G \to G$ given by $x \mapsto g \oplus x$ and right gyrotranslation $R_g\colon G \to G$ given by $x \mapsto x \oplus g$ are indeed the restriction of the binary operation to the set $\{g\} \times G \subseteq G \times G$ and $G \times \{g\} \subseteq G \times G$, respectively, which are smooth maps by definition of diffeological gyrogroups. On the other hand, by [17, theorem 2.22], the equations $a \oplus x = b$ and $x \oplus a = b$ have unique solutions in G, so left and right gyrotranslations are bijective. Similarly, their inverse maps are smooth as they are also gyrotranslations. We have thus proved our claim. □

Remark 3.2. By the above proposition, every quasi-diffeological gyrogroup G gives us a map $L : G \to \mathrm{Diff}(G)$ taking any $a \in G$ to the diffeomorphism L_a. Notice that the map L is not a homomorphism. By the gyrator identity in 2.2 , one can write $gyr[a,b] = L_{\ominus(a \oplus b)} \circ L_a \circ L_b$. Indeed, the gyrator is a measure for which L deviates from to be homomorphism.

Proposition 3.3. Let $(G, \oplus)$ be a quasi-diffeological gyrogroup. Then for every $a, b \in G$, the gyroautomorphism $gyr[a,b]$ is a diffeomorphism.

Proof . Because $gyr[a,b]$ is as a composition of some diffeomorphisms. □ We denote the subgroup $\mathrm{Aut}(G) \bigcap \mathrm{Diff}(G) \subseteq \mathrm{Diff}(G)$ by $\mathrm{DiffAut}(G)$, which is a diffeological group. Thus, the gyrator of a quasi-diffeological gyrogroup G is into $\mathrm{DiffAut}(G)$, i.e., $gyr : G \times G \to \mathrm{DiffAut}(G)$.

Definition 3.2. Let $(G, \oplus)$ be a gyrogroup. A diffeology $\mathcal{D}$ on G is called a *gyrogroup diffeology* if the operation $\oplus : G \times G \to G$ and the map $\ominus : G \to G$ taking $g \mapsto \ominus g$, are both smooth. We call a gyrogroup G along with a gyrogroup diffeology $\mathcal{D}$ on it a *diffeological gyrogroup.*

Thus, any diffeological gyrogroup is a quasi-diffeological gyrogroup. We here show how the converse to this implication depends on the smoothness of the map $L : G \to \mathrm{Diff}(G)$.

Proposition 3.4. A quasi-diffeological gyrogroup $(G, \oplus)$ is a diffeological gyrogroup if and only if the map $L : G \to \mathrm{Diff}(G)$ is smooth, where the group $\mathrm{Diff}(G)$ of diffeomorphisms on G is equipped with the standard diffeology.

Proof . If G is a diffeological gyrogroup, then the smoothness of L follows from the smoothness of $\oplus : G \times G \to G$ and $\ominus : G \to G$, also cartesian closeness property. On the other hand, let the map $L : G \to \mathrm{Diff}(G)$ be smooth. Again, by cartesian closeness property, the operation $\oplus : G \times G \to G$ is smooth. Moreover, one observes that the composition of the following smooth maps

$$G \xrightarrow{L} \mathrm{Diff}(G) \xrightarrow{inv} \mathrm{Diff}(G) \xrightarrow{\mathrm{ev}(0)} G,$$

where inv and $\mathrm{ev}(0)$ are $f \mapsto f^{-1}$ and evaluation map calculated on 0, is the same as the inverse map $\ominus : G \to G$. Because this composition is smooth, the inverse map is smooth as well. □

By the above proposition and a similar argument to [8, 7.7], one can state the following:

Proposition 3.5. If $(G, \oplus)$ is a diffeological gyrogroup then the map $L : G \to \mathrm{Diff}(G)$ is an embedding.

Proposition 3.6. The gyrator of a diffeological gyrogroup is smooth.

Proof . Let $(G, \oplus)$ be a diffeological gyrogroup and let (P_1, P_2) be a plot in $G \times G$. Then $\ominus(P_1 \oplus P_2)$ is a plot in G. Since $L : G \to \mathrm{Diff}(G)$ is smooth, L_{P_1}, L_{P_2}, and $L_{\ominus(P_1 \oplus P_2)}$ are plots in $\mathrm{Diff}(G)$. Now since $(\mathrm{Diff}(G), \circ)$ is a diffeological group, $gyr[P_1, P_2] = L_{\ominus(P_1 \oplus P_2)} \circ L_{P_1} \circ L_{P_2}$ is a plot in $\mathrm{Diff}(G)$. Therefore, $gyr : G \times G \to \mathrm{Diff}(G)$ is smooth. □

One of the most famous examples of gyrogroup is Möbius gyrogroup that is explained in [17]. Here we show that it is also a diffeological gyrogroup.

Example 3.7. Suppose that $\mathbb{D}$ is a complex open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. The Möbius addition on $\mathbb{D}$ is given by

$$a \oplus_M z = \frac{a + z}{1 + \bar{a}z}, \qquad a, z \in \mathbb{D}.$$

Möbius addition on $\mathbb{D}$ is neither commutative nor associative but it is gyroassociative and gyrocommutative such that for all $a, b \in \mathbb{D}$ gyroautomorphism is defined by the formula:

$$gyr[a, b]d = \frac{a \oplus_M b}{b \oplus_M a} d = \frac{1 + a\bar{b}}{1 + \bar{a}b} d, \quad d \in \mathbb{D}.$$

A deeper discussion of verifying gyrogroup axioms can be found in [17]. Thus $(\mathbb{D}, \oplus_M)$ is a gyrogroup which is called Möbius gyrogroup. Because $\mathbb{D}$ is an open subset of an Euclidean space, by example 2.1, it is equipped with standard diffeology. So $\mathbb{D}$ is a diffeological space and we just need to show addition and minus on $\mathbb{D}$ are smooth. Since the equation $1 + \bar{a}z = 0$ has no solution on $\mathbb{D}$, the addition operator is smooth and the inverse of each element $a \in \mathbb{D}$ is $-a$, so the inverse map is also smooth on $\mathbb{D}$. Therefore $\mathbb{D}$ is a diffeological gyrogroup.

As mentioned previously, for a diffeological gyrogroup $(G, \oplus)$, the smooth map $L : G \to (\mathrm{Diff}(G), \circ)$ is not a homomorphism. However, similar to what shown in [16], we can consider another binary operation for $\mathrm{Diff}(G)$ turning it to a gyrogroup such that $L : G \to \mathrm{Diff}(G)$ is a homomorphism. Let f be a diffeomorphism of G. Then f can be written uniquely as $f = L_{f(0)} \circ \alpha$ in which α is diffeomorphism of G with $\alpha(0) = 0$. Define the operation $f \circledcirc g := L_{f(0) \oplus g(0)} \circ \alpha \circ \beta$, where β is the unique diffeomorphism with $g = L_{g(0)} \circ \beta$. An argument like Theorem 3.3 of [16] proves that $(\mathrm{Diff}(G), \circledcirc)$ is a gyrogroup.

Proposition 3.8. Let G be a diffeological gyrogroup. Then $(\mathrm{Diff}(G), \circledcirc)$ is a diffeological gyrogroup.

Proof . We should show that the operation $\circledcirc$ and the inverse map $I : \mathrm{Diff}(G) \to \mathrm{Diff}(G)$ are smooth. First, consider the composition Φ of the following smooth maps

$$\Phi : \mathrm{Diff}(G) \xrightarrow{\mathrm{ev}(0)} G \xrightarrow{L} \mathrm{Diff}(G) \xrightarrow{inv} \mathrm{Diff}(G),$$

which takes a diffeomorphism f to the left gyrotranslation $L_{\ominus f(0)}$. Because the operation $\circ$ is smooth, we conclude that the map

$$\Psi : \mathrm{Diff}(G) \xrightarrow{(\Phi, \mathrm{id})} \mathrm{Diff}(G) \times \mathrm{Diff}(G) \xrightarrow{\circ} \mathrm{Diff}(G),$$

taking a diffeomorphism f to the unique diffeomorphism α with $f = L_{f(0)} \circ \alpha$ is smooth, too. So is the map

$$\Gamma : \mathrm{Diff}(G) \times \mathrm{Diff}(G) \xrightarrow{\Phi\times\Phi} \mathrm{Diff}(G) \times \mathrm{Diff}(G) \xrightarrow{\circ} \mathrm{Diff}(G).$$

Next, the map

$$\Theta : \mathrm{Diff}(G) \times \mathrm{Diff}(G) \xrightarrow{\mathrm{ev}(0)\times\mathrm{ev}(0)} G \times G \xrightarrow{\oplus} G \xrightarrow{L} \mathrm{Diff}(G)$$

which takes the pair (f, g) of diffeomorphisms to the left gyrotranslation $L_{f(0)\oplus g(0)}$, is smooth. Now, the operation $\circledcirc$ that is the composition

$$\circledcirc : \mathrm{Diff}(G) \times \mathrm{Diff}(G) \xrightarrow{(\Gamma,\Theta)} \mathrm{Diff}(G) \times \mathrm{Diff}(G) \xrightarrow{\circ} \mathrm{Diff}(G),$$

is smooth. Finally, let $\overline{\Psi}$ the composition

$$\overline{\Psi} : \mathrm{Diff}(G) \xrightarrow{\Psi} \mathrm{Diff}(G) \xrightarrow{inv} \mathrm{Diff}(G).$$

Then the inverse map I is as

$$I : \mathrm{Diff}(G) \xrightarrow{(\Phi,\overline{\Psi})} \mathrm{Diff}(G) \times \mathrm{Diff}(G) \xrightarrow{\circ} \mathrm{Diff}(G),$$

which is smooth. □

Corollary 3.9. Let G be a diffeological gyrogroup. Then $L : G \to (\mathrm{Diff}(G), \circledcirc)$ is a smooth homomorphism.

Proposition 3.10. Let G be a diffeological gyrogroup. The inverse map $I : G \to G$ which maps g to $\ominus g$ is a diffeomorphism.

Proof . Since in a gyrogroup, the inverse of each element is unique, so the inverse map is bijective. Moreover, by $\ominus(\ominus g) = g$ we have $(I)^{-1} = I$ which is smooth. Thus I is a diffeomorphism. □

Now, let us examine gyrogroup diffeology on sub objects and products and quotients.

Lemma 3.11. Suppose that $\{G_i\}_{i=1}^n$ is a nonempty family of diffeological gyrogroups. By the product diffeology which is defined on 2.2, $G = \prod_{i=1}^n G_i$ is also a diffeological gyrogroup.

Proof . Firstly, we show that by the operation below, G is a gyrogroup:

$$\{a_i\}_{i=1}^n \oplus \{b_i\}_{i=1}^n = \{a_i \oplus b_i\}_{i=1}^n, \qquad \ominus(\{a_i\}_{i=1}^n) = \{\ominus a_i\}_{i=1}^n.$$

Zero element of G is the sequence of zero elements $\{0\}_{i=1}^n$, so we have

$$\ominus\{a_i\}_{i=1}^n \oplus \{a_i\}_{i=1}^n = \{\ominus a_i \oplus a_i\}_{i=1}^n = \{0\}_{i=1}^n.$$

Now suppose that $a_i, b_i, c_i \in G_i$ for all $i = 1, ..., n$. Then we have

$$\begin{aligned}\{a_i\}_{i=1}^n \oplus (\{b_i\}_{i=1}^n \oplus \{c_i\}_{i=1}^n) &= \{a_i \oplus (b_i \oplus c_i)\}_{i=1}^n = \{(a_i \oplus b_i) \oplus gyr[a_i, b_i]c_i\}_{i=1}^n \\ &= (\{a_i\}_{i=1}^n \oplus \{b_i\}_{i=1}^n) \oplus \{gyr[a_i, b_i]c_i\}_{i=1}^n\end{aligned}$$

where, $gyr[a_1, b_1] \in Aut(G_1), ..., gyr[a_n, b_n] \in Aut(G_n)$. We define
$gyr[\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n] := \prod_{i=1}^n gyr[a_i, b_i] \in Aut(\prod_{i=1}^n G_i)$ which maps $\{c_i\}_{i=1}^n$ to $\{gyr[a_i, b_i]c_i\}_{i=1}^n$. So we obtain

$$\{a_i\}_{i=1}^n \oplus (\{b_i\}_{i=1}^n \oplus \{c_i\}_{i=1}^n) = (\{a_i\}_{i=1}^n \oplus \{b_i\}_{i=1}^n) \oplus gyr[\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n]\{c_i\}_{i=1}^n,$$

As we need. For checking the axiom of loop property, we have

$$gyr[\{a_i\}_{i=1}^n \oplus \{b_i\}_{i=1}^n, \{b_i\}_{i=1}^n] = \prod_{i=1}^n gyr[a_i \oplus b_i, b_i] = \prod_{i=1}^n gyr[a_i, b_i] = gyr[\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n].$$

It implies that G is a gyrogroup. Now we must show that $\oplus$ and $\ominus$ maps are smooth on G. Suppose that $r \mapsto \{P_i^1(r)\}_{i=1}^n$, $r \mapsto \{P_i^2(r)\}_{i=1}^n$ are two plots of G. Thus for every $1 \leq i \leq n$, $r \mapsto P_i^1(r)$ and $r \mapsto P_i^2(r)$ are two plots of G_i. Since for all $1 \leq i \leq n$, G_i is a diffeological gyrogroup then $r \mapsto \ominus P_i^1(r)$, $r \mapsto P_i^1(r) \oplus P_i^2(r)$ are smooth. Therefore $r \mapsto \{P_i^1 \oplus P_i^2(r)\}_{i=1}^n$, $r \mapsto \{\ominus P_i^1(r)\}_{i=1}^n$ are smooth on G. $\square$

A nonempty subset H of $(G, \oplus)$ is called a *subgyrogroup* if it makes a gyrogroup under the operation inherited from G and for all $a, b \in H$ the restriction of automorphism $gyr[a, b]$ to H is an automorphism of H. Suksumran and Wiboonton mentioned a criterion for subgyrogroups in [16]:
A nonempty subset H of a gyrogroup $(G, \oplus)$ is a subgyrogroup if and only if $\ominus a \in H$ and $a \oplus b \in H$ for all $a, b \in H$.

Lemma 3.12. Every subgyrogroup H of a diffeological gyrogroup $(G, \oplus)$ is itself a diffeological gyrogroup.

Proof . Suppose that $P_1, P_2 : U \to H$ are two plots of H for the subspace diffeology 2.2. It means that they are plots of G with values in H. Since H is a subgyrogroup, by the above argument $\ominus P_1$, $P_1 \oplus P_2$ are plots of G with values in H. Therefore $\oplus$ and $\ominus$ maps are smooth on H. It implies H is diffeological gyrogroup. $\square$

A subgyrogroup H of a gyrogroup G is called an *L-subgyrogroup* if $gyr[a, h](H) = H$ for all $a \in G$ and $h \in H$ [16]. If H is an L-subgyrogroup of G and $a \in G$, then $a \oplus H = \{a \oplus h \,|\, h \in H\}$ and H have the same cardinality and the coset space $G/H = \{a \oplus H : a \in G\}$ makes a disjoint partition of G.

A map $f : G \to H$ between gyrogroups is called a *gyrogroup homomorphism* if for all $a, b \in G$, $f(a \oplus b) = f(a) \oplus f(b)$ and a bijective gyrogroup homomorphism is called a *gyrogroup isomorphism*. The *kernel* of gyrogroup homomorphism $f : G \to H$ is defined to be the inverse image of the trivial subgyrogroup $\{0\}$. By [16], $\ker f$ is an L-subgyrogroup of G and the following equivalence relation can be considered on it:

$$a \sim_{\ker f} b \quad \text{if and only if} \quad \ominus a \oplus b \in \ker f \quad \text{for all} \quad a, b \in G.$$

Let $G/\ker f$ be the set of left cosets of $\ker f$:

$$G/\ker f = \{a \oplus \ker f \mid a \in G\}.$$

By theorem 5.5 of [16], $G/\ker f$ equipped with the operation below , is a gyrogroup:

$$(a \oplus \ker f) \oplus (b \oplus \ker f) = (a \oplus b) \oplus \ker f, \qquad a, b \in G.$$

Proposition 3.13. Let G be a diffeological gyrogroup and H be a gyrogroup. Let $f : G \to H$ be a gyrogroup homomorphism. The quotient $G/\ker f$ furnished with the quotient diffeology is a diffeological gyrogroup.

Proof . As we mentioned before, the quotient $G/\ker f$ admits a gyrogroup structure and by applying definition 2.3, the proof is straightforward. $\square$ A subgyrogroup N of a gyrogroup G is called *normal* and denoted by $N \trianglelefteq G$ if it is the kernel of some gyrogroup homomorphism of G. This definition comes from the similar property of normal subgroups. As we saw, the quotient of a gyrogroup by the kernel of a gyrogroup homomorphism receives a gyrogroup structure. so we can consider the quotient of a gyrogroup by a normal subgyrogroup and generalize the previous result.

Corollary 3.14. Let G be a diffeological gyrogroup and $N \trianglelefteq G$ be a normal subgyrogroup. The quotient G/N furnished with the quotient diffeology is a diffeological gyrogroup.

Definition 3.3. Let G and G' be two diffeological gyrogroups and $L(G, G')$ be the space of homomorphisms from G to G'. We denote by $L^\infty(G, G') = L(G, G') \cap C^\infty(G, G')$ the space of all *smooth homomorphisms* from G to G' and we denote by $Iso^\infty(G, G') = Iso(G, G') \cap C^\infty(G, G')$ the space of all *smooth isomorphisms* from G to G'.

Corollary 3.15. If $f : G \to G'$ and $g : G' \to G''$ are two smooth homomorphisms between diffeological gyrogroups, the composition $f \circ g$ is also a smooth homomorphism because the composition of two smooth maps (in the sense of diffeology) is smooth and the composition of two homomorphisms is a homomorphism. Therefore, the set of all diffeological gyrogroups make a category with smooth homomorphisms as its morphisms.
As we saw in the above lemmas, this is a rich category which is closed under taking any subgyrogroup and product of objects or even quotient by a normal subgyrogroup.

Now, let us extend the notion of smooth actions by diffeological gyrogroups.

4 Smooth action of a diffeological gyrogroup

In a natural way, the notion of group actions was extended to gyrogroups in [14]. If G is a gyrogroup and X is a nonempty set, gyrogroup action of G on X is the map $G \times X \to X$ sending (g, x) to $g.x$ which satisfies in two following conditions:

1) $0.x = x$ for all $x \in X$,
2) $a.(b.x) = (a \oplus b).x$ for all $a, b \in G$ and $x \in X$.

Now we can apply this notion to define smooth action of a diffeological gyrogroup.

Definition 4.1. Let X be a diffeological space and G be a diffeological gyrogroup. A *diffeological gyrogroup action* of G on X is defined to be a gyrogroup action in which the map $G \times X \to X$ sending (g, x) to $g.x$ is smooth. Every diffeological gyrogroup action can also be represented as a smooth gyrogroup homomorphism. We know $\mathrm{Diff}(X)$ together with the composition of maps is a group of diffeomorphisms on X which is equipped with functional diffeology. By considering gyroautomorphisms as identity maps, $\mathrm{Diff}(X)$ can be regarded as a diffeological gyrogroup. So, smooth action of a diffeological gyrogroup G on a diffeological space X is indeed any smooth gyrogroup homomorphism $\rho : G \to \mathrm{Diff}(X)$, which maps g to $\rho(g) = (x \mapsto g.x)$.

Remark 4.1. *Note that when G is a group, for any $g \in G$, left gyrotranslation $L_g : G \times G \to G$ defined by $(g, x) \mapsto g \oplus x$, makes an action of G on itself but when G is a gyrogroup, this is not an action. Indeed by proposition 4.1 of [14], a gyrogroup G acts on itself by $a.x = a \oplus x$, $a, x \in G$ if and only if $gyr[a, b] = Id_G$ for all $a, b \in G$. That is, G should be a group. The same result can be drawn for right gyrotranslations.*

Suppose that a diffeological gyrogroup G acts on a diffeological space X, we can define a relation on X; $x \sim y$ if and only if there is $g \in G$ such that $g.x = y$.
By theorem 3.4 of [14], this is an equivalence relation. We put the quotient diffeology on the resulted quotient space.

Definition 4.2. *If a diffeological gyrogroup G acts on a diffeological space X and $x \in X$, then as their classical definition*

1) *The stabilizer of x is defined to be the set $G_x = \{g \in G \mid g.x = x\}$.*
2) *The orbit of x is defined to be the set $Gx = \{g.x \mid g \in G\}$.*
3) *The action is said to be effective if the homomorphism $\rho : G \to \mathrm{C}^\infty(X)$ where $\rho(g)(x) = g.x$ is injective. Thus, for any two distinct $g, h \in G$, there exists $x \in X$ such that $g.x \neq h.x$.*
4) *The action is said to be transitive if for given $x, y \in X$, there exists $g \in G$ such that $g.x = y$.*

Theorem 4.2. *Suppose that G is a diffeological gyrogroup which acts on a diffeological space X. For every $x \in X$, the stabilizer of x, G_x is a diffeological subgyrogroup of G.*

Proof . Let $g \in G_x$. Then $\ominus g \in G_x$ because $x = 0.x = (\ominus g \oplus g).x = \ominus g.(g.x) = \ominus g.x$. Now let $g_1, g_2 \in G_x$, then $(g_1 \oplus g_2).x = g_1.(g_2.x) = g_1.x = x$ and we have $g_1 \oplus g_2 \in G_x$. So G_x is a subgyrogroup of G.
Finally, by using lemma 3.12 we deduce that G_x is a diffeological subgyrogroup. □

Corollary 4.3. By the assumption of the previous theorem, for every $x \in X$, the quotient G/G_x is a diffeological gyrogroup.

Proof . By the previous theorem, G_x is a diffeological subgyrogroup of G for every $x \in X$. Moreover it is an L-subgyrogroup and the quotient G/G_x takes a gyrogroup structure. □

Remark 4.4. The concept of diffeological gyrogroup is a generalization of group action. Thus diffeological gyrogroups can be considered as some kind of generalization of dynamical systems. In that sense, all basic dynamical notions have obvious versions for diffeological gyrogroups, like the above concepts of orbit transitiveness and etc.

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