Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 2167–2174
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.6561

Fractional variational iteration method for solving two-dimensional Stefan problem with fractional order derivative

Adnan Yassean Nama[a,b], Fadhel S. Fadhel[c,*]

[a]*Department of Mathematics, College of Education for Pure Science, Ibn Al- Haitham, University of Baghdad, Baghdad, Iraq*

[b]*Department of Mathematics, College of Education for Pure Science, University of Thi-Qar, Thi-Qar, Iraq*

[c]*Department of Mathematics and Computer Applications, College of Science, Al- Nahrian University, Baghdad, Iraq*

(Communicated by Javad Vahidi)

Abstract

In this paper, we present a mathematical model of Stefan problem, this model is ice melting problem where the interface of solid/liquid moves along the y-axis, which is including time fractional derivative in Jumarie sense. The obtained solution of this problem is an approximate solution using fractional variational iteration method. Graphically the results will be compared with an exact solution to the integer order derivatives.

Keywords: Fractional variational iteration method, Two-dimensional problem, Moving boundary value problem, Modified Remann-Lioville fractional derivative, Stefan problem, Parabolic equation
2020 MSC: Primary 90C33, Secondary 26B25.

1 Introduction

Many problems governed by parabolic differential equations in engineering and science, may contains a moving boundary, which means an unknown boundary that depends on time and space variables and must be determined as a part of the problem solution with predetermined condition on it. This such problems are known as moving boundary value problems (MBVPs). Examples of such problems are diffusion of gas , ice melting, crystallization of a melt, shoreline problems etc and In general finding the analytic solutions of such above problems are limited to a very few particular cases [1, 3, 17]. Recently, many attempts have been made to solve these types of problems in which several numerical and approximate methods have been proposed to solve such problems. Rasmussen in [21] discussed an approximate method, which is based on integrating the heat conduction equation and applying the boundary condition on the moving interface for solving two-dimensional Stefan problems and applied this method to two particular cases. Gupta and Banik in [6] presented constrained integral method, which is an approximate method for solving one- dimensional MBVPs based on the various parameters in the choice of a temperature/concentration profile are expressed as functions of the position of the moving boundary plus an additional parameter at the fixed surface. In [7] the same authors used constrained integral method to solve two-dimensional melting problem. Lesnic and Elliot in [16] suggested decomposition approach to solve inverse heat conduction problem. Hon and Wei in 2005 [10] proposed the method of fundamental solution as an approximate method to solving multidimensional inverse conduction problems.

*Corresponding author
Email addresses: adnanyasen.math@utq.edu.iq (Adnan Yassean Nama), dr.fsf67@gmail.com (Fadhel S. Fadhel)

Received: January 2022 *Accepted:* March 2022

Many other authors [2, 19, 23] discussed approximate numerical methods to solve MBVPs, such as Rajeev in 2014 [20] proposed the homotopy perturbation method for solving a particular case of Stefan problem. In [27], authors applied the variational iteration method (VIM) to solve Stefan problem and discussed and used the VIM as a tool for solving partial differential equations, in which they obtain analytical approximate solution of gas dynamics equation and Stefan problem.

As a continuation with the above discussion, in resent years, fractional differential equations (FDEs) have become one of the most exciting and extremely active areas of research because of their potential applications in physics and engineering. These include fluid flow, electrical network, optics etc. [4, 16, 18, 27]. In this topic, different types of fractional integration and differentiation operators are proposed. The most known of them is the Riemann-Liouville definition [22], which has been used successfully in many fields of research and engineering, however it leads to the result that constant function differentiation is not zero. Caputo put definitions which give zero value for fractional differentiation of constant function, but these definitions demand that the function be smooth and differentiable [22]. Recently, Jumarie developed formulations for the fractional integral and derivative known as modified RiemannLiouville [12, 13, 14], which are applicable for continuous and nondifferentiable functions and give differentiation of a constant function equal to zero. The modified Riemann-Liouville fractional definitions are useful in a wide range of applications [8, 25, 26], because some of properties which are realized such as, the $\alpha-$ th derivative of order fractional of a constan, $0 < \alpha < 1$; is zero, fractional Leibniz product law and fractional Leibniz formulation.

In this paper, we will use the fractional variational iteration method (FVIM), which is a modification of the VIM to find the solution of two-dimensional fractional moving boundary problem (Stefan problem) with time-fractional order derivative in Jumarie sense, and compare the results of the solution with the exact results in [24].

2 Basic Concept and Definitions

In this section, some of definitions of fractional calculus with related properties which are used in this paper are presented for formulations purpose.

Definition 2.1. [12] Suppose that $g : R \to R$ be a continuous (not necessarily differentiable) function, let the partition $j > 0$, such that $j \in [0, 1]$. The derivative of Jumarie is defined through the fractional difference:

$$\Delta^{\gamma} = (FW - 1)^{\gamma} g(x) = \sum_{i=0}^{\infty} (-1)^i \begin{pmatrix} \gamma \\ i \end{pmatrix} g[x + (\gamma - i)j], \tag{2.1}$$

where $FWg(x) = g(x + j)$. Then the of fractional derivative of g is given by the following limit:

$$g^{\gamma}(x) = \lim_{j \to 0} \frac{\Delta^{\gamma}[g(x) - g(0)]}{j^{\gamma}}, \quad 0 < \gamma < 1. \tag{2.2}$$

It is notable that the $\gamma - th$ fractional order derivative of a constant function equals zero.

Definition 2.2. [12] The Riemann-Lioville fractional integral of order $\gamma > 0$, of a continuous function $g(x)$ is given by:

$${}_0I_x^{\gamma} g(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x - \xi)^{\gamma - 1} g(\xi) d\xi, \quad x > 0. \tag{2.3}$$

Definition 2.3. [12] The modified Riemann-Lioville fractional derivative of order $\gamma > 0$, of a continuous function $g(x)$ is given by:

$${}_0D_x^{\gamma} g(x) = \frac{1}{\Gamma(n - \gamma)} \frac{d^n}{dx^n} \int_0^x (x - \xi)^{n - \gamma} [g(\xi) - g(0)] d\xi, \tag{2.4}$$

where $x \in [0, 1], n - 1 \leq \gamma < n$, and $0 \leq n$. Some of the must important properties of the fractional modified Riemann-Liouville derivative are given in the following [12]:

(a) Fractional Leibniz product law:

$${}_oD_x^{\gamma}(pq) = p^{(\gamma)} q + pq^{(\gamma)}, \tag{2.5}$$

where $p, q : R \to R$ are two continuous functions.

(b) Fractional Leibniz Formulation:

$$ {}_0I_x^\gamma {}_0D_x^\gamma g(x) = g(x) - g(0), \quad 0 < \gamma \leq 1. \tag{2.6}$$

(c) The formula of fractional integration by parts:

$$ {}_aI_b^\gamma \left(p^{(\gamma)}q\right) = (pq)|_a^b + {}_aI_b^\gamma \left(pq^{(\gamma)}\right). \tag{2.7}$$

In order to use a simpler and appropriate formula to perform the integration calculations, the following lemma can be adopted:

Lemma 2.4. [18] Let $g : R \to R$ be a continuous function, then

$$I^\gamma g(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x-\xi)^{\gamma-1} g(\xi) d\xi = \frac{1}{\Gamma(1+\gamma)} \int_0^x g(\xi)(d\xi)^\gamma, \quad 0 < \gamma \leq 1 \tag{2.8}$$

3 Two-Dimensional Stefan Problem

Consider the two-dimensional single phase Stefan problem:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \text{ in } \Omega(t), 0 \leq t \leq T = 1, \tag{3.1}$$

where $\Omega(t) = \{x, y \mid 0 \leq x \leq 1, 0 \leq y \leq s(x,t)\}, s(x,t)$ denote the unknown moving surface, and with the associated boundary conditions given by:

$$\frac{\partial w}{\partial x} = g_1(y,t), \quad x = 0, x = 1, \tag{3.2}$$

$$w = 1, y = s(x,t), \tag{3.3}$$

$$w = g_2(x,t), \quad y = 0 \tag{3.4}$$

More conditions are needed to find moving boundary conditions which are:

$$\mathrm{s(x,t)} = g_3(x), \quad t = 0, \tag{3.5}$$

$$\left(\frac{\partial s}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) = -\left(\frac{\partial w}{\partial x}\right), y = s(x,t) \tag{3.6}$$

with the initial temperature distribution:

$$w(x,y,0) = e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)}, \tag{3.7}$$

where $w(x,y,t)$ is the temperature at a point (x,y) in a two-dimensional space domain $\Omega(t)$ at any time t, g_1, g_2 and g_3 are known functions.

Many important phenomena in physics are formulated by differential equations of fractional order. These fractional derivatives work more appropriately compared with the standard integer order models. So, the fractional derivatives are regarded as very dominating and useful tool [15, 18, 27]. Stefan problem has many fractional forms. When the derivative is fractional derivative, time- fractional Stefan problem which corresponding to above problem is

$$\frac{\partial^\gamma w}{\partial t^\gamma} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \text{ in } \Omega(t), 0 \leq t \leq T = 1 \text{ and } 0 < \gamma \leq 1 \tag{3.8}$$

Where $\Omega(t) = \{x, y \mid 0 \leq x \leq 1, 0 \leq y \leq s(x,t)\}, \quad s(x,t)$ denoted to unknown moving interface, the associated boundary conditions and initial boundary are the same as the previous boundary.

4 Solution of the Problem by using FVIM

Solution of the equation (3.8) with initial and boundary conditions (3.2)-(3.7) is to find the temperature $w(x,y,t)$ and the moving interface $s(x,t)$. Now to solve presented problem. According to the FVIM, we consider the correction functional in t-direction as follows:

$$w_{n+1}(x,y,t) = w_n(x,y,t) + I_\tau^\gamma \left[\lambda(t,\tau)\left(\frac{\partial^\gamma w_n}{\partial \tau^\gamma} - \frac{\partial^2 \widetilde{w}_n}{\partial x^2} - \frac{\partial^2 \widetilde{w}_n}{\partial y^2}\right)\right] \tag{4.1}$$

where I_τ^γ is the Riemann-Lioville fractional integral operator of order $\gamma > 0$, $\lambda(x,\xi)$ is the general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation, that is, $\delta\tilde{u}_n(x,t) = 0$, where δ is taken as the first variation.

The successive approximation $w_{n+1}(x,y,t)$, for n $= 0, 1, \ldots$ of the solution $w(x,y,t)$ will be readily obtained after determining the Lagrange multiplier and starting with any selective function $w_0(x,y,t)$. Consequently, the solution is

$$w(x,y,t) = \lim_{n\to\infty} w_n(x,y,t) \tag{4.2}$$

In order to find λ, first rewrite the iteration formula (4.1) as:

$$w_{n+1}(x,y,t) = w_n(x,y,t) + \frac{1}{\Gamma(1+\gamma)}\int_0^\tau \lambda(t,\tau)\left(\frac{\partial^\gamma w_n}{\partial \tau^\gamma} - \frac{\partial^2 \widetilde{w}_n}{\partial x^2} - \frac{\partial^2 \widetilde{w}_n}{\partial y^2}\right)(d\tau)^\gamma \tag{4.3}$$

and to make the functional in (4.3) stationary, the following condition can be obtained after taking the first variation with respect to t :

$$\begin{aligned}
\delta w_{n+1}(x,y,t) &= \delta w_n(x,y,t) + \frac{\delta}{\Gamma(1+\gamma)}\int_0^\tau \lambda(t,\tau)\left(\frac{\partial^\gamma w_n}{\partial \tau^\gamma} - \frac{\partial^2 \tilde{w}_n}{\partial x^2} - \frac{\partial^2 \widetilde{w}_n}{\partial y^2}\right)(d\tau)^\gamma \\
&= \delta w_n(x,y,t) + \lambda(t,\tau)\delta w_n(x,y,t)|_{\tau=t} - \frac{1}{\Gamma(1+\gamma)}\int_0^\tau \frac{\partial^\gamma \lambda(t,\tau)}{\partial \tau^\gamma}\delta w_n(x,y,t)(d\tau)^\gamma
\end{aligned}$$

Now, we can set the coefficients of $\delta w_n(x,y,t)$ to zero in equation (4.3):

$$1 + \lambda(t,\tau)|_{\tau=t} = 0 \text{ and } \left.\frac{\partial^\gamma \lambda(t,\tau)}{\partial \tau^\gamma}\right|_{\tau=t} = 0. \tag{4.4}$$

Therefore, $\lambda(t,\tau)$ can be identified as:

$$\lambda(t,\tau) = -1. \tag{4.5}$$

Substituting the value of λ in (4.3), the iteration formulation will be read as follows:

$$w_{n+1}(x,y,t) = w_n(x,y,t) - \frac{1}{\Gamma(1+\gamma)}\int_0^\tau \left(\frac{\partial^\gamma w_n}{\partial \tau^\gamma} - \frac{\partial^2 w_n}{\partial x^2} - \frac{\partial^2 w_n}{\partial y^2}\right)(d\tau)^\gamma \tag{4.6}$$

Choosing initial approximate solution as $w_0(x,y,t) = e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)}$, then we can evaluate:

$$\begin{aligned}
w_1(x,y,t) &= w_0(x,y,t) - \frac{1}{\Gamma(1+\gamma)}\int_0^\tau \left(\frac{\partial^\gamma w_0}{\partial \tau^\gamma} - \frac{\partial^2 w_0}{\partial x^2} - \frac{\partial^2 w_0}{\partial y^2}\right)(d\tau)^\gamma \\
&= e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)} - \frac{1}{\Gamma(1+\gamma)}\int_0^\tau \left[-\frac{1}{4}e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)} - e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)}\right](d\tau)^\gamma \\
&= e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)} - \frac{5}{4\Gamma(1+\gamma)}e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)}t^\gamma \\
w_2(x,y,t) &= w_1(x,y,t) - \frac{1}{\Gamma(1+\gamma)}\int_0^\tau \left(\frac{\partial^\gamma w_1}{\partial \tau^\gamma} - \frac{\partial^2 w_1}{\partial x^2} - \frac{\partial^2 w_1}{\partial y^2}\right)(d\tau)^\gamma \\
&= w_1(x,y,t) - \frac{1}{\Gamma(1+\gamma)}\int_0^\tau \left[-2\frac{5}{4}e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)} + \frac{5^2}{4^2\Gamma(1+\gamma)}e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)}\tau^\gamma\right](d\tau)^\gamma \\
&= e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)} + \frac{5}{4\Gamma(1+\gamma)}e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)}t^\gamma - \frac{5^2}{4^2\Gamma^2(1+\gamma)}e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)}t^{2\gamma}
\end{aligned}$$

$$w_3(x,y,t) = w_2(x,y,t) - \frac{1}{\Gamma(1+\gamma)}\int_0^\tau \left(\frac{\partial^\gamma w_2}{\partial \tau^\gamma} - \frac{\partial^2 w_2}{\partial x^2} - \frac{\partial^2 w_2}{\partial y^2}\right)(d\tau)^\gamma$$
$$= w_2(x,y,t) - \frac{1}{\Gamma(1+\gamma)}\int_0^\tau \left[-\frac{5}{4} + \frac{5}{4\Gamma(1+\gamma)} - \frac{5^2}{4^2\Gamma(1+\gamma)}\tau^\gamma + \frac{5^3}{4^3\Gamma^2(1+\gamma)}\tau^{2\gamma} - \frac{5^2\Gamma(1+2\gamma)}{4^2\Gamma^3(1+\gamma)}\tau^\gamma\right] e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)}(d\tau)^\gamma$$
$$= \left[1 + 2\frac{5}{4\Gamma(1+\gamma)}t^\gamma - \frac{5}{4\Gamma^2(1+\gamma)}t^\gamma + \frac{5^2}{4^2\Gamma^4(1+\gamma)}t^{2\gamma} - \frac{5^3}{4^3\Gamma^3(1+\gamma)}t^{3\gamma}\right] e^{\left(-y+\frac{1}{2}x+\frac{1}{2}\right)}$$

and so on using this procedure for a sufficiently large value of n, to get $w_n(x,y,t)$ as an approximation to the exact solution $w(x,y,t)$ of problem under consideration.

The second part of problem is to find the moving interface $s(x,t)$, which fulfills conditions (3.5) and (3.6). We can consider

$$s(x,t) = Ax + B\Gamma(1+\gamma)t^\gamma + C \tag{4.7}$$

where A, B and C are parameters which must be evaluated. By using (3.6) then we get $A = 1/2$ and the problem is reduced to estimate only the parameters B and C. Condition (3.6) gives the relation between the moving interface $s(x,t)$ and the temperature $w(x,y,t)$. Therefore, the most suitable method that may be used to find the parameters B and C is by using the least square method as follows:

Can be seeing the relation between the temperature $w(x,y,t)$ and the moving interface $s(x,t)$ in equation (3.6), which is the condition of moving interface. From the equation (4.7), may be find:

$\frac{\partial s(x,t)}{\partial x} = \frac{1}{2}$, also find $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ of approximate solution $w_n(x,y,t)$. After that substituting these partial derivatives in equation (3.6). Now by using the least square method on the obtained equation (here $n = 3$ and $\alpha = 1$). We get $C = 1/2$, while the parameter B is changes with initial parameters, implies that:

$$s(x,0) = \frac{1}{2}x + \frac{1}{2}.$$

which is self-exact $s_0(x)$ given in [24].

5 Numerical Simulation

The results for temperature $w(x,y,t)$ and moving interface $s(x,t)$ were simulated calculated and by using MATHCAD computer software and depicted through figures. Also, we discussed in detail the solution of the problem, for different cases of fractional order derivatives $0 < \gamma \leq 1$. It is clear that from Figures 1 and 2 the degree of congruence in the results between the exact and approximate solutions of temperature. Where the temperature was calculated when $t = 0$ and $t = 1$ using the above two solutions, respectively.

Figures 3 and 4 show the results of the analytic approximate solution of temperature when the fractional order $\gamma = 0.9$ and 0.8 at time $t = 0.7$ and $t = 1$, respectively. From Figure 5 it can be seen the change of the moving interface when the parameter B is changed. In this figure the plot of moving surface when $B = 0.8, 1, 1.25$ and $B = 1$ are presented respectively.

6 Conclusion

The variational iteration method has been known as an efficient approximate method to solve linear and nonlinear differential equations, delay differential equation, intgrodifferential equations and many another types of equations. In this paper, a general framework of the FVIM is used and modified Riemann-Lioville fractional derivative as analytic approximate and numerical treatment to solve partial differential equation with moving boundary. It is powerful and effective method to solve this kind of problems, and it is easy to apply. Therefore can be used it to solve another problems in mechanic fluids or engineering.

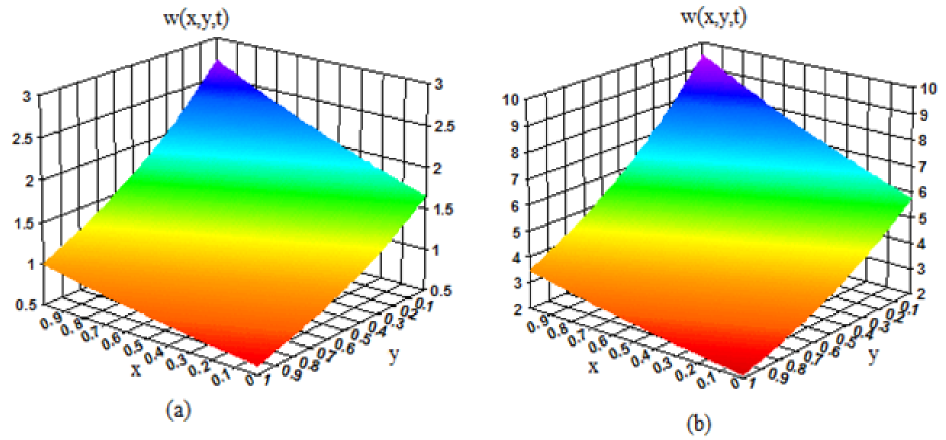

Figure 1: Plot of the exact temperature at $t = 0$ and $t = 1$ respectively [27].

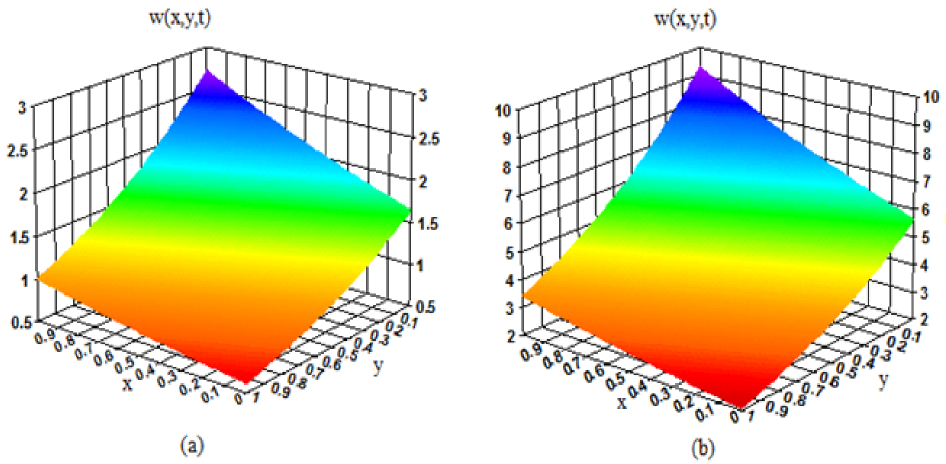

Figure 2: Plot of the approximate temperature at $t = 0$ and $t = 1$ where $\gamma = 1$ respectively.

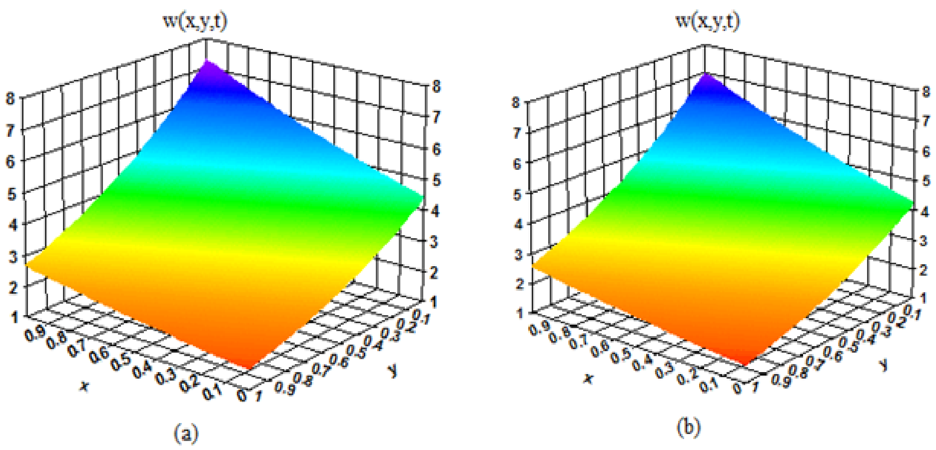

Figure 3: Plot of the approximate temperature at $t = 0.8$ and $\gamma = 0.9, 0.8$, respectively.

References

[1] S.G. Bankoff, *Heat conduction or diffusion with change of phase*, Adv. Chem. Engin. **5** (1964), 75–150.

[2] H. Capart, M. Bellal and D.L. Young, *Self-similar evolution of semi-infinite alluvial channels with moving boundaries*, J. Sedimentary Res. **77** (2007), no. 1, 13–22.

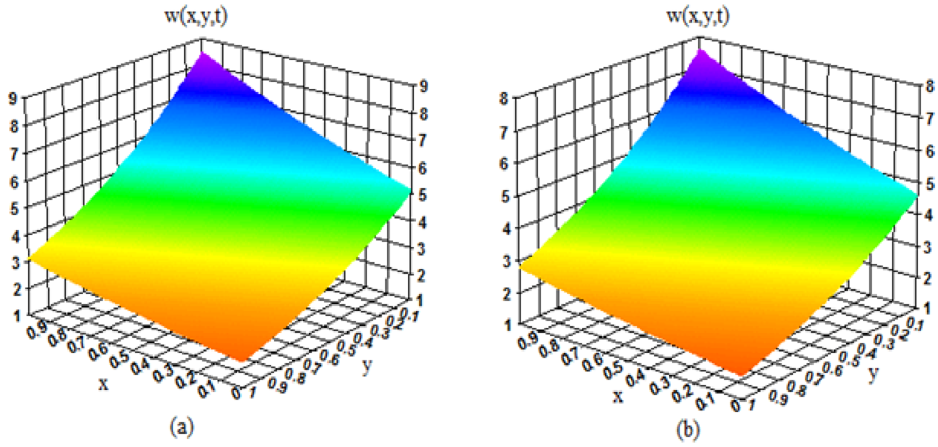

Figure 4: Plot of the approximate temperature at $t = 1$ and $\gamma = 0.9, 0.8$, respectively.

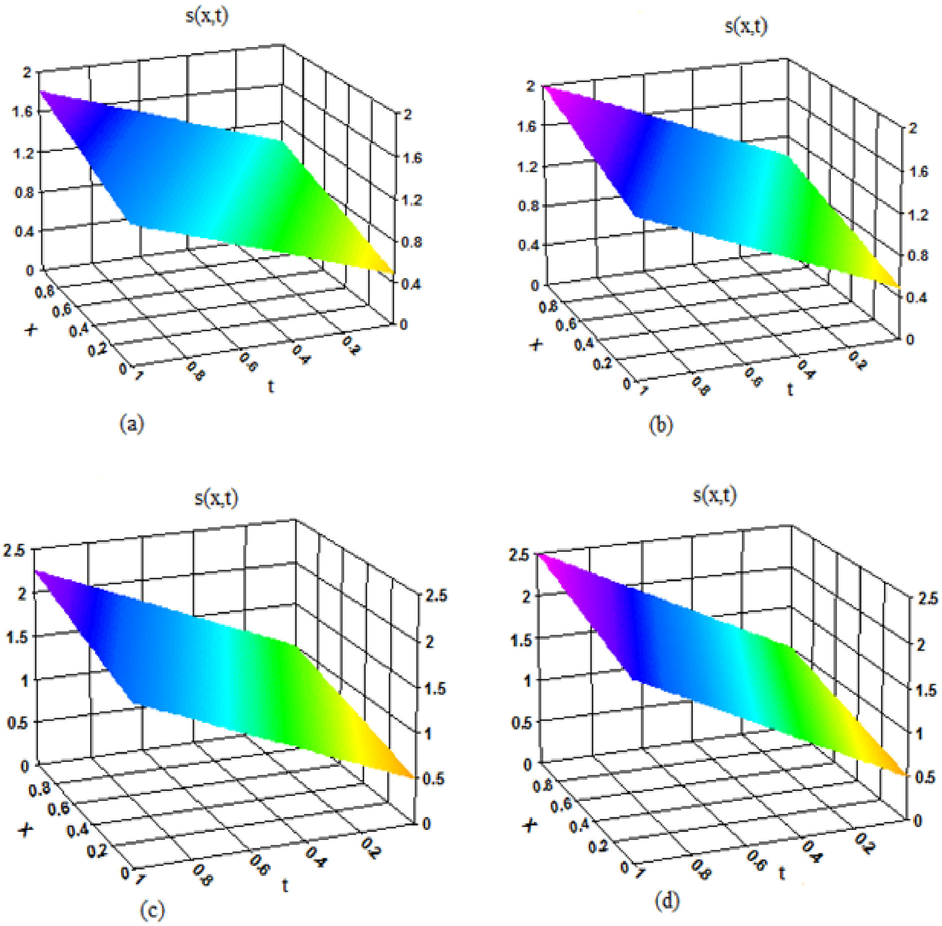

Figure 5: Plot of the approximate moving interface $s(x, t)$ at $\gamma = 1$ for different values of $B = 0.8, 1, 1.25$ and 1.5 respectively.

[3] J. Crank and J. Crank, *Free and moving boundary problems*, Oxford University Press, USA, 1984.

[4] M. Cui, *Compact finite difference method for the fractional diffusion equation*, J. Comput. Phys. **228** (2009), no. 20, 7792–7804.

[5] B.A. Finlayson, *The method of weighted residuals and variational principles*, volume 87 of Mathematics in Science

and Engineering, SIAM, 1972.

[6] R.S. Gupta and N.C. Banik, *Constrained integral method for solving moving boundary problems*, Comput. Meth. Appl. Mmech. Engin. **67** (1988), no. 2, 211–221.

[7] R.S. Gupta and N.C. Banik. *Solution of a weakly two-dimensional melting problem by an approximate method*, J. Comput. Appl. Math. **31** (1990), no. 3, 351–356.

[8] Z. Hammouch and T. Mekkaoui. *Travelling-wave solutions for some fractional partial differential equation by means of generalized trigonometry functions*. Int. J. Appl. Math. Res. **1** (2012), no. 1, 206–212.

[9] J.H. He. *Approximate analytical solution for seepage flow with fractional derivatives in porous media*, Comput. Meth. Appl. Mech. Engin. **167** (1998), no. 1-2, 57–68.

[10] Y.C. Hon and T. Wei, *The method of fundamental solution for solving multidimensional inverse heat conduction problems*, CMES Comput. Model. Eng. Sci. **7** (2005), no. 2, 119–132.

[11] M. Inokuti, H. Sekine and T. Mura, *General use of the Lagrange multiplier in nonlinear mathematical physics*, Var. Method Mech. Solids **33** (1978), no. 5, 156–162.

[12] G. Jumarie, *Table of some basic fractional calculus formulae derived from a modified Riemann–Liouville derivative for non-differentiable functions*, Appl. Math. Lett. **22** (2009), no. 3, 378–385.

[13] G. Jumarie, *An approach via fractional analysis to non-linearity induced by coarse-graining in space*, Nonlinear Anal. Real World Appl. **11** (2010), no. 1, 535–546.

[14] G. Jumarie, *On the fractional solution of the equation* $f(x+y)=f(x)f(y)$ *and its application to fractional Laplace's transform*, Appl. Math. Comput. **219** (2012), no. 4, 1625–1643.

[15] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

[16] D. Lesnic and L. Elliott, *The decomposition approach to inverse heat conduction*, J. Math. Anal. Appl. 232 (1999), no. 1, 82–98.

[17] J.C. Muehlbauer, *Heat conduction with freezing or melting*, Appl. Mech. Rev. **18** (1965), 951.

[18] Z. Odibat and S. Momani, *A generalized differential transform method for linear partial differential equations of fractional order*, Appl. Math. Lett. 21 (2008), no. 2, 194–199.

[19] K.N. Rai and S. Das, *Numerical solution of a moving-boundary problem with variable latent heat*, Int. J. Heat Mass Transfer **52** (2009), no. 7-8, 1913–1917.

[20] Rajeev, *Homotopy perturbation method for a Stefan problem with variable latent heat.* Thermal Sci. 18 (2014), no. 2, 391–398.

[21] H. Rasmussen, *An approximate method for solving two-dimensional Stefan problems*, Lett. Heat Mass Transfer **4** (1977), no. 4, 273–277.

[22] J.J. Trujillo, E. Scalas, K. Diethelm and D. Baleanu. *Fractional calculus: Models and numerical methods*, World Scientific, 2016.

[23] V.R. Voller, J.B. Swenson, W. Kimn and C. Paola, *An enthalpy method for moving boundary problems on the earth's surface*, Int. J. Numerical Meth. Heat Fluid Flow **16** (2006), no. 5, 641–654.

[24] S. Wang and P. Perdikaris. *Deep learning of free boundary and Stefan problems*, J. Comput. Phys. **428** (2021), 109914.

[25] H. Wang and T.C. Xia, *The fractional supertrace identity and its application to the super Jaulent–Miodek hierarchy*, Commun. Nonlinear Sci. Numerical Simul. **18** (2013), no. 10, 2859–2867.

[26] G.W. Wang, X.Q. Liu and Y.Y. Zhang, *Lie symmetry analysis to the time fractional generalized fifth-order KdV equation*, Commun. Nonlinear Sci. Numerical Simul. **18** (2013), no. 9, 2321–2326.

[27] B.J. West, M. Bologna and P. Grigolini, *Physics of fractal operators*, Springer, New York, 2003.