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Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral for Gronwall-Bellman's type lemma on time scales

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Abstract

We discuss some new results for Gronwall's type inequality on time scales. An analysis of the behavior of the solutions of some hyperbolic partial differential equations as applicable to our results is considered.

Keywords: Integral inequalities, Double integral, Henstock-Kurzweil integral, Gronwall type, Time scales
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1 Introduction

The Henstock-Kurzweil-Stieltjes integral is a generalized Riemann-Stieltjes integral with similar properties. In 1988, Hilger [9] introduced the theory of time scales in his Ph.D. Thesis. The Henstock delta integral on time scales was introduced by Peterson and Thompson [15] and Henstock-Kurzweil integrals on time scales was studied by Thompson [16]. The approaches adapted to the time scale setting can be used to derive the majority of the properties of a time scale integral (see [2, 3, 7, 9, 15, 16]).

In 2020, Khan et al.,[10] gives the derivation of dynamical integral inequalities based on two-dimensional time scales theory. Gronwall type inequalities for interval-valued functions on time scales was given by Younus et al.,[17]. Bellman [4] discussed stability of solutions of linear differential equations, and similar result is obtained in Bellman [5]: If the functions $g(t)$ and $u(t)$ are nonnegative for $t \geq 0$, and if $c \geq 0$, then the inequality

$$u(t) \leq c + \int_0^t g(s)u(s)ds, \quad t \geq 0$$

implies that

$$u(t) \leq c\ exp\left(\int_0^t g(s)ds\right), \quad \text{for } t \geq 0.$$

Different applications of the result of Bellman [5] to the study of stability of the solution of linear and nonlinear differential equations can be seen in Bellman [4]. Some other applications to existence and uniqueness theory of

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differential equations can be seen in Bihari [6], Langenhop [12], Nemyckii and Stepanov [14]. But none of these aforementioned researchers has considered the application of Henstock-Kurzweil-Stieltjes-$\Diamond$-double integral on nonlinear integral inequalities of Gronwall type on time scales.

Gronwall-Bellman type inequalities are useful tools to obtain various estimates in the theory of differential equations, see [4]. There are several mathematical models to study the behavior of the real-life situations such as: static or dynamic, linear or nonlinear, continuous or discrete, deterministic or probabilistic. Some of the applications of Henstock-Kurzweil-Stieltjes integral for Gronwall-Bellman's type lemma in finance are found in Kozlowski [11], "Hedging the Black-Scholes call option", and also in elementary stochastic calculus with finance in view by Mikosch [13].

The aim of this paper is to obtain some results of Gronwall-Bellman's type lemma when dealing with Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrals on time scales.

2 Preliminaries

First, we recall some basic concepts used in this paper and also refer interested reader to ([7] , [15, 16]) for detailed theory of time scales.

Definition 2.1. Let $\mathbb{T}$ be a time scale. A function $f : \mathbb{T} \to \mathbb{R}$ is continuous at a point s if there exists $\varepsilon > 0$ such that $|f(t) - f(s)| < \varepsilon$, and for any $\delta > 0$, there is $|f - s| < \delta$.

Right-scattered at $t \in \mathbb{T}$ if $\sigma(t) > t$ and left-scattered at $t \in \mathbb{T}$ if $\rho(t) < t$. It is right-dense at $t \in \mathbb{T}$ if $t < \sup \mathbb{T}$ and $\sigma(t) = t$ and left-dense at $t \in \mathbb{T}$ if $t > \inf \mathbb{T}$ and $\rho(t) = t$.

The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ for all $t \in \mathbb{T}$.

A mapping $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if:
(i) f is continuous at each right-dense point of $\mathbb{T}$
(ii) at each left-dense point $t \in \mathbb{T}$, $\lim_{s\to t^-} g(s) = g(t^-)$ exists.

Let $a, b \in \mathbb{T}_1, c, d \in \mathbb{T}_2$, where $a < d, c < d$, and a rectangle $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} = \{(t, s) : t \in [a, b), s \in [c, d), t \in \mathbb{T}_1, s \in \mathbb{T}_2\}$. Let $g_1, g_2 : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be two non-decreasing functions on $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$, respectively. Let $F : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be bounded on $\mathcal{R}$. Let P_1 and P_2 be two partitions of $[a, b]_{\mathbb{T}_1}$ and $[c, d]_{\mathbb{T}_2}$ such that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ and $P_2 = \{s_0, s_1, ..., s_n\} \subset [c, d]_{\mathbb{T}_2}$. Let $\{\xi_1, \xi_2, ..., \xi_n\}$ denote an arbitrary selection of points from $[a, b]_{\mathbb{T}_1}$ with $\xi_i \in [t_{i-1}, t_i)_{\mathbb{T}_1}, i = 1, 2, ..., n$. Similarly, let $\{\zeta_1, \zeta_2, ..., \zeta_n\}$ denote an arbitrary selection of points from $[c, d]_{\mathbb{T}_2}$ with $\zeta_j \in [s_{j-1}, s_j)_{\mathbb{T}_2}, j = 1, 2, ..., k$.

Definition 2.2. [1] Let $F : [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \to \mathbb{R}$ be a bounded function on $\mathcal{R}$ and let g be a non-decreasing function defined on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ with partitions $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, 2, ..., n$ and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c, d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2}$ for $j = 1, 2, ..., k$. Then

$$S(P_1, P_2, F, g_1, g_2) = \sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_i, \zeta_j)(g_1(t_i) - g_1(t_{i-1}))(g_2(s_j) - g_2(s_{j-1}))$$

is defined as Henstock-Kurweil-Stieltjes-$\Diamond$-double sum of F with respect to functions g_1 and g_2.

The Henstock-Kurweil-Stieltjes-$\Diamond$-double sum of F with respect to functions g_1 and g_2 is denoted by $S(P, F, g_1, g_2)$, where $P = P_1 \times P_2$.

Definition 2.3. Let $F : [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} \to \mathbb{R}$ be a bounded function on $\mathcal{R} = [a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2} : t \in [a, b)_{\mathbb{T}_1}, s \in [c, d)_{\mathbb{T}_1}$. We say that F is Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable with respect to non-decreasing functions g_1, g_2 defined on $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$ if there is a number L, a member of $\mathbb{R}$ such that for every $\varepsilon > 0$, there is a $\Diamond$-gauge δ (or γ) such that

$$|S(P, F, g_1, g_2) - L| < \varepsilon$$

provided that $P_1 = \{t_0, t_1, ..., t_n\} \subset [a, b]_{\mathbb{T}_1}$ with tag points $\xi_i \in [t_{i-1}, t_i]_{\mathbb{T}_1}$ for $i = 1, ..., n$ and $P_2 = \{s_0, s_1, ..., s_k\} \subset [c, d]_{\mathbb{T}_2}$ with tag points $\zeta_j \in [s_{j-1}, s_j]_{\mathbb{T}_2},\ j = 1, 2, ..., k$ are δ-fine (or γ) partitions of $[a, b)_{\mathbb{T}_1} \times [c, d)_{\mathbb{T}_2}$.

We say that L is the Henstock-Kurzweil-Stieltjes-◊-double integral of F with respect to g_1 and g_2 defined on $[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$, and write

$$\int\int_{\mathcal{R}} F(s,t)\Diamond g_1(t)\Diamond g_2(s) = L.$$

Proposition 2.4. If $F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to \mathbb{R}$ is Henstock-Kurzweil-Stieltjes-◊-double integrable with respect to monotone increasing functions g_1 and g_2 defined on $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively, and

$$f(s,t) = \int\int_{\mathcal{R}} F(s,t)\Diamond g_1(t)\Diamond g_2(s)$$

then f is rd-continuous and

$$f(s,t) = F(s,t) \;\; a.e. \in \;\; [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}.$$

Proof . See [1]. Similar proof for functions of single variable also appears in [8]. □

3 The Main Results

Throughout this paper, all the functions used are assumed to be real-valued. $\mathcal{R}$ denotes a rectangle and $\mathbb{R}$ is a set of real numbers with $\mathbb{R}_+ = [0,\infty)$, $I_1 = [x_0, A)$ and $I_2 = [y_0, B)$ be given subsets of $\mathbb{R}$.

Theorem 3.1. Let $G, F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to \mathbb{R}$ be Henstock-Kurzweil-Stieltjes-◊-double integrable with respect to monotone increasing functions g_1 and g_2 on $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively. Let $H(x,y)$ be monotone increasing function in each of the variables for $x_0 \le x,\; y_0 \le y$. Suppose that

$$G(x,y) \le H(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} F(s,t)G^p(t,s)\Diamond g_1(t)\Diamond g_2(s) \tag{3.1}$$

$x_0 \le x,\; y_0 \le y$, where $p \ge 0, p \ne 1$, is a constant. Then

$$G(x,y) \le \left[H^q(x,y) + q\int_{x_0}^{x}\int_{y_0}^{y} F(s,t)\Diamond g_1(t)\Diamond g_2(s)\right]^{\frac{1}{q}} \tag{3.2}$$

for $x \in [x_0, A),\; y \in [y_0, B)$, where $q = 1-p,\;$ A and B are chosen so that the expression between the interval is positive in the subintervals $[x_0, A)$ and $[y_0, B)$.

Proof . Let $x_0 \le A$ and $x \le B$ be fixed. For $x_0 \le x \le A,\; y_0 \le y \le B$ we have

$$G(x,y) \le H(A,B) + \int_{x_0}^{x}\left(\int_{y_0}^{y} F(s,t)G^p(s,t)\Diamond g_1(t)\right)\Diamond g_2(s). \tag{3.3}$$

Let the first order partial derivatives of function $Z(x,y)$ defined for $x, y \in \mathbb{R}$ with respect to x and y be denoted by $Z_x(x,y)$ and $Z_y(x,y)$ respectively. Define a function $J(x,y)$ by the right-hand side of (3.3). Then the function $J(x,y)$ is monotone increasing in each variable x, y, and $J(x_0,y) = H(A,B)$,

$$\frac{\partial J}{\partial b}(x,y) = \int_{y_0}^{y} F(x,t)G^p(x,t)\Diamond g_1(t) \le \int_{y_0}^{y} F(x,t)\Diamond g_1(t)J^p(x,y), \tag{3.4}$$

since $G(x,t) \le J(x,t)$. According to (3.4), the function $Z(x,y) = \frac{J^q(x,y)}{q}$ satisfies

$$\frac{\partial J}{\partial x}(x,y) = J^{q-1}(x,y)\frac{\partial J}{\partial x}(x,y) \le \int_{y_0}^{y} F(x,t)\Diamond g_1(t). \tag{3.5}$$

Integrating (3.5) over s from x_0 to x, and the change of variable gives

$$Z(x,y) \le \frac{1}{q}J^q(x_0,y) + \int_{x_0}^{x}\int_{y_0}^{y} F(s,t)\Diamond g_1(t)\Diamond g_2(s), \tag{3.6}$$

or

$$J^q(x,y) \leq\geq H^q(A,B) + q\int_{x_0}^{x}\int_{y_0}^{y} F(s,t)\Diamond g_1(t)\Diamond g_2(s), \tag{3.7}$$

where $(\leq,\ \geq)$ holds for $(q>0,\ q<0)$ respectively. Considering both cases, this estimate implies

$$J(x,y) \leq \left[H^q(A,B) + q\int_{x_0}^{x}\int_{y_0}^{y} F(s,t)\Diamond g_1(t)\Diamond g_2(s)\right]^{\frac{1}{q}} \tag{3.8}$$

for $x_0 \leq x \leq A,\ y_0 \leq y \leq B$. Now, let $x = A$ and $y = B$ and changing notation, we have

$$G(x,y) \leq \left[H^q(x,y) + q\int_{x_0}^{y}\int_{y_0}^{y} F(s,t)\Diamond g_1(t)\Diamond g_2(s)\right]^{\dfrac{1}{q}}$$

which ends the proof. □

We shall consider a special case of Theorem 3.1 as follows:

Corollary 3.2. Let $G(x,y), H(x,y), F(x,y)$ be rd-continuous functions in a rectangle $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ for $x_0 \leq x,\ y_0 \leq y$ and let $H(x)$ be monotone increasing in $x,\ x_0 \leq x,\ y_0 \leq y$, and $K(y)$ be monotone increasing in $y,\ y_0 \leq y$, functions g_1 and g_2 are non-decreasing functions respectively. Suppose that

$$G(x,y) \leq H(x) + K(y) + \int_{0}^{x}\int_{y}^{\infty} F(s,t)G^p(s,t)\Diamond g_1(t)\Diamond g_2(s), \quad x_0 \leq x,\ y_0 \leq y, \tag{3.9}$$

where $p \geq 0,\ p \neq 1$, is a constant. Then

$$G(x,y) \leq \left[(H(x)+K(y))^q + q\int_{x_0}^{x}\int_{x}^{y} F(s,t)\Diamond g_1(t)\Diamond g_2(s)\right]^{\dfrac{1}{q}} \tag{3.10}$$

for $x \in [x_0, A),\ y \in [y_0, B)$, where $q = 1-p,\ A$ and B are chosen so that the expression between the interval is positive in the subintervals $[x_0, A)$ and $[y_0, B)$.

Theorem 3.3. Let $G, F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to \mathbb{R}$ be Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable with respect to monotone increasing functions g_1 and g_2 on $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively. Let $G(x,y), H(x,y)$ be rd-continuous functions in a rectangle $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ for $x_0 \leq x,\ y_0 \leq y$ and let $H(x,y)$ be monotone increasing in each of the variables for x, y. Suppose that

$$G(x,y) \leq H(x,y) + \int_{x}^{\infty}\int_{y}^{\infty} F(s,t)G^p(t,s)\Diamond g_1(t)\Diamond g_2(s), \quad x \geq 0,\ y \geq 0, \tag{3.11}$$

where $p \geq 0,\ p \neq 1$, is a constant and

$$\int_{x}^{\infty}\int_{y}^{\infty} F(s,t)\Diamond g_1(t)\Diamond g_2(s) < \infty, \quad x \geq 0,\ y \geq 0,$$

Then

$$G(x,y) \leq \left[H^q(x,y) + q\int_{x}^{\infty}\int_{y}^{\infty} F(s,t)\Diamond g_1(t)\Diamond g_2(s)\right]^{\dfrac{1}{q}} \tag{3.12}$$

for $x \in [0, A),\ y \in [0, B)$, where $q = 1-p,\ A$ and B are chosen so that the expression between the interval is positive in the subintervals $[0, A)$ and $[0, B)$.

Proof . The proof of Theorem 3.3 is straightforward from the details of the proof of Theorem 3.1. Therefore, we omit the proof. □

Theorem 3.4. Let $G, F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to \mathbb{R}$ be Henstock-Kurzweil-Stieltjes-◊-double integrable with respect to monotone increasing functions g_1 and g_2 on $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively. Let $G(x,y), H(x,y)$ be rd-continuous functions in a rectangle $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ for $x_0 \le x,\ y_0 \le y$ and let $H(x,y)$ be monotone increasing function in x and monotone decreasing in y. Suppose that

$$G(x,y) \le H(x,y) + \int_0^x \int_y^\infty F(s,t) G^p(t,s) \Diamond g_1(t) \Diamond g_2(s), \quad x \ge 0,\ y \ge 0, \tag{3.13}$$

where $p \ge 0,\ p \ne 1$, is a constant and

$$\int_0^x \int_y^\infty F(s,t) \Diamond g_1(t) \Diamond g_2(s) < \infty, \quad x \ge 0,\ y \ge 0,$$

Then

$$G(x,y) \le \left[H^q(x,y) + q \int_x^\infty \int_y^\infty F(s,t) \Diamond g_1(t) \Diamond g_2(s) \right]^{\dfrac{1}{q}} \tag{3.14}$$

for $x \in [0,A),\ y \in [0,B)$, where $q = 1-p,\ A$ and B are chosen so that the expression between the interval is positive in the subintervals $[0,A)$ and $[0,B)$.

Proof . The proof of Theorem 3.4 follows from the proof of Theorem 3.1. Thus, we omit the proof. □

We shall give the following theorems for the generalizations of the theorems stated earlier.

Theorem 3.5. Let $G, F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to \mathbb{R}$ be Henstock-Kurzweil-Stieltjes-◊-double integrable with respect to monotone increasing functions g_1 and g_2 on $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively. Let $G(x,y), H(x,y), K(x,y)$ be rd-continuous functions in a rectangle $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ for $x_0 \le x,\ y_0 \le y$ and let $H(x,y)$ be monotone increasing in each of the variables for $x_0 \le x,\ y_0 \le y$. Suppose that

$$G(x,y) \le H(x,y) + \int_{x_0}^x K(s,y) G(s,y) \Diamond g_2(s) + \int_{x_0}^x \int_{y_0}^y F(s,t) G^p(s,t) \Diamond g_1(t) \Diamond g_2(s) \tag{3.15}$$

for $x_0 \le x,\ y_0 \le y$, where $p \ge 0,\ p \ne 1$, is a constant. Then

$$\begin{aligned} G(x,y) &\le exp\left(\int_{x_0}^x K(\theta,y) d\theta \right) \\ &\times \left[H^q(x,y) + q \int_{x_0}^x \int_{y_0}^y F(s,t) exp\left(\int_{x_0}^s K(\theta,y) d\theta \right) \Diamond g_1(t) \Diamond g_2(s) \right]^{\dfrac{1}{q}} \end{aligned}$$

for $x \in [x_0,A),\ y \in [y_0,B)$, where $q = 1-p,\ A$ and B are chosen so that the expression between the interval is positive in the subintervals $[x_0,A)$ and $[y_0,B)$.

Proof . Let a function $Z(x,y)$ be defined by

$$Z(x,y) \le H(x,y) + \int_{x_0}^x \int_{y_0}^y F(s,t) G^p(s,t) \Diamond g_1(t) \Diamond g_2(s). \tag{3.16}$$

Then $Z(x,y)$ is monotonic increasing in each variables x, y, and we restated (3.15) as

$$G(x,y) \le Z(x,y) + \int_{x_0}^x K(s,y) G(s,y) \Diamond g_2(s). \tag{3.17}$$

Let us further define a function $J(x,y)$ by

$$J(x,y) = \int_{x_0}^x K(s,y) G(s,y) \Diamond g_2(s).$$

Then $J(x_0,y) = 0$, we have

$$\frac{\partial J}{\partial x}(x,y) \le K(x,y) Z(x,y) + K(x,y) J(x,y), \tag{3.18}$$

since $G(x,y) \le Z(x,y) + J(x,y)$. The inequality (3.18) implies that

$$\left[\frac{\partial J}{\partial s}(s,y) - (s,y)J(s,y)\right] exp\left(\int_s^x K(\theta,y)d\theta\right) \le K(s,y)Z(s,y)exp\left(\int_s^x K(\theta,y)d\theta\right)$$

for $x_0 \le s$, or

$$\frac{\partial}{\partial s}\left[J(s,y)exp\left(\int_s^x K(\theta,y)d\theta\right)\right] \le K(s,y)Z(s,y)exp\left(\int_s^x K(\theta,y)d\theta\right).$$

Now integrating over s from x_0 to x, we have

$$J(x,y) \le \int_{x_0}^x K(s,y)Z(s,y)exp\left(\int_s^x K(\theta,y)d\theta\right)\Diamond g_2(s),$$

which implies that

$$J(x,y) \le Z(x,y)\int_{x_0}^x K(s,y)exp\left(\int_s^x K(\theta,y)d\theta\right)\Diamond g_2(s), \tag{3.19}$$

since $J(x_0,y) = 0$. From (3.17) and (3.19), we have

$$G(x,y) \le Z(x,y)exp\left(\int_{x_0}^x K(\theta,y)d\theta\right). \tag{3.20}$$

By using the definition of $Z(x,y)$ and (3.20), we get the estimate

$$Z(x,y) \le H(x,y) + \int_{x_0}^x\int_{y_0}^y F(s,t)exp\left(p\int_{x_0}^s K(\theta,y)d\theta\right)Z^p(s,t)\Diamond g_1(t)\Diamond g_2(s).$$

Now by Theorem 3.1, we have

$$Z(x,y) \le \left[H^q(x,y) + q\int_{x_0}^x\int_{y_0}^y F(s,t)exp\left(p\int_{x_0}^s K(\theta,y)d\theta\right)Z^p(s,t)\Diamond g_1(t)\Diamond g_2(s)\right]^{\frac{1}{q}}. \tag{3.21}$$

for $x \in [x_0, A),\ y \in [y_0, B)$, where $q = 1-p,\ A$ and B are chosen so that the expression between the interval is positive in the subintervals $[x_0, A)$ and $[y_0, B)$. By using (3.20) and (3.21), we have the desired inequality

$$G(x,y) \le exp\left(\int_{x_0}^x K(\theta,y)d\theta\right)$$
$$\times\left[H^q(x,y) + q\int_{x_0}^x\int_{y_0}^y F(s,t)exp\left(\int_{x_0}^s K(\theta,y)d\theta\right)\Diamond g_1(t)\Diamond g_2(s)\right]^{\frac{1}{q}}.$$

That ends the proof. □

Theorem 3.6. Let $G, F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to \mathbb{R}$ be Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable with respect to monotone increasing functions g_1 and g_2 on $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively. Let $G(x,y), H(x,y), K(x,y)$ be rd-continuous functions in a rectangle $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ for $x_0 \le x,\ y_0 \le y$ and let $H(x,y)$ be monotone decreasing in each of the variables for x, y. Suppose that

$$G(x,y) \le H(x,y) + \int_x^\infty K(s,y)G(s,y)\Diamond g_2(s) + \int_x^\infty\int_y^\infty F(s,t)G^p(s,t)\Diamond g_1(t)\Diamond g_2(s)$$

for $x_0 \le x,\ y_0 \le y$, where $p \ge 0,\ p \ne 1$, is a constant, and

$$\int_x^\infty K(s,y)\Diamond g_2(s) < \infty, \quad \int_x^\infty\int_y^\infty F(s,t)\Diamond g_1(t)\Diamond g_2(s) < \infty$$

for $x \geq 0, \quad y \geq 0$. Then

$$G(x,y) \leq exp\left(\int_x^\infty K(\theta,y)d\theta\right)$$
$$\times\left[H^q(x,y)+q\int_x^\infty\int_y^\infty F(s,t)exp\left(\int_s^\infty K(\theta,y)d\theta\right)\Diamond g_1(t)\Diamond g_2(s)\right]^{\frac{1}{q}}$$

for $x \in [0,A),\ y \in [0,B)$, where $q = 1-p,\ A$ and B are chosen so that the expression between the interval is positive in the subintervals $[0,A)$ and $[0,B)$.

Proof . The proof of Theorem 3.6 follows from Theorem 3.1 with little changes in notations of Theorem 3.4. Therefore, we omit the proof. $\square$

Theorem 3.7. Let $G,F : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to \mathbb{R}$ be Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrable with respect to monotone increasing functions g_1 and g_2 on $[a,b)_{\mathbb{T}_1}$ and $[c,d)_{\mathbb{T}_2}$ respectively. Let $G(x,y),H(x,y),K(x,y)$ be rd-continuous functions in a rectangle $\mathcal{R}=[a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$ for $x_0 \leq x,\ y_0 \leq y$ and let $H(x,y)$ be monotone increasing in x and monotone decreasing in y. Suppose that

$$G(x,y) \leq H(x,y)+\int_x^\infty K(s,y)G(s,y)\Diamond g_2(s)+\int_x^\infty\int_y^\infty F(s,t)G^p(s,t)\Diamond g_1(t)\Diamond g_2(s)$$

for $x_0 \leq x,\ y_0 \leq y$, where $p \geq 0,\ p \neq 1$, is a constant, and

$$\int_0^x\int_y^\infty F(s,t)\Diamond g_1(t)\Diamond g_2(s) < \infty$$

for $x \geq 0, \quad y \geq 0$. Then

$$G(x,y) \leq exp\left(\int_0^x K(\theta,y)d\theta\right)$$
$$\times\left[H^q(x,y)+q\int_0^x\int_y^\infty F(s,t)exp\left(\int_0^s K(\theta,y)d\theta\right)\Diamond g_1(t)\Diamond g_2(s)\right]^{\frac{1}{q}}$$

for $x \in [0,A),\ y \in [0,B)$, where $q = 1-p,\ A$ and B are chosen so that the expression between the interval is positive in the subintervals $[0,A)$ and $[0,B)$.

Proof . By the assertions of Theorem 3.5 with similar reasoning, then Theorem 3.7 holds. Therefore, we omit the proof. $\square$

4 Applications

We shall give some applications of Theorem 3.5 and Theorem 3.6 in hyperbolic partial differential equation. Some properties of solutions of terminal value problem will be considered.

Consider the following:

$$G_{xy}(x,y) - U(x,y,G(x,y)) + V(x,y), \tag{4.1}$$
$$G(x,\infty)=\psi_\infty(x), G(\infty,y)=\theta_\infty(y), G(\infty,\infty)=F, \tag{4.2}$$

where $U : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to \mathbb{R},\ V : [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2} \to \mathbb{R},\ \psi_\infty,\theta_\infty : \mathbb{R}_+ \to \mathbb{R}$ are continuous functions and F is a real constant.

Let us consider an example that estimates the solution of the partial differential equation (4.1) with condition (4.2).

Example 4.1. Suppose that the function U satisfies the condition

$$|U(x,y,G)| \leq F(x,y)|G|^p, \tag{4.3}$$

and

$$\left|\psi_\infty(x) + \theta_\infty(y) - F + \int_x^\infty \int_y^\infty V(s,t)\Diamond g_1(t)\Diamond g_2(s)\right| \leq H(x,y) + \int_x^\infty K(s,y)G(s,y)\Diamond g_2(s), \tag{4.4}$$

where $G(x,y), H(x,y), K(x,y)$ are rd-continuous functions in a rectangle $\mathcal{R} = [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. If $G(x,y)$ is a solution of (4.1) with the condition (4.2), then we have

$$G(x,y) = \psi_\infty(x) + \theta_\infty(y) - F + \int_x^\infty \int_y^\infty (U(s,t,G(s,t)) + V(s,t))\Diamond g_1(t)\Diamond g_2(s) \tag{4.5}$$

for $s,t \in [a,b)_{\mathbb{T}_1} \times [c,d)_{\mathbb{T}_2}$. Then from (4.3), (4.4), (4.5), we have

$$|G(x,y)| \leq H(x,y) + \int_x^\infty K(s,t)|G|\Diamond g_2(s) + \int_x^\infty \int_y^\infty F(x,y)|G|^p \Diamond g_1(t)\Diamond g_2(s). \tag{4.6}$$

By applying Theorem 3.6 in (4.6), we have the required estimate

$$|G(x,y)| \leq exp\left(\int_x^\infty K(\theta,y)d\theta\right)$$
$$\times \left[H^q(x,y) + q\int_x^\infty \int_y^\infty F(s,t)exp\left(\int_s^\infty K(\theta,y)d\theta\right)\Diamond g_1(t)\Diamond g_2(s)\right]^{\dfrac{1}{q}}$$

for $x \in [0,A),\ y \in [0,B)$, where $q = 1-p,\ A$ and B are chosen so that the expression between the interval is positive in the subintervals $[0,A)$ and $[0,B)$. The right hand side of the estimate yields the bound on the solution $G(x,y)$ of (4.1)and (4.2) in terms of the known functions.

Conclusion

In this paper, we have obtained some Gronwall-Bellman's type lemma when dealing with Henstock-Kurzweil-Stieltjes-$\Diamond$-double integrals on time scales. An analysis of the behavior of the solutions of some hyperbolic partial differential equations as an application to our results is considered.

Open Problem

Can analogue of the results in this paper be obtain for multidimensional case?

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