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Convergence theorems of new three-step iterations scheme for I-asymptotically nonexpansive mappings

Seyit Temir

Department of Mathematics, Art and Science Faculty, Adıyaman University, 02040, Adıyaman, Turkey

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Abstract

The purpose of this paper is to establish weak and strong convergence theorems of new three-step iterations for I-asymptotically nonexpansive mappings in Banach space.Also we introduce and study convergence theorems of the three-step iterative sequence for three I-asymptotically nonexpansive mappings in an uniformly convex Banach space. The results obtained in this paper extend and improve the recent ones announced by Chen and Guo [1], S. Temir [14], Yao and Noor[16] and many others.

Keywords: I-Asymptotically nonexpansive, common fixed point, iteration process, convergence theorems
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1 Introduction

Let K be a nonempty closed convex subset of a real normed space X. Let $T : K \to K$ be a mapping . Let $F(T) = \{x \in K : Tx = x\}$ be denoted as the set of fixed points of a mapping T.

$T : K \to K$ is called *asymptotically nonexpansive* mapping if there exist a sequence $\{\kappa_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \kappa_n = 1$ such that

$$\|T^n x - T^n y\| \leq \kappa_n \|x - y\|$$

for all $x, y \in K$ and $n \geq 1$. The mapping $T : K \to K$ is said to be uniformly Lipschitz with a Lipschitzian constant $L > 0$ if

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

holds for all $x, y \in K$ and $n \geq 1$. Note that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian with $L = sup\{\kappa_n : n \geq 1\}$.

In [2], Goebel and Kirk proved that, if K is a nonempty closed convex bounded subset of a uniformly convex Banach space X and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point in K.

Recently, in [9], [13] and [14], the convergence theorems for I-nonexpansive and I-asymptotically quasi-nonexpansive mapping defined for some iterative schemes in Banach spaces were proved. In [17], Yao and Wang established the

Email address: seyittemir@adiyaman.edu.tr (Seyit Temir)

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strong convergence of an iterative scheme with errors involving I-asymptotically quasi-nonexpansive mappings in a uniformly convex Banach space. Recently, in [13] and [14] I-asymptotically nonexpansive mapping was introduced. Namely, T is called *I- asymptotically nonexpansive* on K if there exists a sequence $\{\upsilon_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} \upsilon_n = 1$ such that

$$\|T^n x - T^n y\| \leq \upsilon_n \|I^n x - I^n y\|,$$

for all $x, y \in K$ and $n \geq 1$. The mapping $T, I : K \to K$ is said to be I-uniformly Lipschitz with a Lipschitzian constant $\Gamma > 0$ if

$$\|T^n x - T^n y\| \leq \Gamma \|I^n x - I^n y\|$$

holds for all $x, y \in K$ and $n \geq 1$. It is obvious that, an I-asymptotically nonexpansive mapping is I-uniformly Lipschitz with Lipschitz constant $\Gamma = sup\{\upsilon_n : n \geq 1\}$.

The class of asymptotically nonexpansive maps which an important generalization of the class nonexpansive maps was introduced by Goebel and Kirk [2]. In 2000, Noor [7] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [3] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [3] that the three-step iterative scheme gives better numerical results than the Mann-type[6](one-step) and the Ishikawa-type[5] (two-step) approximate iterations. Xu and Noor [15] introduced and studied a three-step iterative for asymptotically nonexpansive mappings and they proved weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space.

Recently, Suantai [11] introduced the following iterative scheme which is an extension of Xu and Noor [15] iterations and used it for the weak and strong convergence of fixed points in an uniformly convex Banach space. The scheme is defined as follows.

$$\begin{cases} x_1 = x \in K \\ z_n = a_n T^n x_n + (1 - a_n) x_n \\ y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n) x_n \\ x_{n+1} = \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n) x_n, \forall n \geq 1, \end{cases} \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ in $[0,1]$ satisfy certain conditions. The iterative scheme (1.1) is called the modified Noor iterative scheme for asymptotically nonexpansive mappings. If $\{c_n\} = \{\beta_n\} = 0$, then (1.1) reduces to Noor iterations defined by Xu and Noor [15] as follows:

$$\begin{cases} x_1 = x \in K \\ z_n = a_n T^n x_n + (1 - a_n) x_n \\ y_n = b_n T^n z_n + (1 - b_n) x_n \\ x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n) x_n, \forall n \geq 1, \end{cases} \tag{1.2}$$

If $\{a_n\} = \{c_n\} = \{\beta_n\} = 0$, then (1.1) reduces to Ishikawa iterations[5] as follows:

$$\begin{cases} x_1 = x \in K \\ y_n = b_n T^n x_n + (1 - b_n) x_n \\ x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n) x_n, \forall n \geq 1, \end{cases} \tag{1.3}$$

If $\{a_n\} = \{b_n\} = \{c_n\} = \{\beta_n\} = 0$, then (1.1) reduces to Mann iterative process [6] as follows:

$$\begin{cases} x_1 = x \in K \\ x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n) x_n, \forall n \geq 1, \end{cases} \tag{1.4}$$

Inspired by the preceding iteration schemes, we define a new iteration scheme as follows. Let X be a real uniformly convex Banach space and K be a nonempty closed, bounded and convex subset of X. Let $T : K \to K$ be a I-asymptotically nonexpansive mapping and $I : K \to K$ be an asymptotically nonexpansive mapping. We shall consider the following iteration scheme:

$$\begin{cases} x_1 = x \in K \\ z_n = a_n T^n x_n + (1-a_n) I^n x_n \\ y_n = b_n T^n z_n + c_n T^n x_n + (1-b_n-c_n) I^n x_n \\ x_{n+1} = \alpha_n T^n y_n + \beta_n T^n z_n + (1-\alpha_n-\beta_n) I^n x_n, \forall n \geq 1, \end{cases} \tag{1.5}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$,$\{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ are appropriate sequences in $[0, 1]$.

The iterative scheme (1.5) is called the modified Noor iterative scheme for asymptotically nonexpansive mappings. If I is identity mapping then (1.5) reduces to the (1.1) defined by [11].

The aim of this paper is to introduce and study convergence problem of iterative process (1.5) to a common fixed point of T and I. Also we introduce and study convergence problem of three-step iterative sequence for three I-asymptotically nonexpansive mappings in an uniformly convex Banach space. The convergence theorems presented in this paper improve and generalize many results in the current literature.

2 Preliminaries and Notations

Let X be a Banach space with dimension $X \geq 2$. The modulus of X is function $\delta_X : (0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = inf\{1 - \frac{\|x+y\|}{2} : \|x\| = 1, \|y\| = 1, \|x-y\| = \varepsilon\}.$$

A Banach space X is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Recall that a Banach space X is said to satisfy Opial's condition [8] if, for each sequence $\{x_n\}$ in X, the condition $x_n \rightharpoonup x$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$.

A mapping $T : K \to K$ is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in K such that $x_n \to x* \in K$ and $Tx_n \to p$ then $Tx* = p$.

A mapping $T : K \to K$ is said to be semi-compact if, for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly $x* \in K$.

A mapping $T : K \to K$ is said to be completely continuous if for every bounded sequence $\{x_n\}$ in K converges weakly $x*$ implies that Tx_n converges to strongly to $Tx*$.

Let $\{u_n\}$ in K be a given sequence. $T : K \to X$ with the nonempty fixed point set F(T) in K is said to satisfy Condition(A)[10] with respect to the $\{u_n\}$ if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|u_n - Tu_n\| \geq f(d(u_n, F(T)))$ for all $n \geq 1$. Senter and Dotson [10] pointed out that every continuous and demi-compact must satisfying Condition (A). In order to obtain strong convergence of common fixed points of I- asymptotically nonexpansive mappings and finite numbers of these mappings, we introduce the following condition (B): The mappings $T_i, I_i, (i = 1, 2, 3)$ are said to satisfy condition (B) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $max_{1\leq i\leq 3}\{\frac{1}{2}(\|x - T_i x\| + \|x - I_i x\|)\} \geq f(d(x, F(T_i \cap I_i)))$ for all $x \in K$, where $F(T_i \cap I_i) \neq \emptyset$ and $d(x, F(T_i \cap I_i)) = inf\{d(x, p) : p \in F(T_i \cap I_i)\}$.

In what follows, we shall make use of the following lemmas.

Lemma 2.1. [4]Let X be a uniformly convex Banach space, K a nonempty closed convex subset of X and $T : K \longrightarrow K$ be a asymptotically nonexpansive mapping with a sequence $k_n \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$, Then $E - T$(E is identity mapping) is *demiclosed* at zero, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$.

Lemma 2.2. [12] Let $\{s_n\}$, $\{t_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real sequences satisfying the following conditions: $\forall n \geq 1$, $s_{n+1} \leq (1 + \sigma_n)s_n + t_n$, where $\sum_{n=0}^{\infty} \sigma_n < \infty$ and $\sum_{n=0}^{\infty} t_n < \infty$. Then $\lim_{n\to\infty} s_n$ exists.

Lemma 2.3. [10] Let X be a uniformly convex Banach space and b,c be two constants with $0 < b < c < 1$. suppose that t_n is a sequence in $[b, c]$ and x_n and y_n are two sequences of X such that $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = d$, $\limsup_{n\to\infty} \|x_n\| \le d$, $\limsup_{n\to\infty} \|y_n\| \le d$, holds some $d \ge 0$, Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.4. [16] Let X be a uniformly convex Banach space. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in (0,1) satisfying $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < \lim_{n\to\infty} \alpha_n < \liminf_{n\to\infty}(\alpha_n + \beta_n) \le \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$. Suppose that x_n , y_n and z_n are three sequences in X. Then

$$\begin{aligned} \limsup_{n\to\infty} \|x_n\| &\le d, \\ \limsup_{n\to\infty} \|y_n\| &\le d, \\ \limsup_{n\to\infty} \|z_n\| &\le d, \\ \lim_{n\to\infty} \|\alpha_n x_n + \beta_n y_n + \gamma_n z_n\| &= d, \end{aligned}$$

imply that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0, \lim_{n\to\infty} \|y_n - z_n\| = 0, \lim_{n\to\infty} \|z_n - y_n\| = 0,$$

where $d \ge 0$ is some constant.

Lemma 2.5. (See [11],Lemma 2.7) Let X be a Banach space which satisfies Opial's condition and let x_n be a sequence in X. Let $q_1, q_2 \in X$ be such that $\lim_{n\to\infty} \|x_n - q_1\|$ and $\lim_{n\to\infty} \|x_n - q_2\|$ exist.If $\{x_{n_k}\}$, $\{x_{n_j}\}$ are the subsequences of $\{x_n\}$ which converge weakly to $q_1, q_2 \in X$, respectively. Then $q_1 = q_2$.

3 Convergence Theorems For *I*-Asymptotically Nonexpansive

Lemma 3.1. Let X be a real uniformly convex Banach space and K be a nonempty closed, bounded and convex subset of X. Let $T : K \to K$ be a I-asymptotically nonexpansive mapping with $\{k_n\}$ a sequence of real numbers such that $k_n \ge 1$ and $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ and $I : K \to K$ be an asymptotically nonexpansive mapping with $\{\ell_n\}$ a sequence of real numbers such that $\ell_n \ge 1$ and $\sum_{n=1}^{\infty}(\ell_n - 1) < \infty$. Suppose further that the set $F(T) \cap F(I)$ (i.e., $F(T) := \{x \in K : x = Tx\}, F(I) := \{x \in K : x = Ix\}$) is nonempty. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[0, 1]$ such that $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ in $[0, 1]$ for all $n \ge 1$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences in K defined by (1.5). If q is a common fixed point of T and I, then $\lim_{n\to\infty} \|x_n - q\|$ exists.

Proof . Let $q \in F(T) \cap F(I)$. Using (1.5), we have

$$\begin{aligned} \|z_n - q\| &= \|a_n T^n x_n + (1 - a_n)I^n x_n - q\| \\ &= \|a_n(T^n x_n - q) + (1 - a_n)(I^n x_n - q)\| \\ &\le a_n\|T^n x_n - q\| + (1 - a_n)\|I^n x_n - q\| \\ &\le a_n k_n\|I^n x_n - q\| + (1 - a_n)\ell_n\|x_n - q\| \\ &\le a_n k_n \ell_n\|x_n - q\| + (1 - a_n)\ell_n\|x_n - q\| \\ &\le \ell_n(1 + a_n(k_n - 1))\|x_n - q\| \end{aligned} \tag{3.1}$$

$$\begin{aligned}
\|y_n - q\| &= \|\Big(b_n T^n z_n + c_n T^n x_n + (1-b_n-c_n) I^n x_n\Big) - q\| \\
&\le b_n\|T^n z_n - q\| + c_n\|T^n x_n - q\| + (1-b_n-c_n)\|I^n x_n - q\| \\
&\le b_n k_n\|I^n z_n - q\| + c_n k_n\|I^n x_n - q\| + (1-b_n-c_n)\ell_n\|x_n - q\| \\
&\le b_n k_n \ell_n\|z_n - q\| + c_n k_n \ell_n\|x_n - q\| + (1-b_n-c_n)\ell_n\|x_n - q\| \\
&\le \Big(b_n k_n \ell_n^2\Big(1 + a_n\Big(k_n - 1\Big)\Big) + c_n k_n \ell_n + (1-b_n-c_n)\ell_n\Big)\|x_n - q\| \\
&\le \ell_n\Big(1 + b_n a_n \ell_n\Big(k_n-1\Big) + b_n k_n\Big(k_n-1\Big) + b_n\Big(\ell_n-1\Big) + c_n\Big(k_n-1\Big)\Big)\|x_n - q\|
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n T^n y_n + \beta_n T^n z_n + (1-\alpha_n-\beta_n) I^n x_n - q\| \\
&\le \alpha_n\|T^n y_n - q\| + \beta_n\|T^n z_n - q\| + (1-\alpha_n-\beta_n)\|I^n x_n - q\| \\
&\le \alpha_n k_n\|I^n y_n - q\| + \beta_n k_n\|I^n z_n - q\| + (1-\alpha_n-\beta_n)\ell_n\|x_n - q\| \\
&\le \alpha_n k_n \ell_n\|y_n - q\| + \beta_n k_n \ell_n\|z_n - q\| + (1-\alpha_n-\beta_n)\ell_n\|x_n - q\|
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\|x_{n+1} - q\| &\le \ell_n\Big(1 + \alpha_n b_n a_n k_n \ell_n^2\Big(k_n-1\Big) + \alpha_n k_n \ell_n^2\Big(k_n-1\Big) \\
&+ \alpha_n k_n \ell_n b_n\Big(\ell_n-1\Big) + \alpha_n k_n\Big(k_n-1\Big) + \beta_n a_n k_n \ell_n\Big\{k_n-1\Big\} \\
&+ \alpha_n \ell_n\Big(k_n-1\Big) + \beta_n \ell_n\Big(k_n-1\Big) + \alpha_n\Big(\ell_n-1\Big) + \beta_n\Big(\ell_n-1\Big)\Big\}\|x_n - q\|
\end{aligned} \tag{3.3}$$

Since $\sum_{n=0}^{\infty}(k_n-1) < \infty$, $\sum_{n=1}^{\infty}(\ell_n-1) < \infty$, it follows from Lemma 2.2 that $\lim_{n\to\infty}\|x_n - q\|$ exists. □

Lemma 3.2. Under assumptions of Lemma 3.1, if $\lim_{n\to\infty}\|I^n x_n - x_n\| = 0$, then $\lim_{n\to\infty}\|Tx_n - x_n\| = \lim_{n\to\infty}\|Ix_n - x_n\| = 0$.

Proof . By Lemma 3.1, we can assume that $\lim_{n\to\infty}\|x_n - q\| = d$

for $q \in F(T \cap I)$. If $d = 0$ by continuity T and I then the proof is completed. Now suppose $d > 0$.

$$\limsup_{n\to\infty}\|I^n x_n - q\| \le \limsup_{n\to\infty}\ell_n\|x_n - q\| \le d, \tag{3.4}$$

$$\limsup_{n\to\infty}\|T^n x_n - q\| \le \limsup_{n\to\infty}k_n\ell_n\|x_n - q\| \le d, \tag{3.5}$$

From (3.2), we have

$$\limsup_{n\to\infty}\|y_n - q\| \le d, \tag{3.6}$$

and from (3.1), we have

$$\limsup_{n\to\infty}\|z_n - q\| \le d, \tag{3.7}$$

$$\|T^n y_n - q\| \le k_n\|I^n y_n - q\| \le k_n\ell_n\|y_n - q\|,$$

taking the limsup on both sides in this inequality, we have

$$\limsup_{n\to\infty}\|T^n y_n - q\| \le d. \tag{3.8}$$

$$\|T^n z_n - q\| \le k_n\|I^n z_n - q\| \le k_n\ell_n\|z_n - q\|,$$

taking the limsup on both sides in this inequality, we have

$$\limsup_{n\to\infty}\|T^n z_n - q\| \leq d. \tag{3.9}$$

From (1.5) ,we have

$$\begin{aligned} d &= \lim_{n\to\infty}\|x_{n+1}-q\| \leq \lim_{n\to\infty}\|\alpha_n T^n y_n + \beta_n T^n z_n + (1-\alpha_n-\beta_n)I^n x_n - q\| \\ &= \lim_{n\to\infty}\|\alpha_n(T^n y_n - q) + \beta_n(T^n z_n - q) + (1-\alpha_n-\beta_n)(I^n x_n - q)\| \end{aligned}$$

From (3.4),(3.8),(3.9) and Lemma 2.4 , we have

$$\begin{cases} \lim\limits_{n\to\infty}\|T^n y_n - T^n z_n\| = 0 \\ \lim\limits_{n\to\infty}\|T^n z_n - I^n x_n\| = 0 \\ \lim\limits_{n\to\infty}\|I^n x_n - T^n y_n\| = 0 \end{cases} \tag{3.10}$$

From (1.5), we have

$$\begin{aligned} \|x_{n+1}-q\| &\leq \|\alpha_n T^n y_n + \beta_n T^n z_n + (1-\alpha_n-\beta_n)I^n x_n - q\| \\ &\leq \|\alpha_n(T^n y_n - I^n x_n) + \beta_n(T^n z_n - I^n x_n) + (I^n x_n - q)\| \end{aligned}$$

Taking the liminf on both sides in this inequality and using (3.4) we have

$$\lim_{n\to\infty}\|I^n x_n - q\| = d. \tag{3.11}$$

$$\begin{aligned} \|I^n x_n - q\| &\leq \|I^n x_n - T^n y_n\| + \|T^n y_n - q\| \\ &\leq \|I^n x_n - T^n y_n\| + k_n \ell_n \|y_n - q\| \end{aligned}$$

Taking the liminf on both sides in this inequality and using (3.6) we have

$$\lim_{n\to\infty}\|y_n - q\| = d. \tag{3.12}$$

Also, from (1.5) ,we have

$$\begin{aligned} d &= \lim_{n\to\infty}\|y_n - q\| \leq \lim_{n\to\infty}\|b_n T^n z_n + c_n T^n x_n + (1-b_n-c_n)I^n x_n - q\| \\ &= \lim_{n\to\infty}\|b_n(T^n z_n - q) + c_n(T^n x_n - q) + (1-b_n-c_n)(I^n x_n - q)\| \end{aligned}$$

From (3.4),(3.5),(3.9) and Lemma 2.4 , we have

$$\begin{cases} \lim\limits_{n\to\infty}\|T^n z_n - T^n x_n\| = 0 \\ \lim\limits_{n\to\infty}\|T^n x_n - I^n x_n\| = 0 \\ \lim\limits_{n\to\infty}\|I^n x_n - T^n z_n\| = 0 \end{cases} \tag{3.13}$$

From (3.13) and by assumption we have

$$\begin{aligned} \|y_n - x_n\| &\leq \|b_n T^n z_n + c_n T^n x_n + (1-b_n-c_n)I^n x_n - x_n\| \\ &\leq b_n\|T^n z_n - I^n x_n\| + c_n\|T^n x_n - I^n x_n\| + \|I^n x_n - x_n\| \underset{n\to\infty}{\to} 0\ . \end{aligned} \tag{3.14}$$

Next,

$$\begin{aligned} \|I^n x_n - q\| &\leq \|I^n x_n - T^n z_n\| + \|T^n z_n - q\| \\ &\leq \|I^n x_n - T^n z_n\| + k_n \ell_n \|z_n - q\|. \end{aligned}$$

Taking the liminf on both sides in this inequality and using (3.7), (3.13) we have

$$\lim_{n\to\infty}\|z_n - q\| = d. \tag{3.15}$$

From (3.13) and by assumption we have

$$\begin{aligned}\|z_n - x_n\| &\leq \|a_n T^n x_n + (1-a_n) I^n x_n - x_n\| \\ &\leq a_n\|T^n x_n - I^n x_n\| + \|I^n x_n - x_n\| \underset{n\to\infty}{\to 0}\ .\end{aligned} \tag{3.16}$$

Also from (1.5), (3.13), (3.16) and by assumption

$$\begin{aligned}\|y_n - z_n\| &\leq \|b_n T^n z_n + c_n T^n x_n + (1-b_n-c_n) I^n x_n - z_n\| \\ &\leq b_n\|T^n z_n - I^n x_n\| + c_n\|T^n x_n - I^n x_n\| + \|I^n x_n - x_n\| + \|x_n - z_n\| \underset{n\to\infty}{\to 0}\end{aligned} \tag{3.17}$$

Using (1.5), (3.10) and by assumption,

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \|\alpha_n T^n y_n + \beta_n T^n z_n + (1-\alpha_n-\beta_n) I^n x_n - x_n\| \\ &\leq \alpha_n\|T y_n - I^n x_n\| + \beta_n\|T^n z_n - I^n x_n\| + \|I^n x_n - x_n\| \underset{n\to\infty}{\to 0}\ .\end{aligned} \tag{3.18}$$

If $\lim\limits_{n\to\infty}\|I^n x_n - x_n\| = 0$, then we have

$$\lim_{n\to\infty}\|T^n x_n - x_n\| \leq \lim_{n\to\infty}\|T^n x_n - I^n x_n\| + \lim_{n\to\infty}\|I^n x_n - x_n\| = 0. \tag{3.19}$$

We consider

$$\begin{aligned}\|x_n - I x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - I^{n+1} x_{n+1}\| \\ &+ \|I^{n+1} x_{n+1} - I^{n+1} x_n\| + \|I^{n+1} x_n - I x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - I^n x_{n+1}\| \\ &+ \Gamma\|x_{n+1} - x_n\| + \Gamma\|I^n x_n - x_n\|,\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\|x_n - T x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1} x_{n+1}\| \\ &+ \|T^{n+1} x_{n+1} - T^{n+1} x_n\| + \|T^{n+1} x_n - T x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^n x_{n+1}\| \\ &+ L\Gamma\|x_{n+1} - x_n\| + \Gamma\|I^n x_n - x_n\|.\end{aligned} \tag{3.21}$$

Since $\|x_n - I^n x_n\| \to 0 \ \ asn \to \infty$ and $\|x_{n+1} - x_n\| \to 0 \ \ asn \to \infty$, by continuity of I and T, together with (3.20) and (3.21), we have

$$\lim_{n\to\infty}\|x_n - I x_n\| = 0 \tag{3.22}$$

and

$$\lim_{n\to\infty}\|x_n - T x_n\| = 0. \tag{3.23}$$

□

Theorem 3.3. Let the conditions of Lemma 3.2 be satisfied. If at least one of the mappings T and I is completely continuous and $F(T \cap I) \neq \emptyset$, then $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of T and I.

Proof . By Lemma 3.2, we have $\lim\limits_{n\to\infty}\|x_n - T x_n\| = \lim\limits_{n\to\infty}\|x_n - I x_n\| = 0$. It follows by our assumption that T is completely continuous,and $\{x_n\} \subseteq K$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{T x_{n_k}\}$

converges. Therefore from (3.23), $\{x_{n_k}\}$ converges. Let $\lim_{k\to\infty} x_{n_k} = q$. By continuity of T and (3.23) we have that $Tq = q$. On the other hand, according to (3.22) and continuity of I, we obtain that $Iq = q$, so q is a common fixed point T and I. By Lemma 3.1 $\lim_{n\to\infty} \|x_n - q\|$ exists. But $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$. Thus $\lim_{n\to\infty} \|x_n - q\| = 0$, that is, $\{x_n\}$ converges strongly to a common fixed point q of T and I.

Also, from (3.14) and (3.16), it follows that $\lim_{n\to\infty} \|y_n - q\| = 0$ and $\lim_{n\to\infty} \|z_n - q\| = 0$ that is, $\{y_n\}$, $\{z_n\}$ converges strongly to a common fixed point q of T and I. □

Theorem 3.4. Let the conditions of Lemma 3.2 be satisfied. If one of the mappings T and I is semi-compact and $F(T \cap I) \neq \emptyset$, then $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of T and I.

Proof . Since one of the mappings T and I is semi-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to a $q \in K$. Therefore from (3.22) and (3.23), $\lim_{k\to\infty} \|x_{n_k} - Ix_{n_k}\| = \|q - Iq\| = 0$ and $\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = \|q - Tq\| = 0$. It follows that $q \in F(T \cap I)$. Since $\lim_{n\to\infty} \|x_n - q\|$ exists and the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to q, then $\{x_n\}$ converges to common fixed point $q \in F(T \cap I)$. Also, from (3.14) and (3.16), it follows that $\lim_{n\to\infty} \|y_n - q\| = 0$ and $\lim_{n\to\infty} \|z_n - q\| = 0$ that is, $\{y_n\}$, $\{z_n\}$ converges strongly to a common fixed point q of T and I. The proof is completed. □

In the next result, we prove the strong convergence of the scheme (1.5) under condition (B) which is weaker than the compactness of the domain of the mappings.

Theorem 3.5. Let the conditions of Lemma 3.2 be satisfied. If T, I satisfy condition (B) and $F(T \cap I) \neq \emptyset$, then $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of T and I.

Proof . By Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - q\|$ exists and so $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in F(T \cap I)$. Also by Lemma 3.2, $\lim_{n\to\infty} \|x_n - Ix_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = 0$. It follows from condition (B) that $\lim_{n\to\infty} f(d(x_n, F(T \cap I))) \leq \lim_{n\to\infty} \{\frac{1}{2}(\|x_n - Tx_n\| + \|x_n - Ix_n\|)\}$. That is, $\lim_{n\to\infty} f(d(x_n, F(T \cap I))) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, we have $\lim_{n\to\infty} d(x_n, F(T \cap I)) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence in K. for given $\epsilon > 0$, there exists a natural number n_0 such that $d(x_n, F(T \cap I)) < \frac{\epsilon}{2}$. We can find $q* \in F(T \cap I)$ such that $\|x_n - q * \| < \frac{\epsilon}{2}$. For $n, m \geq n_0$, we have

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - q * \| + \|x_m - q * \| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Thus shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since X complete. Suppose $\lim_{n\to\infty} \{x_n\} = q$. Since K is closed , we get $q \in K$. Now we prove that $q \in F(T \cap I)$. Since $\lim_{n\to\infty} \{x_n\} = q$ and $\lim_{n\to\infty} d(x_n, F(T \cap I)) = 0$, we obtain $d(q, F(T \cap I)) = 0$. Thus $q \in F(T \cap I)$. Also, from (3.14) and (3.16), it follows that $\lim_{n\to\infty} \|y_n - q\| = 0$ and $\lim_{n\to\infty} \|z_n - q\| = 0$ that is, $\{y_n\}$, $\{z_n\}$ converges strongly to a common fixed point q of T and I. The proof is completed. □

Finally, we prove the weak convergence of the iterative scheme (1.5) for I-asymptotically nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.6. Let X be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed, bounded and convex subset of X. Let $T : K \to K$ be a I-asymptotically nonexpansive mapping with $\{k_n\}$ a sequence of real numbers such that $k_n \geq 1$ and $\sum_{n=0}^{\infty} (k_n - 1) < \infty$ and $I : K \to K$ be an asymptotically nonexpansive mapping with $\{\ell_n\}$ a sequence of real numbers such that $\ell_n \geq 1$ and $\sum_{n=1}^{\infty} (\ell_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$,$\{\alpha_n\}$, $\{\beta_n\}$ be sequences of real numbers in $[0, 1]$, such that $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ in $[0, 1]$ for all $n \geq 1$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences in K defined by (1.5). If $F(T) \cap F(I) \neq \emptyset$, then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge weakly to a common fixed point of T and I.

Proof . Let $q \in F(T) \cap F(I)$. Then as in Lemma 3.1, $\lim_{n\to\infty} \|x_n - q\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)\cap F(I)$. We assume that q_1 and q_2 are weak limits of the subsequences $\{x_{n_k}\}$, $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By (3.22) and (3.23), $\lim_{n\to\infty} \|x_n - Ix_n\| = 0$, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $E - T$ and $E - I$ are demiclosed by Lemma 2.1, $Tq_1 = q_1$, $Iq_1 = q_1$ and in the same way, $Tq_2 = q_2$, $Iq_2 = q_2$. Therefore, we have $q_1, q_2 \in F(T) \cap F(I)$. It follows from Lemma 2.5 that $q_1 = q_2$. This completes the proof. □

4 Convergence Theorems For Three I-Asymptotically Nonexpansive Mappings

Here we give the theorems for three $I_i, (i = 1, 2, 3)$-asymptotically nonexpansive mapping which can be proved in similar way as the above theorems.

Let X be a real uniformly convex Banach space and K be a nonempty closed, bounded and convex subset of X. Let $T_i : K \to K, (i = 1, 2, 3)$ be $I_i, (i = 1, 2, 3)$-asymptotically nonexpansive mapping with $k_n = max\{k_n^1, k_n^2, k_n^3\}$ a sequence of real numbers such that $k_n \geq 1$ and $\sum_{n=0}^{\infty} (k_n - 1) < \infty$ and $I_i : K \to K, (i = 1, 2, 3)$ be an asymptotically nonexpansive mapping with $\ell_n = max\{\ell_n^1, \ell_n^2, \ell_n^3\}$ a sequence of real numbers such that $\ell_n \geq 1$ and $\sum_{n=1}^{\infty} (\ell_n - 1) < \infty$. We shall consider the following iteration scheme:

$$\begin{cases} x_1 = x \in K \\ z_n = a_n T_1^n x_n + (1 - a_n) I_1^n x_n \\ y_n = b_n T_2^n z_n + c_n T_2^n x_n + (1 - b_n - c_n) I_2^n x_n \\ x_{n+1} = \alpha_n T_3^n y_n + \beta_n T_3^n z_n + (1 - \alpha_n - \beta_n) I_3^n x_n, \forall n \geq 1, \end{cases} \tag{4.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$,$\{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ are appropriate sequences in $[0, 1]$.

The iterative scheme (4.1) is called the modified Noor iterative scheme for asymptotically nonexpansive mappings. If $T_i = T, (i = 1, 2, 3)$, and $I_i, (i = 1, 2, 3)$, are identity mappings then (4.1) reduces to the (1.1) defined by [11].

Lemma 4.1. Let X be a real uniformly convex Banach space and K be a nonempty closed, bounded and convex subset of X. Let $T_i : K \to K, (i = 1, 2, 3)$ be $I_i, (i = 1, 2, 3)$-asymptotically nonexpansive mappings with $k_n = \max\{k_n^1, k_n^2, k_n^3\}$ a sequence of real numbers such that $k_n \geq 1$ and $\sum_{n=0}^{\infty} (k_n - 1) < \infty$ and $I_i : K \to K, (i = 1, 2, 3)$ be asymptotically nonexpansive mappings with $\ell_n = max\{\ell_n^1, \ell_n^2, \ell_n^3\}$ a sequence of real numbers such that $\ell_n \geq 1$ and $\sum_{n=1}^{\infty} (\ell_n - 1) < \infty$. Suppose further that the set $\bigcap_{i=1}^{3} F(T_i) \cap F(I_i)$ is nonempty. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[0, 1]$ such that $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ in $[0, 1]$ for all $n \geq 1$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences in K defined by (4.1). If q is a common fixed point of T_i and $I_i, (i = 1, 2, 3)$, then

(1) $\lim_{n\to\infty} \|x_n - q\|$ exists.

(2) For $i = 1, 2, 3$, if $\lim_{n\to\infty} \|I_i^n x_n - x_n\| = 0$, then $\lim_{n\to\infty} \|T_i x_n - x_n\| = \lim_{n\to\infty} \|I_i x_n - x_n\| = 0$.

Proof . Let $q \in \bigcap_{i=1}^{3} F(T_i) \cap F(I_i)$. Using (4.1), Similar way as Lemma 3.1

$$\begin{aligned} \|z_n - q\|\| &\leq a_n T_1^n x_n + (1 - a_n) I_1^n x_n - q\| \\ &\leq \ell_n (1 + a_n (k_n - 1)) \|x_n - q\| \end{aligned} \tag{4.2}$$

$$\begin{aligned} \|y_n - q\| &\leq \|b_n T_2^n z_n + c_n T_2^n x_n + (1 - b_n - c_n) I_2^n x_n - q\| \\ &\leq \ell_n \Big(1 + b_n a_n \ell_n \Big(k_n - 1\Big) + b_n k_n \Big(k_n - 1\Big) + b_n \Big(\ell_n - 1\Big) + c_n \Big(k_n - 1\Big)\Big) \|x_n - q\| \end{aligned} \tag{4.3}$$

Thus we obtain

$$\begin{aligned}\|x_{n+1}-q\| &\leq \|\alpha_n T_3^n y_n+\beta_n T_3^n z_n+(1-\alpha_n-\beta_n)I_3^n x_n\|\\ &\leq \ell_n\Big(1+\alpha_n b_n a_n k_n \ell_n^2\Big(k_n-1\Big)+\alpha_n k_n \ell_n^2\Big(k_n-1\Big)\\ &+ \alpha_n k_n \ell_n b_n\Big(\ell_n-1\Big)+\alpha_n k_n\Big(k_n-1\Big)+\beta_n a_n k_n \ell_n\Big\{k_n-1\Big\}\\ &+ \alpha_n \ell_n\Big(k_n-1\Big)+\beta_n \ell_n\Big(k_n-1\Big)+\alpha_n\Big(\ell_n-1\Big)+\beta_n\Big(\ell_n-1\Big)\Big\}\|x_n-q\|\end{aligned} \tag{4.4}$$

Since $\sum_{n=0}^{\infty}(k_n-1)<\infty$, $\sum_{n=1}^{\infty}(\ell_n-1)<\infty$, it follows from Lemma 2.2 that $\lim_{n\to\infty}\|x_n-q\|$ exists and the first part of lemma is over.

Next, we prove that for $i=1,2,3$, $\lim_{n\to\infty}\|T_i x_n-x_n\|=\lim_{n\to\infty}\|I_i x_n-x_n\|=0$. We can assume that $\lim_{n\to\infty}\|x_n-q\|=d$, for $q\in F(T\cap I)$. If $d=0$ by continuity T and I then the proof is completed. Now suppose $d>0$. For $i=1,2,3$

$$\limsup_{n\to\infty}\|I_i^n x_n-q\|\leq \limsup_{n\to\infty}\ell_n\|x_n-q\|\leq d, \tag{4.5}$$

$$\limsup_{n\to\infty}\|T_1^n x_n-q\|\leq \limsup_{n\to\infty}k_n\ell_n\|x_n-q\|\leq d, \tag{4.6}$$

and

$$\limsup_{n\to\infty}\|T_2^n x_n-q\|\leq \limsup_{n\to\infty}k_n\ell_n\|x_n-q\|\leq d, \tag{4.7}$$

From (4.2), we have

$$\limsup_{n\to\infty}\|z_n-q\|\leq d, \tag{4.8}$$

and from (4.3), we have

$$\limsup_{n\to\infty}\|y_n-q\|\leq d, \tag{4.9}$$

Further,

$$\|T_3^n y_n-q\|\leq k_n\|I_3^n y_n-q\|\leq k_n\ell_n\|y_n-q\|,$$

taking the limsup on both sides in this inequality, we have

$$\limsup_{n\to\infty}\|T_3^n y_n-q\|\leq d. \tag{4.10}$$

$$\|T_3^n z_n-q\|\leq k_n\|I_3^n z_n-q\|\leq k_n\ell_n\|z_n-q\|,$$

taking the limsup on both sides in this inequality, we have

$$\limsup_{n\to\infty}\|T_3^n z_n-q\|\leq d. \tag{4.11}$$

From (4.1) ,we have

$$\begin{aligned}d &= \lim_{n\to\infty}\|x_{n+1}-q\|\leq \lim_{n\to\infty}\|\alpha_n T_3^n y_n+\beta_n T_3^n z_n+(1-\alpha_n-\beta_n)I_3^n x_n-q\|\\ &= \lim_{n\to\infty}\|\alpha_n(T_3^n y_n-q)+\beta_n(T_3^n z_n-q)+(1-\alpha_n-\beta_n)(I_3^n x_n-q)\|\end{aligned}$$

From (4.5),(4.10),(4.11) and Lemma 2.4 , we have

$$\begin{cases} \lim_{n\to\infty} \|T_3^n y_n - T_3^n z_n\| = 0 \\ \lim_{n\to\infty} \|T_3^n z_n - I_3^n x_n\| = 0 \\ \lim_{n\to\infty} \|I_3^n x_n - T_3^n y_n\| = 0 \end{cases} \tag{4.12}$$

From (4.1), we have

$$\begin{aligned} \|x_{n+1} - q\| &\leq \|\alpha_n T_3^n y_n + \beta_n T_3^n z_n + (1-\alpha_n-\beta_n) I_3^n x_n - q\| \\ &\leq \|\alpha_n (T_3^n y_n - I_3^n x_n) + \beta_n (T_3^n z_n - I^n x_n) + (I_3^n x_n - q)\| \end{aligned}$$

Taking the liminf on both sides in this inequality and using (4.5) we have

$$\lim_{n\to\infty} \|I_3^n x_n - q\| = d. \tag{4.13}$$

$$\begin{aligned} \|I_3^n x_n - q\| &\leq \|I_3^n x_n - T_3^n y_n\| + \|T_3^n y_n - q\| \\ &\leq \|I_3^n x_n - T_3^n y_n\| + k_n \ell_n \|y_n - q\| \end{aligned}$$

Taking the liminf on both sides in this inequality and using (4.8) we have

$$\lim_{n\to\infty} \|y_n - q\| = d. \tag{4.14}$$

$$\begin{aligned} d &= \lim_{n\to\infty} \|y_n - q\| \leq \lim_{n\to\infty} \|b_n T_2^n z_n + c_n T_2^n x_n + (1-b_n-c_n) I_2^n x_n - q\| \\ &= \lim_{n\to\infty} \|b_n (T_2^n z_n - q) + c_n (T_2^n x_n - q) + (1-b_n-c_n)(I_2^n x_n - q)\| \end{aligned}$$

$$\|T_2^n z_n - q\| \leq k_n \|I_2^n z_n - q\| \leq k_n \ell_n \|z_n - q\|,$$

taking the limsup on both sides in this inequality, we have

$$\limsup_{n\to\infty} \|T_2^n z_n - q\| \leq d. \tag{4.15}$$

From (4.5),(4.7),(4.15) and Lemma 2.4 , we have

$$\begin{cases} \lim_{n\to\infty} \|T^n z_n - T^n x_n\| = 0 \\ \lim_{n\to\infty} \|T^n x_n - I^n x_n\| = 0 \\ \lim_{n\to\infty} \|I^n x_n - T^n z_n\| = 0 \end{cases} \tag{4.16}$$

From (4.16) and by assumption we have

$$\begin{aligned} \|y_n - x_n\| &\leq \|b_n T_2^n z_n + c_n T_2^n x_n + (1-b_n-c_n) I_2^n x_n - x_n\| \\ &\leq b_n \|T_2^n z_n - I_2^n x_n\| + c_n \|T_2^n x_n - I_2^n x_n\| + \|I_2^n x_n - x_n\| \underset{n\to\infty}{\to} 0 \end{aligned} \tag{4.17}$$

Next,

$$\begin{aligned} \|I_2^n x_n - q\| &\leq \|I_2^n x_n - T_2^n z_n\| + \|T_2^n z_n - q\| \\ &\leq \|I_2^n x_n - T^n z_n\| + k_n \ell_n \|z_n - q\| \end{aligned}$$

Taking the liminf on both sides in this inequality and using (4.9), (4.16) we have

$$\lim_{n\to\infty} \|z_n - q\| = d. \tag{4.18}$$

$$\begin{aligned}\|z_n - q\| &\leq \|a_n T_1^n x_n + (1-a_n) I_1^n x_n - q\| \\ &\leq \|a_n (T_1^n x_n - q) + (1-a_n)(I_1^n x_n - q)\| \end{aligned} \tag{4.19}$$

By Lemma 2.3 we have

$$\lim_{n\to\infty} \|T_1^n x_n - I_1^n x_n\| = 0, \tag{4.20}$$

Thus by assumption and from (4.20), we have

$$\begin{aligned}\|z_n - x_n\| &\leq \|a_n T_1^n x_n + (1-a_n) I_1^n x_n - x_n\| \\ &\leq a_n \|T_1^n x_n - I_1^n x_n\| + \|I_1^n x_n - x_n\| \underset{n\to\infty}{\to} 0 \end{aligned} \tag{4.21}$$

Also from (4.1), (4.16), (4.21) and by assumption

$$\begin{aligned}\|y_n - z_n\| &\leq \|b_n T_2^n z_n + c_n T_2^n x_n + (1-b_n-c_n) I_2^n x_n - z_n\| \\ &\leq b_n \|T_2^n z_n - I_2^n x_n\| + c_n \|T_2^n x_n - I_2^n x_n\| + \|I_2^n x_n - x_n\| + \|x_n - z_n\| \underset{n\to\infty}{\to} 0 \end{aligned} \tag{4.22}$$

Using (4.1), (4.12) and by assumption,

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \|\alpha_n T_3^n y_n + \beta_n T_3^n z_n + (1-\alpha_n-\beta_n) I^n x_n - x_n\| \\ &\leq \alpha_n \|T y_n - I^n x_n\| + \beta_n \|T^n z_n - I^n x_n\| + \|I^n x_n - x_n\| \underset{n\to\infty}{\to} 0 \end{aligned} \tag{4.23}$$

If for $i = 1, 2, 3$, $\lim_{n\to\infty} \|I_i^n x_n - x_n\| = 0$, then we have

$$\lim_{n\to\infty} \|T_1^n x_n - x_n\| \leq \lim_{n\to\infty} \|T_1^n x_n - I_1^n x_n\| + \lim_{n\to\infty} \|I_1^n x_n - x_n\| = 0. \tag{4.24}$$

$$\lim_{n\to\infty} \|T_2^n x_n - x_n\| \leq \lim_{n\to\infty} \|T_2^n x_n - I_2^n x_n\| + \lim_{n\to\infty} \|I_2^n x_n - x_n\| = 0. \tag{4.25}$$

$$\begin{aligned}\lim_{n\to\infty} \|T_3^n x_n - x_n\| &\leq \lim_{n\to\infty} (\|T_3^n x_n - T_3^n y_n\| + \|T_3^n y_n - I_3^n x_n\| + \|I_3^n x_n - x_n\|) \\ &= \lim_{n\to\infty} k_n \ell_n \|x_n - y_n\| + \lim_{n\to\infty} \|T_3^n y_n - I_3^n x_n\| + \lim_{n\to\infty} \|I_3^n x_n - x_n\| = 0. \end{aligned} \tag{4.26}$$

Thus,For $i = 1, 2, 3$, we get

$$\lim_{n\to\infty} \|T_i^n x_n - x_n\| = 0, \tag{4.27}$$

We consider

$$\begin{aligned}\|x_n - I_1 x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - I_1^{n+1} x_{n+1}\| \\ &+ \|I_1^{n+1} x_{n+1} - I_1^{n+1} x_n\| + \|I_1^{n+1} x_n - I_1 x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - I_1^n x_{n+1}\| \\ &+ \Gamma \|x_{n+1} - x_n\| + \Gamma \|I_1^n x_n - x_n\| \end{aligned} \tag{4.28}$$

and

$$\begin{aligned}\|x_n - T_1 x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_1^{n+1} x_{n+1}\| \\ &+ \|T_1^{n+1} x_{n+1} - T_1^{n+1} x_n\| + \|T_1^{n+1} x_n - T_1 x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_1^n x_{n+1}\| \\ &+ L\Gamma \|x_{n+1} - x_n\| + \Gamma \|I_1^n x_n - x_n\| \end{aligned} \tag{4.29}$$

Since $\|I_1^n x_n - x_n\| \to 0 \ \ as \ n \to \infty$, $\|T_1^n x_n - x_n\| \to 0 \ \ as \ n \to \infty$ and $\|x_{n+1} - x_n\| \to 0 \ \ as \ n \to \infty$, by continuity of T_1 and I_1, together with (4.28) and (4.29), we have

$$\lim_{n\to\infty} \|x_n - I_1 x_n\| = 0 \tag{4.30}$$

and

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0. \tag{4.31}$$

Similarly, we can show that

$$\lim_{n\to\infty} \|x_n - I_2 x_n\| = 0. \tag{4.32}$$

$$\lim_{n\to\infty} \|x_n - I_3 x_n\| = 0. \tag{4.33}$$

$$\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0. \tag{4.34}$$

$$\lim_{n\to\infty} \|x_n - T_3 x_n\| = 0. \tag{4.35}$$

□

Theorem 4.2. Let the conditions of Lemma 4.1 be satisfied. If for $i = 1, 2, 3$, at least one of the mappings T_i and I_i is completely continuous and $\bigcap_{i=1}^{3} F(T_i) \cap F(I_i) \neq \emptyset$, then $\{x_n\}$ defined by (4.1) converges strongly to a common fixed point of T_i and I_i.

Proof . By Lemma 4.1, we have $\lim_{n\to\infty} \|x_n - T_i x_n\| = \lim_{n\to\infty} \|x_n - I_i x_n\| = 0$. It follows by our assumption that T_1 is completely continuous,and $\{x_n\} \subseteq K$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{T_1 x_{n_k}\}$ converges. Therefore from (4.31), $\{x_{n_k}\}$ converges. Let $\lim_{k\to\infty} x_{n_k} = q$. By continuity of T_1 and (4.31) we have that $T_1 q = q$. On the other hand, according to (4.30)-(4.35) and for $i = 1, 2, 3$ continuity of T_i and I_i, we obtain that $T_2 q = q$,$T_3 q = q$, $I_1 q = q$,$I_2 q = q$ and $I_3 q = q$, so for $i = 1, 2, 3$, q is a common fixed point T_i and I_i. By Lemma 4.1(1), $\lim_{n\to\infty} \|x_n - q\|$ exists. But $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$. Thus $\lim_{n\to\infty} \|x_n - q\| = 0$, that is, $\{x_n\}$ converges strongly to a common fixed point $q \in \bigcap_{i=1}^{3} F(T_i) \cap F(I_i)$.

Also, from (4.17) and (4.21), it follows that $\lim_{n\to\infty} \|y_n - q\| = 0$ and $\lim_{n\to\infty} \|z_n - q\| = 0$ that is, $\{y_n\}$, $\{z_n\}$ converges strongly to a common fixed point $q \in \bigcap_{i=1}^{3} F(T_i) \cap F(I_i)$. □

Theorem 4.3. Let the conditions of Lemma 4.1 be satisfied. If one of the mappings T_i and I_i, $(i = 1, 2, 3)$, is semi-compact and $\bigcap_{i=1}^{3} F(T_i) \cap F(I_i) \neq \emptyset$, for $i = 1, 2, 3$, then $\{x_n\}$ defined by (4.1) converges strongly to a common fixed point of T_i and I_i

Proof . Since, for $i = 1, 2, 3$, one of the mappings T_i and I_i is semi-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to a $q \in K$. Suppose that T_1 is semi-compact. Therefore from (4.31), we obtain $\lim_{k\to\infty} \|x_{n_k} - T_1 x_{n_k}\| = \|q - T_1 q\| = 0$. Now Lemma 4.1 guarantees that $\lim_{n\to\infty} \|T_2 x_{n_k} - x_{n_k}\| = 0$, $\lim_{n\to\infty} \|T_3 x_{n_k} - x_{n_k}\| = 0$ and so $\|T_1 q * - q * \| = 0$, $\|T_2 q * - q * \| = 0$, $\|T_3 q * - q * \| = 0$, and $\lim_{n\to\infty} \|I_1 x_{n_k} - x_{n_k}\| = 0$, $\lim_{n\to\infty} \|I_2 x_{n_k} - x_{n_k}\| = 0$, $\lim_{n\to\infty} \|I_3 x_{n_k} - x_{n_k}\| = 0$ and so $\|I_1 q * - q * \| = 0$, $\|I_2 q * - q * \| = 0$, $\|I_3 q * - q * \| = 0$. It follows that $q \in \bigcap_{i=1}^{3} F(T_i) \cap F(I_i)$. Since $\lim_{n\to\infty} \|x_n - q\|$ exists and the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to q, then $\{x_n\}$ converges to common fixed point $q \in \bigcap_{i=1}^{3} F(T_i) \cap F(I_i)$. Also, from (4.17) and (4.21), it follows that $\lim_{n\to\infty} \|y_n - q\| = 0$ and $\lim_{n\to\infty} \|z_n - q\| = 0$ that is, $\{y_n\}$, $\{z_n\}$ converges strongly to a common fixed point q of T_i and I_i, $(i = 1, 2, 3)$. The proof is completed. □

In the next result, we prove the strong convergence of the scheme (4.1) under condition (B) which is weaker than the compactness of the domain of the mappings.

Theorem 4.4. Let the conditions of Lemma 4.2 be satisfied. If, for $i = 1,2,3$, T_i and I_i satisfy condition (B) and $\bigcap_{i=1}^{3} F(T_i) \cap F(I_i) \neq \emptyset$, then $\{x_n\}$ defined by (4.1) converges strongly to a common fixed point of T_i and I_i, $(i = 1,2,3)$.

Proof . By Lemma 4.1(1), we have $\lim_{n\to\infty} \|x_n - q\|$ exists and so $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in F(T \cap I)$. Also by Lemma 4.1(2), $\lim_{n\to\infty} \|x_n - I_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$. It follows from condition (B) that $\lim_{n\to\infty} f(d(x_n, \bigcap_{i=1}^{3} F(T_i) \cap F(I_i))) \leq \lim_{n\to\infty} \{\frac{1}{2}(\|x_n - T_i x_n\| + \|x_n - I_i x_n\|)\}$. That is, $\lim_{n\to\infty} f(d(x_n, \bigcap_{i=1}^{3} F(T_i) \cap F(I_i))) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$, we have $\lim_{n\to\infty} d(x_n, \bigcap_{i=1}^{3} F(T_i) \cap F(I_i)) = 0$. By the same method given in the proof of Theorem 3.5, the proof is completed. □

Finally, we prove the weak convergence of the iterative scheme (4.1) for three I-asymptotically nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 4.5. Let X be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed, bounded and convex subset of X. Let $T_i : K \to K, (i = 1,2,3)$ be a I-asymptotically nonexpansive mapping with $\{k_n\}$ a sequence of real numbers such that $k_n \geq 1$ and $\sum_{n=0}^{\infty} (k_n - 1) < \infty$ and $I_i : K \to K, (i = 1,2,3)$ be an asymptotically nonexpansive mapping with $\{\ell_n\}$ a sequence of real numbers such that $\ell_n \geq 1$ and $\sum_{n=1}^{\infty} (\ell_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$,$\{\alpha_n\}$, $\{\beta_n\}$ be sequences of real numbers in $[0,1]$, such that $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ in $[0,1]$ for all $n \geq 1$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ be the sequences in K defined by (4.1). If $\bigcap_{i=1}^{3} F(T_i) \cap F(I_i) \neq \emptyset$, then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge weakly to a common fixed point of T_i and I_i, $(i = 1,2,3)$.

Proof . Let $q \in \bigcap_{i=1}^{3} F(T_i) \cap F(I_i)$. Then as in Lemma 4.1(1), $\lim_{n\to\infty} \|x_n - q\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in $\bigcap_{i=1}^{3} F(T_i) \cap F(I_i)$. We assume that q_1 and q_2 are weak limits of the subsequences $\{x_{n_k}\}$, $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By (4.30)-(4.35), for $i = 1,2,3$, $\lim_{n\to\infty} \|x_n - I_i x_n\| = 0$, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ and $E - T_i$ and $E - I_i$ are demiclosed by Lemma 2.1, for $i = 1,2,3$, $Tq_1 = q_1$, $I_i q_1 = q_1$ and in the same way, $T_i q_2 = q_2$, $I_i q_2 = q_2$. Therefore, we have $q_1, q_2 \in \bigcap_{i=1}^{3} F(T_i) \cap F(I_i)$. It follows from Lemma 2.5 that $q_1 = q_2$. This completes the proof. □

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