Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 1249–1259
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.25857.3146

Several comparisons between matrix means

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(Communicated by Ali Jabbari)

Abstract

In this note, we present some operator inequalities via convexity property. In the end, refinements of mixed mean inequalities are given.

Keywords: Hilbert-Schmidt norm, Convex function, Positive operator, Heinz-Heron means, Logarithmic-Harmonic means
2020 MSC: Primary: 15A45; Secondary: 15A60, 47A63

1 Introduction

For the non-negative real numbers a, b and $0 \leq \nu \leq 1$, let

$$a\nabla_\nu b = (1-\nu)a + \nu b, a\#_\nu b = a^{1-\nu}b^\nu \text{ and } a!_\nu b = ((1-\nu)a^{-1} + \nu b^{-1})^{-1}$$

be the weighted arithmetic, geometric and harmonic means, respectively. When $\nu = \frac{1}{2}$, we write $a\nabla b, a\sharp b$ and $a!b$ for brevity, respectively. Then, the Heron and Heinz means are defined, respectively, as follows

$$F_\nu(a,b) = (1-\nu)(a\#b) + \nu(a\nabla b) \text{ and } H_\nu(a,b) = \frac{a\#_\nu b + a\#_{1-\nu} b}{2}.$$

For the arithmetic, geometric and harmonic means, we have the simple inequalities

$$a!_\nu b \leq a\#_\nu b \leq a\nabla_\nu b, a, b > 0, 0 \leq \nu \leq 1.$$

On the other hand, we have the Heinz inequalities

$$a\#b \leq H_\nu(a,b) \leq a\nabla b. \tag{1.1}$$

The Heron means interpolate between the geometric and arithmetic means via the inequality

$$a\#b \leq F_\nu(a,b) \leq a\nabla b. \tag{1.2}$$

Now (1.1) and (1.2) invite the question about the relation between $F_\nu(a,b)$ and $H_\nu(a,b)$. An interesting reference about this relation is [2], where the inequality

$$H_\nu(a,b) \leq F_{\alpha(\nu)}(a,b),\ \alpha(\nu) = 1 - 4(\nu - \nu^2), 0 \leq \nu \leq 1$$

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Received: January 2022 *Accepted:* May 2022

was proved, with some interesting matrix versions.

Direct numerical experiments show that neither $H_\nu(a,b) \le F_\nu(a,b)$ nor $F_\nu(a,b) \le H_\nu(a,b)$ is valid for all $0 \le \nu \le 1$. Our motivation behind this work is the possible comparison between these means. We shall prove that

$$F_\tau(a,b) \le \left(\frac{a\nabla b}{a\# b}\right)^2 H_\nu(a,b), a,b > 0, 0 \le \nu,\tau \le 1. \tag{1.3}$$

Moreover, a reversed version is presented. The factor $\left(\frac{a\nabla b}{a\# b}\right)^2$ is called the Kantorovich constant, and has appeared in recent studies of mean inequalities. Letting $K(h) = \frac{(h+1)^2}{4h}$, we have $\left(\frac{a\nabla b}{a\# b}\right)^2 = K(a/b) = K(b/a) \ge 1$.

Then, we present the matrix version of (1.3) for (Frobenious) the Hilbert-Schmidt norm. To state the desired matrix inequality, we introduce our notations.
For a positive integer n, let $\mathbb{M}_n$ denote the algebra of all $n \times n$ complex matrices, $\mathbb{M}_n^+$ denote the cone of positive semidefinite matrices in $\mathbb{M}_n$ and $\mathbb{M}_n^{++}$ be the cone of strictly positive definite matrices in $\mathbb{M}_n$. For two Hermitian matrices $A, B \in \mathbb{M}_n$, we write $A \le B$ or $B \ge A$ to mean $B - A \in \mathbb{M}_n^+$. This is called the Löwener partial ordering on Hermitian matrices.
The unitarily invariance of the norm $\|.\|$ means that $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. Example of unitarily invariant norm is the Hilbert-Schmidt (Frobenious) norm defined by

$$\|A\|_2 = \sqrt{\sum_{j=1}^{n} s_j^2(A)},$$

where $s_1(A) \ge s_2(A) \ge .... \ge s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity.
The matrix means corresponding to the numerical ones are defined as follows, for $A, B \in \mathbb{M}_n^{++}$ and $0 \le \nu \le 1$,

$$A\nabla_\nu B = (1-\nu)A + \nu B, A\#_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\nu A^{\frac{1}{2}},$$

$$A!_\nu B = \left((1-\nu)A^{-1} + \nu B^{-1}\right)^{-1}, H_\nu(A,B) = \frac{A\#_\nu B + A\#_{1-\nu}B}{2}$$

and

$$F_\nu(A,B) = (1-\nu)(A\# B) + \nu(A\nabla B).$$

A standard functional calculus argument applied on (1.3) will imply the following Heron-Heinz inequality for $A, B \in \mathbb{M}_n^+$ satisfying $mI \le A, B \le MI$ for some positive numbers m, M,

$$F_\tau(A,B) \le K(M/m)H_\nu(A,B), 0 \le \nu,\tau \le 1. \tag{1.4}$$

Note that the simple inequality $a\# b \le a\nabla b$ means $ab \le \frac{(a+b)^2}{4}$. It is clear that

$$a!b \le H_\nu(a,b), \tag{1.5}$$

for $0 \le \nu \le 1$. The another aim in this is to obtain a matrix version and a reversed version of (1.5).
If $\{e_i\}$ is an orthonormal basis of $H, V : H \to H \otimes H$ is the isometry defined by $Ve_i = e_i \otimes e_i$ and $A \otimes B$ is the tensor product of operators A and B, then Hadamard product $A \circ B$ regarding $\{e_i\}$ is expressed by $A \circ B = V^*(A \otimes B)V$. For $A = (a_{ij})$ and $B = (b_{ij})$, $A \circ B = (a_{ij}b_{ij})$ denotes the Hadamard (Schur's) product of A and B.

2 Main results

We will present our main results in three sections.

2.1 Hilbert-Schmidt (Frobenius) norm

In this section, we first present some numerical inequalities between Heron-Heinz mean and arbitrary means. Then on the base of them, corresponding matrix inequalities for the Hilbert-Schmidt norm $\| \ \|_2$ were established. In the following proof, we will use the observation

$$a!b \leq \frac{a!_\nu b + a!_{1-\nu} b}{2} \leq H_\nu(a,b) \Rightarrow H_\nu^{-1}(a,b) \leq (a!b)^{-1}. \tag{2.1}$$

Lemma 2.1. Let $a, b > 0$ and $0 \leq \nu, \tau \leq 1$. Then

$$F_\tau(a,b) \leq K(a/b) H_\nu(a,b) \text{ and } H_\nu(a,b) \leq K(a/b) F_\tau(a,b). \tag{2.2}$$

Proof . Notice that, by the arithmetic-geometric mean inequality,

$$\begin{aligned} F_\tau(a,b) \cdot ab\ H_\nu^{-1}(a,b) &\leq \frac{1}{4}\left(F_\tau(a,b) + ab\ H_\nu^{-1}(a,b)\right)^2 \\ &\leq \frac{1}{4}\left(a\nabla b + ab(a!b)^{-1}\right)^2 \\ &= \frac{1}{4}\left(a\nabla b + a\nabla b\right)^2 = (a\nabla b)^2. \end{aligned}$$

That is

$$F_\tau(a,b) \leq \left(\frac{a\nabla b}{a\#b}\right)^2 H_\nu(a,b),$$

which proves the first inequality. For the second inequality, notice that $F_\tau(a,b) \geq a\#b \geq a!b$, hence $F_\tau^{-1}(a,b) \leq (a!b)^{-1}$. Therefore,

$$\begin{aligned} ab\ F_\tau^{-1}(a,b) \cdot H_\nu(a,b) &\leq \frac{1}{4}\left(ab\ F_\tau^{-1}(a,b) + H_\nu(a,b)\right)^2 \\ &\leq \frac{1}{4}\left(ab\ (a!b)^{-1} + a\nabla b\right)^2 = (a\nabla b)^2, \end{aligned}$$

proving the second inequality. □

As mentioned in the introduction, we know that neither $F_\nu(a,b) \leq H_\nu(a,b)$ nor the opposite inequality holds for all $0 \leq \nu \leq 1$. Thus, the above Lemma provides a significant achievement; where the factor $K(a/b)$ makes both inequalities valid, even with different parameters ν and τ.

In fact the proof of Lemma 2.1 was based on the fact that both the Heron $F_\nu(a,b)$ and the Heinz $H_\nu(a,b)$ mean lie between the harmonic and arithmetic means. Thus, using the same steps, we deduce the more general result about any two means σ, τ satisfying $! \leq \sigma, \tau \leq \nabla$.

Proposition 2.2. Let $a, b > 0$ and let σ, τ be two arbitrary means between the harmonic and arithmetic means. Then

$$a\ \tau b \leq K(a/b)(a\ \sigma b) \text{ and } a\ \sigma b \leq K(a/b)(a\ \tau b).$$

The numerical inequalities of Lemma 2.1 can be used to prove the following matrix versions based on the Hilbert-Schmidt norm $\| \ \|_2$, as a complete comparison between the Heron and Heinz means.

Theorem 2.3. Let $A, B \in \mathbb{M}_n^{++}$ be such that $0 < mI \leq A, B \leq MI$ and let $X \in \mathbb{M}_n$. Then for $0 \leq \nu, \tau \leq 1$,

$$\left\|(1-\tau)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \tau\frac{AX+XB}{2}\right\|_2 \leq K(M/m)\left\|\frac{A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu}{2}\right\|_2,$$

and

$$\left\|\frac{A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu}{2}\right\|_2 \leq K(M/m)\left\|(1-\tau)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \tau\frac{AX+XB}{2}\right\|_2.$$

Proof . We prove the first inequality, leaving the similar proof of the second inequality to the reader. Since every positive definite matrix is unitarily diagonalizable, it follows that there are unitary matrices $U, V \in \mathbb{M}_n$ such that $A = U\Gamma_1 U^*$ and $B = V\Gamma_2 V^*$, where $\Gamma_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $\Gamma_2 = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ and $\lambda_i, \mu_i \geq 0$ are the eigenvalues of A and B, respectively. Furthermore, let $Y = U^*XV = [y_{ij}]$. Then standard computations show that

$$(1-\tau) \quad A^{\frac{1}{2}}XB^{\frac{1}{2}} + \tau\frac{AX+XB}{2} = U\left(\left[(1-\tau)\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}} + \tau\frac{\lambda_i+\mu_j}{2}\right] \circ Y\right)V^*,$$

and

$$\frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} = U\left(\left[\frac{\lambda_i^{\nu}\mu_j^{1-\nu} + \lambda_i^{1-\nu}\mu_j^{\nu}}{2}\right] \circ Y\right)V^*,$$

where $\circ$ is the (Hadamard)Schur's product operation.

Since the Hilbert-Schmidt norm $\| \ \|_2$ is a unitarily invariant norm, we have

$$\begin{aligned}
\left\|(1-\tau)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \tau\frac{AX+XB}{2}\right\|_2^2 &= \left\|U\left(\left[(1-\tau)\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}} + \tau\frac{\lambda_i+\mu_j}{2}\right] \circ Y\right)V^*\right\|_2^2 \\
&= \left\|\left[(1-\tau)\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}} + \tau\frac{\lambda_i+\mu_j}{2}\right] \circ Y\right\|_2^2 \\
&= \sum_{i,j=1}^{n}\left((1-\tau)\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}} + \tau\frac{\lambda_i+\mu_j}{2}\right)^2 |y_{ij}|^2 \\
&= \sum_{i,j=1}^{n} F_\tau^2(\lambda_i,\mu_j)|y_{ij}|^2 \\
&\leq \sum_{i,j=1}^{n} K^2(\lambda_i/\mu_j)H_\nu^2(\lambda_i,\mu_j)|y_{ij}|^2 \\
&\leq K^2(M/m)\sum_{i,j=1}^{n}\left(\frac{\lambda_i^{\nu}\mu_j^{1-\nu} + \lambda_i^{1-\nu}\mu_j^{\nu}}{2}\right)^2 |y_{ij}|^2 \\
&= K^2(M/m)\left\|\frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2}\right\|_2^2.
\end{aligned}$$

Note that the fact that $0 < mI \leq A, B \leq MI$ has been used in the above proof in the following way: For such A, B, we have $\lambda_i \leq M$ and $\mu_j \geq m$, or $\frac{\lambda_i}{\mu_j} \leq \frac{M}{m}$. Now since $K = K(t)$ is an increasing function, we have $K(\lambda_i/\mu_j) \leq K(M/m)$. $\square$

In [11], it was proved that

$$\left\|\frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2}\right\|_2 \leq \left\|(1-\alpha(\nu))\sqrt{A}X\sqrt{B} + \alpha(\nu)\frac{AX+XB}{2}\right\|_2,$$

for $\alpha(\nu) = 1 - 4(\nu - \nu^2), 0 \leq \nu \leq 1$. The above theorem presents a generalization and a reverse for this inequality, at the cost of the extra coefficient $K(M/m)$.

Integral inequalities of these means have been also of some interests to several researchers in the literature. For example, it is proved in [4] that for any unitarily invariant norm $|||\ |||$, one has

$$|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \leq \left|\left|\left|\int_0^1 A^tXB^{1-t}dt\right|\right|\right| \leq \frac{1}{2}|||AX+XB|||$$

and

$$\left|\left|\left|\int_0^1 A^tXB^{1-t}dt\right|\right|\right| \leq \frac{1}{2}\left|\left|\left|A^{\frac{1}{2}}XB^{\frac{1}{2}} + \frac{AX+XB}{2}\right|\right|\right|$$

for $A, B \in \mathbb{M}_n^{++}$ and $X \in \mathbb{M}_n$. Moreover, it is proved in [2] that

$$\left|\left|\left|\int_0^1 A^t X B^{1-t} dt\right|\right|\right| \leq \left|\left|\left|(1-\alpha) A^{\frac{1}{2}} X B^{\frac{1}{2}} + \alpha \frac{AX + XB}{2}\right|\right|\right|,$$

for $\frac{1}{2} \leq \alpha \leq 1$. Then, motivated by these results and the known Pólya inequality [10, Equ. 12]

$$\int_0^1 a^t b^{1-t} dt \leq \frac{1}{3}\left(2\sqrt{ab} + \frac{a+b}{2}\right),$$

the following integral version was given in [12]

$$\left\|\int_0^1 A^t X B^{1-t} dt\right\|_2 \leq \frac{1}{3}\left\|2 A^{\frac{1}{2}} X B^{\frac{1}{2}} + \frac{AX + XB}{2}\right\|_2.$$

In our next result, we present two-sided integral versions that obtain our results, for the Hilbert-Schmidt norm, but again with an extra coefficient.

Theorem 2.4. Let $A, B \in \mathbb{M}_n^{++}$ be such that $0 < mI \leq A, B \leq MI$, for some positive numbers m, M and let $X \in \mathbb{M}_n$. Then for $0 \leq \nu, \tau \leq 1$,

$$\left\|\int_0^1 A^{1-\tau} X B^{\tau} d\tau\right\|_2 \leq K(m/M)\left\|(1-\nu) A^{\frac{1}{2}} X B^{\frac{1}{2}} + \nu \frac{AX + XB}{2}\right\|_2$$

and

$$\left\|(1-\nu) A^{\frac{1}{2}} X B^{\frac{1}{2}} + \nu \frac{AX + XB}{2}\right\|_2 \leq K(m/M)\left\|\int_0^1 A^{\tau} X B^{1-\tau} d\tau\right\|_2.$$

Proof . Following the notations of Theorem 2.3, we have

$$A^{1-\tau} X B^{\tau} = U\left([\lambda_i^{1-\tau} \mu_j^{\tau}] \circ Y\right) V^*$$

and

$$\begin{aligned}\int_0^1 A^{1-\tau} X B^{\tau}\, d\tau &= U\left(\left[\int_0^1 \lambda_i^{1-\tau} \mu_j^{\tau} d\tau\right] \circ Y\right) V^* \\ &= U\left([L(\lambda_i, \mu_j)] \circ Y\right) V^*,\end{aligned}$$

where $L(\lambda_i, \mu_j) = \frac{\lambda_i - \mu_j}{\log \lambda_i - \log \mu_j}$ is the logarithmic mean of λ_i and μ_j. Now since $\lambda_i ! \mu_j \leq L(\lambda_i, \mu_j) \leq \lambda_i \nabla \mu_j$, we may apply Proposition 2.2 to get

$$\begin{aligned}\left\|\int_0^1 A^{1-\tau} X B^{\tau} d\tau\right\|_2^2 &= \left\|U\left(\left[\int_0^1 \lambda_i^{1-\tau} \mu_j^{\tau} d\tau\right] \circ Y\right) V^*\right\|_2^2 \\ &= \left\|U\left([L(\lambda_i, \mu_j)] \circ Y\right) V^*\right\|_2^2 \\ &= \sum_{i,j=1}^n L^2(\lambda_i, \mu_j) |y_{ij}|^2 \\ &\leq \sum_{i,j=1}^n K^2(\lambda_i/\mu_j) F_\nu^2(\lambda_i, \mu_j) |y_{ij}|^2 \\ &\leq K^2(m/M)\left\|(1-\nu) A^{\frac{1}{2}} X B^{\frac{1}{2}} + \nu \frac{AX + XB}{2}\right\|_2^2,\end{aligned}$$

which proves the first inequality. The second inequality follows in a similar way. □

Theorem 2.4 presents a generalization and a reverse of the obtained results in [2], [12] and [7]. Now, we are going to present a matrix version for the Hilber-Schmidt norm with coefficient X of (1.5).

Theorem 2.5. Let $A, B \in \mathbb{M}_n^{++}$ be such that $0 < mI \leq A, B \leq MI$, for some positive numbers m, M and let $X \in \mathbb{M}_n$. Then for $0 \leq \nu \leq 1$,

$$\left\| \left(\frac{A^{-1}X + XB^{-1}}{2} \right)^{-1} \right\|_2^2 \leq K(m/M) \left\| \frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} \right\|_2^2$$

and

$$\left\| \frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} \right\|_2^2 \leq K(m/M) \left\| \left(\frac{A^{-1}X + XB^{-1}}{2} \right)^{-1} \right\|_2^2.$$

Proof . Since $A, B > 0$, by the spectral theorem, there exists unitary matrices $U, V \in M_n$ such that $A = U\Gamma_1 U^*$ and $B = V\Gamma_2 V^*$, where $\Gamma_1 = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, $\Gamma_2 = \operatorname{diag}(\mu_1, \ldots, \mu_n)$ and $\lambda_i, \mu_i \geq 0$ are the eigenvalues of A and B, respectively. To obtain our results, let $Y = U^{-1}X(V^*)^{-1}$. Then

$$\begin{aligned} \left(\frac{A^{-1}X + XB^{-1}}{2} \right)^{-1} &= \left(\frac{(U\Gamma_1 U^*)^{-1}X + X(V\Gamma_2 V^*)^{-1}}{2} \right)^{-1} \\ &= \left(\frac{((U^*)^{-1}\Gamma_1^{-1}U^{-1})X + X((V^*)^{-1}\Gamma_2^{-1}V^{-1})}{2} \right)^{-1} \\ &= \left(\frac{(U^*)^{-1}[\Gamma_1^{-1}U^{-1}X(V^*)^{-1} + U^{-1}X(V^*)^{-1}\Gamma_2^{-1}]V^{-1}}{2} \right)^{-1} \\ &= V\left[\left(\frac{\Gamma_1^{-1}Y + Y\Gamma_2^{-1}}{2} \right)^{-1} \right] U^* \end{aligned}$$

So,

$$\begin{aligned} \left\| \left(\frac{A^{-1}X + XB^{-1}}{2} \right)^{-1} \right\|_2^2 &= \left\| V\left[\left(\frac{\Gamma_1^{-1}Y + Y\Gamma_2^{-1}}{2} \right)^{-1} \right] U^* \right\|_2^2 \\ &= \sum_{i,j=1}^{n} \left(\left(\frac{\lambda_i^{-1} + \mu_j^{-1}}{2} \right)^{-1} \right)^2 |y_{ij}|^2, \end{aligned}$$

and

$$\left\| \frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} \right\|_2^2 = \sum_{i,j=1}^{n} \left(\frac{\lambda_i^{\nu}\mu_j^{1-\nu} + \lambda_i^{\nu}\mu_j^{1-\nu}}{2} \right)^2 |y_{ij}|^2.$$

By applying Proposition 2.2 and a way similar to Theorem 2.3 , we can get the desired results.

□

Note that by integrating of both sides of the inequalities in Theorem 2.3 and Theorem 2.5, we get

$$\begin{aligned} K^{-1}(m/M) &\left\| (1-\tau)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \tau\frac{AX + XB}{2} \right\|_2 \\ &\leq \int_0^1 \left\| \frac{A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}}{2} \right\|_2 d\nu \\ &\leq K(m/M) \left\| (1-\tau)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \tau\frac{AX + XB}{2} \right\|_2 \end{aligned}$$

and

$$\begin{aligned} K^{-1}(m/M) \left\| \left(\frac{A^{-1}X + XB^{-1}}{2} \right)^{-1} \right\|_2^2 &\leq \int_0^1 \left\| A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu} \right\|_2^2 \\ &\leq K(m/M) \left\| \left(\frac{A^{-1}X + XB^{-1}}{2} \right)^{-1} \right\|_2^2. \end{aligned}$$

We conclude this section by presenting a full matrix comparison for means between the harmonic and arithmetic means. For $a, b > 0$, let $N(a, b)$ be a mean such that $a!b \le N(a, b) \le a\nabla b$. Now if $A, B \in \mathbb{M}_n^+$ have eigenvalues $\{\lambda_i\}$ and $\{\mu_j\}$ respectively, let $N(A, B)$ denote the matrix whose $ij-$entry is $N(\lambda_i, \mu_j)$. Following the same steps of Theorem 2.4, we have the following.

Theorem 2.6. Let $A, B \in \mathbb{M}_n^+$ be such that $0 < mI \le A, B \le MI$, for some positive numbers m, M and let $X \in \mathbb{M}_n$. Then for $0 \le \nu, \tau \le 1$,

$$\|N_1(A, B) \circ X\|_2 \le K(M/m)\|N_2(A, B) \circ X\|_2$$

and

$$\|N_2(A, B) \circ X\|_2 \le K(M/m)\|N_1(A, B) \circ X\|_2,$$

for any two means N_1 and N_2 between the harmonic and arithmetic means.

2.2 Löwener partial ordering

In this section, we present some ordering relations between different matrix means. In the first result, we apply a standard functional calculus argument that has been used extensively in the literature. We present the result for the Heron and Heinz means, however it is still valid for any mean between ! and ∇.
Here, we need to state the well-known monotonicity principle for bounded hermitian operators ([8]). If X be a Hermitian operator with a spectrum $Sp(X)$, then

$$f(t) \ge g(t), \quad t \in Sp(X) \quad \Rightarrow f(X) \ge g(X), \tag{2.3}$$

provided that f and g are real-valued continuous functions.

Theorem 2.7. Let $A, B \in \mathbb{M}_n^{++}$ be such that $0 < mI \le A, B \le MI$, for some positive numbers m, M and let $0 \le \nu, \tau \le 1$. Then

$$\begin{aligned} F_\tau(A, B) &\le max\{K(M/m), K(m/M)\}H_\nu(A, B), \\ H_\nu(A, B) &\le max\{K(M/m), K(m/M)\}F_\tau(A, B). \end{aligned} \tag{2.4}$$

Proof . We prove the first inequality. Let $a = 1$ in the first inequality of Lemma 2.1 to get

$$(1-\tau)\sqrt{b} + \tau\frac{1+b}{2} \le K(b)\frac{b^\nu + b^{1-\nu}}{2}. \tag{2.5}$$

Notice that if $b \in \left[\frac{m}{M}, \frac{M}{m}\right]$, then $K(b) \le K(M/m)$, since K is decreasing when $0 < h < 1$ and increasing when $h > 1$. Therefore, (2.5) implies

$$(1-\tau)\sqrt{b} + \tau\frac{1+b}{2} \le K(M/m)\frac{b^\nu + b^{1-\nu}}{2}. \tag{2.6}$$

Now when $mI \le A, B \le MI$, we have $\frac{m}{M}I \le A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le \frac{M}{m}I$, and $\mathrm{Sp}(X) \subset \left[\frac{m}{M}, \frac{M}{m}\right]$, where $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Therefore, letting

$$f(b) = (1-\tau)\sqrt{b} + \tau\frac{1+b}{2}$$

and

$$g(b) = K(M/m)\frac{b^\nu + b^{1-\nu}}{2},$$

by 2.3, we have $f(X) \le g(X)$. That is,

$$(1-\tau)X^{\frac{1}{2}} + \tau\frac{I+X}{2} \le K(M/m)\frac{X^\nu + X^{1-\nu}}{2}.$$

Conjugating both sides of this inequality with $A^{\frac{1}{2}}$ implies the desired inequality. The second inequality proves similarly. $\square$

2.3 Refinements of mixed mean inequalities

Sagae and Tanabe ([9]) gave a mixed arithmetic-geometric mean inequality for two invertible positive operators. Mond and Pecaric ([6]) presented a mixed arithmetic-geometric and geometric-Harmonic mean inequality for two invertible positive operators in following form: For two invertible positive operators A and B, we have

$$A\sharp(A\nabla B) \geq A\nabla(A\sharp B),$$

$$A\sharp(A!B) \leq A!(A\sharp B).$$

In this section, we give refinements of above inequalities.

If $f : [a, b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

The above inequality is known as the Hermite-Hadamard integral inequality for convex functions see [5]. It is a interesting issue to know that whether for a convex function f on an interval I there exist numbers l and L such that

$$f\left(\frac{a+b}{2}\right) \leq l(\lambda) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq L(\lambda) \leq \frac{f(a)+f(b)}{2}.$$

A. El Farissi in ([3]) gave a positive reply to this question. If $f : [a, b] \to \mathbb{R}$ is a convex function, then for $0 \leq \lambda \leq 1$

$$f\left(\frac{a+b}{2}\right) \leq l \leq \frac{1}{b-a}\int_a^b f(t)dt \leq L \leq \frac{f(a)+f(b)}{2}, \tag{2.7}$$

where

$$l(\lambda) = \lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right) + (1-\lambda)f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)$$

and

$$L(\lambda) = \frac{1}{2}(f(\lambda b+(1-\lambda)a) + \lambda f(a) + (1-\lambda)f(b)).$$

Remark 2.8. The case $\lambda = \frac{1}{2}$ in (2.7) is simplified to the following case

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \leq \frac{1}{b-a}\int_a^b f(t)dt \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{2.8}$$

Now, we ready to give the following theorems:

Theorem 2.9. Let A and B be two invertible positive operators. Then

$$\begin{aligned} A\sharp(A\nabla B) &\geq \frac{1}{2}\left[A\sharp\left(\frac{3A}{2}\nabla\frac{B}{2}\right) + A\sharp\left(\frac{A}{2}\nabla\frac{3B}{2}\right)\right] \\ &\geq \frac{1}{3}\left[4A\nabla(A\sharp B) - (A\sharp B)(A\nabla(A\sharp B))^{-1}A\right] \\ &\geq \frac{1}{2}\left[A\sharp(A\nabla B) + A\nabla(A\sharp B)\right] \\ &\geq A\nabla(A\sharp B). \end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
A\sharp(A!B) &\leq 2\left[\left(A\sharp\left(\frac{2}{3}A!2B\right)\right)^{-1}+\left(A\sharp\left(2A!\frac{2}{3}B\right)\right)^{-1}\right]^{-1}\\
&\leq 3\left[4(A!(A\sharp B))^{-1}-(A\sharp B)^{-1}(A!(A\sharp B))A^{-1}\right]^{-1}\\
&\leq 2\left[\left(A\sharp(A!B)\right)^{-1}+\left(A!(A\sharp B)\right)^{-1}\right]^{-1}\\
&\leq A!(A\sharp B).
\end{aligned}\tag{2.10}$$

Proof . Considering inequalities (2.8) with the convex function $f(t)=-t^{\frac{1}{2}}, t>0$, it follows that

$$\begin{aligned}
\left(\frac{a+b}{2}\right)^{\frac{1}{2}} &\geq \frac{1}{2}\left[\left(\frac{\frac{3a}{2}+\frac{b}{2}}{2}\right)^{\frac{1}{2}}+\left(\frac{\frac{a}{2}+\frac{3b}{2}}{2}\right)^{\frac{1}{2}}\right]\\
&\geq \frac{1}{3}\left[4\left(\frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{2}\right)-(ab)^{\frac{1}{2}}\left(\frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{2}\right)^{-1}\right]\\
&\geq \frac{1}{2}\left[\left(\frac{a+b}{2}\right)^{\frac{1}{2}}+\frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{2}\right]\\
&\geq \frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{2}
\end{aligned}\tag{2.11}$$

Insertting $a=1$ in (2.11), it follows that

$$\begin{aligned}
\left(\frac{1+b}{2}\right)^{\frac{1}{2}} &\geq \frac{1}{2}\left[\left(\frac{\frac{3}{2}+\frac{b}{2}}{2}\right)^{\frac{1}{2}}+\left(\frac{\frac{1}{2}+\frac{3b}{2}}{2}\right)^{\frac{1}{2}}\right]\\
&\geq \frac{1}{3}\left[4\left(\frac{1+b^{\frac{1}{2}}}{2}\right)-(b)^{\frac{1}{2}}\left(\frac{1+b^{\frac{1}{2}}}{2}\right)^{-1}\right]\\
&\geq \frac{1}{2}\left[\left(\frac{1+b}{2}\right)^{\frac{1}{2}}+\frac{1+b^{\frac{1}{2}}}{2}\right]\\
&\geq \frac{1+b^{\frac{1}{2}}}{2}.
\end{aligned}$$

Since $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\geq 0$, monotonicity principle (2.3) for operator functions yeilds inequality

$$\begin{aligned}
\left(\frac{I+A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2}\right)^{\frac{1}{2}} &\geq \frac{1}{2}\left[\left(\frac{\frac{3}{2}I+\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2}}{2}\right)^{\frac{1}{2}}+\left(\frac{\frac{1}{2}I+\frac{3A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2}}{2}\right)^{\frac{1}{2}}\right]\\
&\geq \frac{1}{3}\left[4\left(\frac{I+(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}}{2}\right)-\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}\right.\\
&\qquad\left.\left(\frac{I+\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}}{2}\right)^{-1}\right]\\
&\geq \frac{1}{2}\left[\left(\frac{I+A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2}\right)^{\frac{1}{2}}+\frac{I+(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}}{2}\right]\\
&\geq \frac{I+\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}}{2}.
\end{aligned}$$

Moreover, multiplying both sides of the previous series of inequalities by $A^{\frac{1}{2}}$, we have (2.9), as claimed. Replacing A and B by A^{-1} and B^{-1} in inequality (2.9), respectively, and then taking inverse of both sides (2.9), we yeild (2.10). □

If $f : [a, b] \to\to R$ is a integrable function, then a simple computation shows that (2.8) is equivalent to the following inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \int_0^1 f((1-t)a+tb)dt \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{2.12}$$

Theorem 2.10. Let A and B be two invertible positive operators. Then

$$\begin{aligned} A\sharp(A\nabla B) &\geq \frac{1}{2}\left[A\sharp\left(\frac{3A}{2}\nabla\frac{B}{2}\right)+A\sharp\left(\frac{A}{2}\nabla\frac{3B}{2}\right)\right] \\ &\geq \int_0^1 A\sharp(A\nabla_t B)dt \\ &\geq \frac{1}{2}\Big[A\nabla(A\sharp B)+A\nabla(A\sharp B)\Big] \\ &\geq A\nabla(A\sharp B). \end{aligned} \tag{2.13}$$

Proof . Considering inequalities (2.12) with the convex function $f(t) = -t^{\frac{1}{2}}, t > 0$, it follows that

$$\begin{aligned} \left(\frac{a+b}{2}\right)^{\frac{1}{2}} &\geq \frac{1}{2}\left[\left(\frac{\frac{3a}{2}+\frac{b}{2}}{2}\right)^{\frac{1}{2}}+\left(\frac{\frac{a}{2}+\frac{3b}{2}}{2}\right)^{\frac{1}{2}}\right] \\ &\geq \int_0^1 ((1-t)a+tb)^{\frac{1}{2}}dt \\ &\geq \frac{1}{2}\left[\left(\frac{a+b}{2}\right)^{\frac{1}{2}}+\frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{2}\right] \\ &\geq \frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{2}. \end{aligned}$$

Using a method as in Theorem (2.9), we can obtain (2.13). □

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