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A study on hyperbolic numbers with generalized Jacobsthal numbers components

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Abstract

In this paper, we introduce the generalized hyperbolic Jacobsthal numbers. As special cases, we deal with hyperbolic Jacobsthal and hyperbolic Jacobsthal-Lucas numbers. We present Binet's formulas, generating functions and the summation formulas for these numbers. Moreover, we give Catalan's, Cassini's, d'Ocagne's, Gelin-Cesàro's, Melham's identities and present matrices related with these sequences.

Keywords: Jacobsthal numbers, Jacobsthal-Lucas numbers, hyperbolic numbers, hyperbolic Jacobsthal numbers, Cassini identity
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1 Introduction

Jacobsthal sequence $\{J_n\}_{n\geq 0}$ (OEIS: A001045, [30]) and Jacobsthal-Lucas sequence $\{K_n\}_{n\geq 0}$ (OEIS: A014551, [30]) are defined by the second-order recurrence relations

$$J_n = J_{n-1} + 2J_{n-2}, \qquad J_0 = 0, J_1 = 1 \tag{1.1}$$

and

$$K_n = K_{n-1} + 2K_{n-2}, \quad K_0 = 2, K_1 = 1. \tag{1.2}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{K_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$J_{-n} = -\frac{1}{2}J_{-(n-1)} + \frac{1}{2}J_{-(n-2)}$$

and

$$K_{-n} = -\frac{1}{2}K_{-(n-1)} + \frac{1}{2}K_{-(n-2)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.1) and (1.2) hold for all integer n.

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A generalized Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$V_n = V_{n-1} + 2V_{n-2}; \quad V_0 = a, \; V_1 = b, \quad (n \geq 2) \tag{1.3}$$

with the initial values V_0, V_1 not all being zero. The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} + \frac{1}{2}V_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.3) holds for all integer n.

Note that if we set $V_0 = 0, V_1 = 1$ then $\{V_n\}$ is the well-known Jacobsthal sequence and if we set $V_0 = 2, V_1 = 1$ then $\{V_n\}$ is the well-known Jacobsthal-Lucas sequence. The first few generalized Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Jacobsthal numbers

n	V_n	V_{-n}
0	V_0	...
1	V_1	$-\frac{1}{2}V_0 + \frac{1}{2}V_1$
2	$2V_0 + V_1$	$\frac{3}{4}V_0 - \frac{1}{4}V_1$
3	$2V_0 + 3V_1$	$-\frac{5}{8}V_0 + \frac{3}{8}V_1$
4	$6V_0 + 5V_1$	$\frac{11}{16}V_0 - \frac{5}{16}V_1$
5	$10V_0 + 11V_1$	$-\frac{21}{32}V_0 + \frac{11}{32}V_1$
6	$22V_0 + 21V_1$	$\frac{43}{64}V_0 - \frac{21}{64}V_1$

Jacobsthal sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [2], [3], [7], [8], [9], [12], [13], [16], [18], [19], [24], [25], [27], [37].

We can list some important properties of generalized Jacobsthal numbers that are needed.

- Binet's formula of generalized Jacobsthal sequence can be calculated using its characteristic equation which is given as
$$t^2 - t - 2 = 0.$$
The roots of characteristic equation are
$$\alpha = 2, \quad \beta = -1$$
and the roots satisfy the following
$$\alpha + \beta = 1, \; \alpha\beta = -2, \; \alpha - \beta = 3.$$
Using these roots and the recurrence relation, Binet formula can be given as
$$V_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} = \frac{A2^n - B(-1)^n}{3} \tag{1.4}$$
where $A = V_1 - V_0\beta = V_1 + V_0$ and $B = V_1 - V_0\alpha = V_1 - 2V_0$.

- Binet's formula of Jacobsthal and Jacobsthal-Lucas sequences are
$$J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{2^n - (-1)^n}{3}$$
and
$$K_n = \alpha^n + \beta^n = 2^n + (-1)^n$$
respectively.

- The generating function for generalized Jacobsthal numbers is
$$g(t) = \frac{V_0 + (V_1 - V_0)\, t}{1 - t - 2t^2}. \tag{1.5}$$

- The Cassini identity for generalized Jacobsthal numbers is
$$V_{n+1}V_{n-1} - V_n^2 = 2^{n-1}(V_0V_1 - V_1^2 - 2V_0^2) \tag{1.6}$$

•

$$\begin{aligned} A\alpha^n &= \alpha V_n + V_{n-1}, && (1.7)\\ B\beta^n &= \beta V_n + V_{n-1}. && (1.8) \end{aligned}$$

There are some extensions (generalizations) of real numbers into real algebras of dimension 2 which are the followings: complex numbers,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\},$$

hyperbolic (double, split-complex) numbers, [31],

$$\mathbb{H} = \{h = a + hb : a, b \in \mathbb{R}, h^2 = 1\},$$

and dual numbers, [15],

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0\}.$$

In fact, each possible system can be reduced to one of the above and there exist essentially three possible ways to generalize real numbers into real algebras of dimension 2 (see, for example, [22] for details).

There are also other extensions (generalizations) of real numbers into real algebras of higher dimension. The hypercomplex numbers systems, [22], are extensions of real numbers. Some commutative examples of hypercomplex number systems are complex numbers, hyperbolic numbers, [31], and dual numbers, [15]. Some non-commutative examples of hypercomplex number systems are quaternions [17], octonions [5] and sedenions [33]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n-ions (see for example [6], [20], [28]).

Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) [17] as an extension to the complex numbers. Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848, [11]. H. H. Cheng and S. Thompson [10] introduced dual numbers with complex coefficients and called complex dual numbers. Akar, Yüce and Şahin [1] introduced dual hyperbolic numbers.

Here we use the set of hyperbolic numbers. The set of hyperbolic numbers $\mathbb{H}$ can be described as

$$\mathbb{H} = \{z = x + hy \mid h \notin \mathbb{R},\ h^2 = 1, x, y \in \mathbb{R}\}.$$

The hyperbolic ring $\mathbb{H}$ is a bidimensional Clifford algebra, see [23] for details. Hyperbolic numbers has been called in the mathematical literature with different names: Lorentz numbers, double numbers, duplex numbers, split complex numbers and perplex numbers. Hyperbolic numbers are useful for measuring distances in the Lorentz space-time plane (see Sobczyk [31]). For more information on hyperbolic numbers, see also [21], [26], [29], [32].

Addition, substraction and multiplication of any two hyperbolic numbers z_1 and z_2 are defined by

$$\begin{aligned} z_1 \pm z_2 &= (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h\,(y_1 \pm y_2)\,,\\ z_1 \times z_2 &= (x_1 + hy_1) \times (x_2 + hy_2) = x_1x_2 + y_1y_2 + h\,(x_1y_2 + y_1x_2)\,. \end{aligned}$$

and the division of two hyperbolic numbers are given by

$$\frac{z_1}{z_2} = \frac{x_1 + hy_1}{x_2 + hy_2} = \frac{(x_1 + hy_1)\,(x_2 - hy_2)}{(x_2 + hy_2)\,(x_2 - hy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 - y_2^2} + h\frac{x_1y_2 + y_1x_2}{x_2^2 - y_2^2}.$$

It is easy to see that this algebra of hyperbolic numbers is commutative and contains zero divisors. The hyperbolic conjugation of $z = x + hy$ is defined by

$$\overline{z} = z^{\dagger} = x - hy.$$

Note that $\overline{\overline{z}} = z$. Note also that for any hyperbolic numbers z_1, z_2, z we have

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2},\\ \overline{z_1 \times z_2} &= \overline{z_1} \times \overline{z_2},\\ \|z\|^2 &= z \times \overline{z} = x^2 - y^2. \end{aligned}$$

In this paper, we define the hyperbolic generalized Jacobsthal numbers in the next section and give some properties of them.

2 Hyperbolic Generalized Jacobsthal Numbers and their Generating Functions and Binet's Formulas

In this section, we define hyperbolic generalized Jacobsthal numbers and present generating functions and Binet formulas for them.

In [4], the author defined hyperbolic Fibonacci numbers and Dikmen [14] defined hyperbolic Jacobsthal numbers. Soykan [36], defined hyperbolic generalized Fibonacci numbers.

We now define hyperbolic generalized Jacobsthal numbers over $\mathbb{H}_{\mathbb{D}}$. The nth hyperbolic generalized Jacobsthal number is

$$\widetilde{V}_n = V_n + hV_{n+1}. \tag{2.1}$$

As special cases, the nth hyperbolic Jacobsthal numbers and the nth hyperbolic Jacobsthal-Lucas numbers are given as

$$\widetilde{J}_n = J_n + hJ_{n+1}$$

and

$$\widetilde{K}_n = K_n + hK_{n+1}$$

respectively. It can be easily shown that

$$\widetilde{V}_n = \widetilde{V}_{n-1} + 2\widetilde{V}_{n-2}. \tag{2.2}$$

The sequence $\{\widetilde{V}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\widetilde{V}_{-n} = -\frac{1}{2}\widetilde{V}_{-(n-1)} + \frac{1}{2}\widetilde{V}_{-(n-2)}.$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n. Note that

$$\widetilde{V}_n h = V_{n+1} + V_n h.$$

The first few hyperbolic generalized Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few hyperbolic generalized Jacobsthal numbers

n	$\widetilde{V}_n$	$\widetilde{V}_{-n}$
0	$V_0 + hV_1$	...
1	$V_1 + h(2V_0 + V_1)$	$-\frac{1}{2}a + \frac{1}{2}b = \frac{1}{2}V_1 - \frac{1}{2}V_0 + hV_0$
2	$2V_0 + V_1 + h(2V_0 + 3V_1)$	$\frac{3}{4}V_0 - \frac{1}{4}V_1 + h(-\frac{1}{2}V_0 + \frac{1}{2}V_1)$
3	$2V_0 + 3V_1 + h(6V_0 + 5V_1)$	$\frac{3}{8}V_1 - \frac{5}{8}V_0 + h(\frac{3}{4}V_0 - \frac{1}{4}V_1)$
4	$6V_0 + 5V_1 + h(10V_0 + 11V_1)$	$\frac{11}{16}V_0 - \frac{5}{16}V_1 + h(-\frac{5}{8}V_0 + \frac{3}{8}V_1)$
5	$10V_0 + 11V_1 + h(22V_0 + 21V_1)$	$\frac{11}{32}V_1 - \frac{21}{32}V_0 + h(\frac{11}{16}V_0 - \frac{5}{16}V_1)$
6	$22V_0 + 21V_1 + h(42V_0 + 43V_1)$	$\frac{43}{64}V_0 - \frac{21}{64}V_1 + h(-\frac{21}{32}V_0 + \frac{11}{32}V_1)$

Note that

$$\begin{aligned} \widetilde{V}_0 &= V_0 + hV_1 = V_0 + hV_1, \\ \widetilde{V}_1 &= V_1 + hV_2 = V_1 + h(2V_0 + V_1). \end{aligned}$$

For hyperbolic Jacobsthal numbers (taking $V_n = J_n$, $J_0 = 0, J_1 = 1$) we get

$$\begin{aligned} \widetilde{J}_0 &= h, \\ \widetilde{J}_1 &= 1 + h. \end{aligned}$$

and for hyperbolic Jacobsthal-Lucas numbers (taking $V_n = K_n$, $K_0 = 2, K_1 = 1$) we get

$$\begin{aligned} \widetilde{K}_0 &= 2 + h, \\ \widetilde{K}_1 &= 1 + 5h. \end{aligned}$$

A few hyperbolic Jacobsthal numbers and hyperbolic Jacobsthal-Lucas numbers with positive subscript and negative subscript are given in the following Table 3 and Table 4.

Table 3. Hyperbolic Jacobsthal numbers

n	$\widetilde{J}_n$	$\widetilde{J}_{-n}$
0	h	...
1	$1+h$	$\frac{1}{2}$
2	$1+3h$	$-\frac{1}{4}+\frac{1}{2}h$
3	$3+5h$	$\frac{3}{8}-\frac{1}{4}h$
4	$5+11h$	$-\frac{5}{16}+\frac{3}{8}h$
5	$11+21h$	$\frac{11}{32}-\frac{5}{16}h$
6	$21+43h$	$-\frac{21}{64}+\frac{11}{32}h$

Table 4. Hyperbolic Jacobsthal-Lucas numbers

n	$\widetilde{K}_n$	$\widetilde{K}_{-n}$
0	$2+h$	...
1	$1+5h$	$-\frac{1}{2}+2h$
2	$5+7h$	$\frac{5}{4}-\frac{1}{2}h$
3	$7+17h$	$-\frac{7}{8}+\frac{5}{4}h$
4	$17+31h$	$\frac{17}{16}-\frac{7}{8}h$
5	$31+65h$	$-\frac{31}{32}+\frac{17}{16}h$
6	$65+127h$	$\frac{65}{64}-\frac{31}{32}h$

Now, we will state Binet's formula for the hyperbolic generalized Jacobsthal numbers and in the rest of the paper, we fix the following notations:

$$\begin{aligned}
\widetilde{\alpha} &= 1+h\alpha = 1+2h, \\
\widetilde{\beta} &= 1+h\beta = 1-h.
\end{aligned}$$

Note that we have the following identities:

$$\begin{aligned}
\widetilde{\alpha} &= 1+2h, \\
\widetilde{\beta} &= 1-h, \\
\widetilde{\alpha}\widetilde{\beta} &= -1+h, \\
\widetilde{\alpha}^2 &= 5+4h, \\
\widetilde{\beta}^2 &= 2-2h, \\
\widetilde{\alpha}^2\widetilde{\beta} &= 1-h, \\
\widetilde{\alpha}\widetilde{\beta}^2 &= -2+2h, \\
\widetilde{\alpha}^3 &= 13+14h, \\
\widetilde{\beta}^3 &= 4-4h, \\
\widetilde{\alpha}^2\widetilde{\beta}^2 &= 2-2h, \\
\widetilde{\alpha}^3\widetilde{\beta} &= -1+h, \\
\widetilde{\alpha}\widetilde{\beta}^3 &= -4+4h.
\end{aligned}$$

Theorem 2.1. (Binet's Formula) For any integer n, the nth hyperbolic generalized Jacobsthal number is

$$\widetilde{V}_n = \frac{A\widetilde{\alpha}\alpha^n - B\widetilde{\beta}\beta^n}{\alpha-\beta} = \frac{A\widetilde{\alpha}2^n - B\widetilde{\beta}(-1)^n}{3} \tag{2.3}$$

where

$$\begin{aligned}
A &= V_1 - V_0\beta = V_1 + V_0, \\
B &= V_1 - V_0\alpha = V_1 - 2V_0.
\end{aligned}$$

Proof . Using Binet's formula (1.4) and the recurrence relation (2.1)

$$V_n = \frac{A\alpha^n - B\beta^n}{\alpha-\beta} = \frac{A2^n - B(-1)^n}{3}$$

we obtain

$$\begin{aligned}
\widetilde{V}_n &= V_n + hV_{n+1} = \frac{A2^n - B(-1)^n}{3} + h\frac{A2^{n+1} - B(-1)^{n+1}}{3} \\
&= \frac{A(1+2h)2^n - B(1-h)(-1)^n}{3}.
\end{aligned}$$

This proves (2.3). □

As special cases, for any integer n, the Binet Formula of nth hyperbolic Jacobsthal number is

$$\widetilde{J}_n = \frac{\widetilde{\alpha}\alpha^n - \widetilde{\beta}\beta^n}{\alpha - \beta} = \frac{\widetilde{\alpha}2^n - \widetilde{\beta}(-1)^n}{3} \tag{2.4}$$

and the Binet Formula of nth hyperbolic Jacobsthal-Lucas number is

$$\widetilde{K}_n = \widetilde{\alpha}\alpha^n + \widetilde{\beta}\beta^n = \widetilde{\alpha}2^n + \widetilde{\beta}(-1)^n. \tag{2.5}$$

Next, we present generating function.

Theorem 2.2. The generating function for the hyperbolic generalized Jacobsthal numbers is

$$\sum_{n=0}^{\infty} \widetilde{V}_n x^n = \frac{\widetilde{V}_0 + (\widetilde{V}_1 - \widetilde{V}_0)x}{1 - x - 2x^2}. \tag{2.6}$$

Proof . Let

$$g(x) = \sum_{n=0}^{\infty} \widetilde{V}_n x^n$$

be generating function of the hyperbolic generalized Jacobsthal numbers. Then, using the definition of the hyperbolic generalized Jacobsthal numbers, and substracting $xg(x)$ and $2x^2g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1 - x - 2x^2)g(x) &= \sum_{n=0}^{\infty} \widetilde{V}_n x^n - x\sum_{n=0}^{\infty} \widetilde{V}_n x^n - 2x^2\sum_{n=0}^{\infty} \widetilde{V}_n x^n \\
&= \sum_{n=0}^{\infty} \widetilde{V}_n x^n - \sum_{n=0}^{\infty} \widetilde{V}_n x^{n+1} - 2\sum_{n=0}^{\infty} \widetilde{V}_n x^{n+2} \\
&= \sum_{n=0}^{\infty} \widetilde{V}_n x^n - \sum_{n=1}^{\infty} \widetilde{V}_{n-1} x^n - 2\sum_{n=2}^{\infty} \widetilde{V}_{n-2} x^n \\
&= (\widetilde{V}_0 + \widetilde{V}_1 x) - \widetilde{V}_0 x + \sum_{n=2}^{\infty} (\widetilde{V}_n - \widetilde{V}_{n-1} - 2\widetilde{V}_{n-2})x^n \\
&= (\widetilde{V}_0 + \widetilde{V}_1 x) - \widetilde{V}_0 x = \widetilde{V}_0 + (\widetilde{V}_1 - \widetilde{V}_0)x.
\end{aligned}$$

Note that we used the recurrence relation $\widetilde{V}_n = \widetilde{V}_{n-1} + 2\widetilde{V}_{n-2}$. Rearranging above equation, we get

$$g(x) = \frac{\widetilde{V}_0 + (\widetilde{V}_1 - \widetilde{V}_0)x}{1 - x - 2x^2}.$$

□

As special cases, the generating functions for the hyperbolic Jacobsthal and hyperbolic Jacobsthal-Lucas numbers are

$$\sum_{n=0}^{\infty} \widetilde{J}_n x^n = \frac{h + x}{1 - x - 2x^2}$$

and

$$\sum_{n=0}^{\infty} \widetilde{K}_n x^n = \frac{(2 + h) + (-1 + 4h)x}{1 - x - 2x^2}$$

respectively.

3 Obtaining Binet's Formula From Generating Function

We next find Binet's formula of hyperbolic generalized Jacobsthal number $\{\widetilde{V}_n\}$ by the use of generating function for $\widetilde{V}_n$.

Theorem 3.1. (Binet's formula of hyperbolic generalized Jacobsthal numbers)

$$\widetilde{V}_n = \frac{d_1\alpha^n}{3} - \frac{d_2\beta^n}{3} \tag{3.1}$$

where

$$\begin{aligned} d_1 &= \widetilde{V}_0\alpha + (\widetilde{V}_1 - \widetilde{V}_0), \\ d_2 &= \widetilde{V}_0\beta + (\widetilde{V}_1 - \widetilde{V}_0). \end{aligned}$$

Proof . Let

$$h(x) = 1 - x - 2x^2.$$

Then for $\alpha = 2$ and $\beta = -1$ we write

$$h(x) = (1-\alpha x)(1-\beta x)$$

i.e.,

$$1 - x - 2x^2 = (1-\alpha x)(1-\beta x) \tag{3.2}$$

Hence $\frac{1}{\alpha} = \frac{1}{2}$ and $\frac{1}{\beta} = -1$ are the roots of $h(x)$. This gives α and β as the roots of

$$h(\frac{1}{x}) = 1 - \frac{1}{x} - \frac{2}{x^2} = 0.$$

This implies $x^2 - x - 2 = 0$. Now, by (2.6) and (3.2), it follows that

$$\sum_{n=0}^{\infty} \widetilde{V}_n x^n = \frac{\widetilde{V}_0 + (\widetilde{V}_1 - \widetilde{V}_0)x}{(1-\alpha x)(1-\beta x)}.$$

Then we write

$$\frac{\widetilde{V}_0 + (\widetilde{V}_1 - \widetilde{V}_0)x}{(1-\alpha x)(1-\beta x)} = \frac{A_1}{(1-\alpha x)} + \frac{A_2}{(1-\beta x)}. \tag{3.3}$$

So

$$\widetilde{V}_0 + (\widetilde{V}_1 - \widetilde{V}_0)x = A_1(1-\beta x) + A_2(1-\alpha x).$$

If we consider $x = \frac{1}{\alpha}$, we get $\widetilde{V}_0 + (\widetilde{V}_1 - \widetilde{V}_0)\frac{1}{\alpha} = A_1(1-\beta\frac{1}{\alpha})$. This gives

$$A_1 = \frac{\widetilde{V}_0\alpha + (\widetilde{V}_1 - \widetilde{V}_0)}{(\alpha-\beta)} = \frac{d_1}{3}.$$

Similarly, we obtain

$$\widetilde{V}_0 + (\widetilde{V}_1 - \widetilde{V}_0)\frac{1}{\beta} = A_2(1-\alpha\frac{1}{\beta}) \Rightarrow \widetilde{V}_0\beta + (\widetilde{V}_1 - \widetilde{V}_0) = A_2(\beta-\alpha)$$

and so

$$A_2 = -\frac{\widetilde{V}_0\beta + (\widetilde{V}_1 - \widetilde{V}_0)}{(\alpha-\beta)} = -\frac{d_2}{3}.$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} \widetilde{V}_n x^n = A_1(1-\alpha x)^{-1} + A_2(1-\beta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} \widetilde{V}_n x^n = A_1\sum_{n=0}^{\infty} \alpha^n x^n + A_2\sum_{n=0}^{\infty} \beta^n x^n = \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$\widetilde{V}_n = A_1\alpha^n + A_2\beta^n$$

and then we get (3.1). □

Note that from (2.3) and (3.1) we have

$$\begin{aligned}(V_1 - V_0\beta)\widetilde{\alpha} &= \widetilde{V}_0\alpha + (\widetilde{V}_1 - \widetilde{V}_0),\\ (V_1 - V_0\alpha)\widetilde{\beta} &= \widetilde{V}_0\beta + (\widetilde{V}_1 - \widetilde{V}_0),\end{aligned}$$

i.e.,

$$\begin{aligned}(V_1 + V_0)\widetilde{\alpha} &= 2\widetilde{V}_0 + (\widetilde{V}_1 - \widetilde{V}_0),\\ (V_1 - 2V_0)\widetilde{\beta} &= -\widetilde{V}_0 + (\widetilde{V}_1 - \widetilde{V}_0).\end{aligned}$$

Next, using Theorem 3.1, we present the Binet formulas of hyperbolic Jacobsthal and hyperbolic Jacobsthal-Lucas numbers.

Corollary 3.2. Binet's formulas of hyperbolic Jacobsthal and hyperbolic Jacobsthal-Lucas numbers are

$$\widetilde{J}_n = \frac{\widetilde{\alpha}2^n - \widetilde{\beta}(-1)^n}{3}$$

and

$$\widetilde{K}_n = \widetilde{\alpha}2^n + \widetilde{\beta}(-1)^n$$

respectively.

4 Some Identities

We now present a few special identities for the hyperbolic generalized Jacobsthal sequence $\{\widetilde{V}_n\}$. The following theorem presents the Catalan identity for the hyperbolic generalized Jacobsthal numbers.

Theorem 4.1. (Catalan identity) For all integers n and m, the following identity holds

$$\widetilde{V}_{n+m}\widetilde{V}_{n-m} - \widetilde{V}_n^2 = \frac{1}{9}AB2^{n-m}(-1)^{n-m+1}((-1)^m - 2^m)^2(-1+h).$$

Proof . Using the Binet formula (2.3)

$$\widetilde{V}_n = \frac{A\widetilde{\alpha}2^n - B\widetilde{\beta}(-1)^n}{3} \tag{4.1}$$

where

$$\begin{aligned}A &= V_1 + V_0,\\ B &= V_1 - 2V_0,\end{aligned}$$

we get

$$\begin{aligned}\widetilde{V}_{n+m}\widetilde{V}_{n-m} - \widetilde{V}_n^2 &= \frac{A\widetilde{\alpha}2^{n+m} - B\widetilde{\beta}(-1)^{n+m}}{3}\frac{A\widetilde{\alpha}2^{n-m} - B\widetilde{\beta}(-1)^{n-m}}{3}\\ &\quad -\left(\frac{A\widetilde{\alpha}2^n - B\widetilde{\beta}(-1)^n}{3}\right)^2\\ &= \frac{1}{9}AB2^{n-m}(-1)^{n-m+1}((-1)^m - 2^m)^2\widetilde{\alpha}\widetilde{\beta}\\ &= \frac{1}{9}AB2^{n-m}(-1)^{n-m+1}((-1)^m - 2^m)^2(-1+h).\end{aligned}$$

□

As special cases of the above theorem, we give Catalan's identity of hyperbolic Jacobsthal and hyperbolic Jacobsthal-Lucas numbers. Firstly, we present Catalan's identity of hyperbolic Jacobsthal numbers.

Corollary 4.2. (Catalan's identity for the hyperbolic Jacobsthal numbers) For all integers n and m, the following identity holds

$$\widetilde{J}_{n+m}\widetilde{J}_{n-m}-\widetilde{J}_{n}^{2}=\frac{1}{9}2^{n-m}\left(-1\right)^{n-m+1}\left(\left(-1\right)^{m}-2^{m}\right)^{2}\left(-1+h\right).$$

Proof . Taking $V_n = J_n$ in Theorem 4.1 we get the required result. □

Secondly, we give Catalan's identity of hyperbolic Jacobsthal-Lucas numbers.

Corollary 4.3. (Catalan's identity for the hyperbolic Jacobsthal-Lucas numbers) For all integers n and m, the following identity holds

$$\widetilde{K}_{n+m}\widetilde{K}_{n-m}-\widetilde{K}_{n}^{2}=2^{n-m}\left(-1\right)^{n-m}\left(\left(-1\right)^{m}-2^{m}\right)^{2}\left(-1+h\right).$$

Proof . Taking $V_n = K_n$ in Theorem 4.1, we get the required result. □

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the hyperbolic generalized Jacobsthal sequence.

Corollary 4.4. (Cassini's identity) For all integers n, the following identity holds

$$\widetilde{V}_{n+1}\widetilde{V}_{n-1}-\widetilde{V}_{n}^{2}=AB2^{n-1}\left(-1\right)^{n}\left(-1+h\right).$$

As special cases of Cassini's identity, we give Cassini's identity of hyperbolic Jacobsthal and hyperbolic Jacobsthal-Lucas numbers. Firstly, we present Cassini's identity of hyperbolic Jacobsthal numbers.

Corollary 4.5. (Cassini's identity of hyperbolic Jacobsthal numbers) For all integers n, the following identity holds

$$\widetilde{J}_{n+1}\widetilde{J}_{n-1}-\widetilde{J}_{n}^{2}=2^{n-1}\left(-1\right)^{n}\left(-1+h\right)$$

Secondly, we give Cassini's identity of hyperbolic Jacobsthal-Lucas numbers.

Corollary 4.6. (Cassini's identity of hyperbolic Jacobsthal-Lucas numbers) For all integers n, the following identity holds

$$\widetilde{K}_{n+1}\widetilde{K}_{n-1}-\widetilde{K}_{n}^{2}=9\times2^{n-1}\left(-1\right)^{n+1}\left(-1+h\right).$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using the Binet Formula of the hyperbolic generalized Jacobsthal sequence:

$$\widetilde{V}_{n}=\frac{A\widetilde{\alpha}2^{n}-B\widetilde{\beta}(-1)^{n}}{3}.$$

The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of the hyperbolic generalized Jacobsthal sequence $\{\widetilde{V}_n\}$.

Theorem 4.7. Let n and m be any integers. Then the following identities are true:

(a) (d'Ocagne's identity)

$$\widetilde{V}_{m+1}\widetilde{V}_{n}-\widetilde{V}_{m}\widetilde{V}_{n+1}=\frac{1}{3}AB\left(\left(-1\right)^{m}2^{n}-\left(-1\right)^{n}2^{m}\right)\left(-1+h\right).$$

(b) (Gelin-Cesàro's identity)

$$\begin{aligned}\widetilde{V}_{n+2}\widetilde{V}_{n+1}\widetilde{V}_{n-1}\widetilde{V}_{n-2}-\widetilde{V}_{n}^{4} &= \frac{1}{36}AB2^{n}\left(-1\right)^{n}(2^{2n}A^{2}+4\left(-1\right)^{2n}B^{2}\\ &\quad +13\left(-1\right)^{n}2^{n}AB)\left(-1+h\right).\end{aligned}$$

(c) (Melham's identity)

$$\widetilde{V}_{n+1}\widetilde{V}_{n+2}\widetilde{V}_{n+6}-\widetilde{V}_{n+3}^{3}=\frac{1}{3}AB\left(-1\right)^{n+1}2^{n+2}\left(2\times 2^{n}A-5\left(-1\right)^{n}B\right)\left(1-h\right).$$

Proof .

(a) Using the Binet formula, we get

$$\begin{aligned}
\widetilde{V}_{m+1}\widetilde{V}_{n}-\widetilde{V}_{m}\widetilde{V}_{n+1} &= \frac{A\widetilde{\alpha}2^{m+1}-B\widetilde{\beta}(-1)^{m+1}}{3}\frac{A\widetilde{\alpha}2^{n}-B\widetilde{\beta}(-1)^{n}}{3} \\
&\quad -\frac{A\widetilde{\alpha}2^{m}-B\widetilde{\beta}(-1)^{m}}{3}\frac{A\widetilde{\alpha}2^{n+1}-B\widetilde{\beta}(-1)^{n+1}}{3} \\
&= \frac{1}{3}AB\left(\left(-1\right)^{m}2^{n}-\left(-1\right)^{n}2^{m}\right)(-1+h).
\end{aligned}$$

(b) Using the identities

$$\begin{aligned}
\widetilde{\alpha}^{3}\widetilde{\beta} &= (-1+h) \\
\widetilde{\alpha}\widetilde{\beta}^{3} &= (-4+4h) \\
\widetilde{\alpha}^{2}\widetilde{\beta}^{2} &= (2-2h)
\end{aligned}$$

we obtain

$$\begin{aligned}
\widetilde{V}_{n+2}\widetilde{V}_{n+1}\widetilde{V}_{n-1}\widetilde{V}_{n-2}-\widetilde{V}_{n}^{4} &= \frac{A\widetilde{\alpha}2^{n+2}-B\widetilde{\beta}(-1)^{n+2}}{3}\frac{A\widetilde{\alpha}2^{n+1}-B\widetilde{\beta}(-1)^{n+1}}{3} \\
&\quad \frac{A\widetilde{\alpha}2^{n-1}-B\widetilde{\beta}(-1)^{n-1}}{3}\frac{A\widetilde{\alpha}2^{n-2}-B\widetilde{\beta}(-1)^{n-2}}{3} \\
&\quad -\left(\frac{A\widetilde{\alpha}2^{n}-B\widetilde{\beta}(-1)^{n}}{3}\right)^{4} \\
&= \frac{1}{72}AB2^{n}\left(-1\right)^{n}(2^{2n+1}A^{2}\widetilde{\alpha}^{3}\widetilde{\beta} \\
&\quad +2\left(-1\right)^{2n}B^{2}\widetilde{\alpha}\widetilde{\beta}^{3}-13\left(-1\right)^{n}2^{n}AB\widetilde{\alpha}^{2}\widetilde{\beta}^{2}) \\
&= \frac{1}{36}AB2^{n}\left(-1\right)^{n}(2^{2n}A^{2}(-1+h)+4\left(-1\right)^{2n}B^{2}(-1+h) \\
&\quad -13\left(-1\right)^{n}2^{n}AB(1-h)) \\
&= \frac{1}{36}AB2^{n}\left(-1\right)^{n}(2^{2n}A^{2}+4\left(-1\right)^{2n}B^{2}+13\left(-1\right)^{n}2^{n}AB)\left(-1+h\right).
\end{aligned}$$

(c) Taking account of the identity

$$\begin{aligned}
\widetilde{\alpha}^{2}\widetilde{\beta} &= 1-h, \\
\widetilde{\alpha}\widetilde{\beta}^{2} &= -2+2h,
\end{aligned}$$

and Binet formula of $\widetilde{V}_{n}$, we get

$$\begin{aligned}
\widetilde{V}_{n+1}\widetilde{V}_{n+2}\widetilde{V}_{n+6}-\widetilde{V}_{n+3}^{3} &= \frac{A\widetilde{\alpha}2^{n+1}-B\widetilde{\beta}(-1)^{n+1}}{3}\frac{A\widetilde{\alpha}2^{n+2}-B\widetilde{\beta}(-1)^{n+2}}{3} \\
&\quad \frac{A\widetilde{\alpha}2^{n+6}-B\widetilde{\beta}(-1)^{n+6}}{3}-\left(\frac{A\widetilde{\alpha}2^{n+3}-B\widetilde{\beta}(-1)^{n+3}}{3}\right)^{3} \\
&= -\frac{1}{3}AB\left(-1\right)^{n}2^{n+1}\left(2^{n+2}A\widetilde{\alpha}^{2}\widetilde{\beta}+5\left(-1\right)^{n}B\widetilde{\alpha}\widetilde{\beta}^{2}\right) \\
&= \frac{1}{3}AB\left(-1\right)^{n+1}2^{n+2}(2^{n+1}A(1-h)+5\left(-1\right)^{n}B(-1+h)) \\
&= \frac{1}{3}AB\left(-1\right)^{n+1}2^{n+2}\left(2\times 2^{n}A-5\left(-1\right)^{n}B\right)\left(1-h\right).
\end{aligned}$$

□

As special cases of the above theorem, we give the d'Ocagne's, Gelin-Cesàro's and Melham' identities of hyperbolic Jacobsthal and hyperbolic Jaco-bsthal-Lucas numbers. Firstly, we present the d'Ocagne's, Gelin-Cesàro's and Melham' identities of hyperbolic Jacobsthal numbers.

Corollary 4.8. Let n and m be any integers. Then, for the hyperbolic Jacobsthal numbers, the following identities are true:

(a) (d'Ocagne's identity)

$$\widetilde{J}_{m+1}\widetilde{J}_n - \widetilde{J}_m\widetilde{J}_{n+1} = \frac{1}{3}\left((-1)^m 2^n - (-1)^n 2^m\right)(-1+h).$$

(b) (Gelin-Cesàro's identity)

$$\widetilde{J}_{n+2}\widetilde{J}_{n+1}\widetilde{J}_{n-1}\widetilde{J}_{n-2} - \widetilde{J}_n^4 = \frac{1}{36}2^n(-1)^n\left(4(-1)^{2n} + 2^{2n} + 13(-1)^n 2^n\right)(-1+h).$$

(c) (Melham's identity)

$$\widetilde{J}_{n+1}\widetilde{J}_{n+2}\widetilde{J}_{n+6} - \widetilde{J}_{n+3}^3 = \frac{1}{3}(-1)^{n+1}2^{n+2}(2^{n+1} - 5(-1)^n)(1-h).$$

Secondly, we present the d'Ocagne's, Gelin-Cesàro's and Melham' identities of hyperbolic Jacobsthal-Lucas numbers.

Corollary 4.9. Let n and m be any integers. Then, for the hyperbolic Jacobsthal-Lucas numbers, the following identities are true:

(a) (d'Ocagne's identity)

$$\widetilde{K}_{m+1}\widetilde{K}_n - \widetilde{K}_m\widetilde{K}_{n+1} = 3\left((-1)^n 2^m - (-1)^m 2^n\right)(-1+h).$$

(b) (Gelin-Cesàro's identity)

$$\widetilde{K}_{n+2}\widetilde{K}_{n+1}\widetilde{K}_{n-1}\widetilde{K}_{n-2} - \widetilde{K}_n^4 \quad = \quad 9\times 2^{n-2}(-1)^{n+1}\left(4(-1)^{2n} + 2^{2n} - 13(-1)^n 2^n\right)(-1+h).$$

(c) (Melham's identity)

$$\widetilde{K}_{n+1}\widetilde{K}_{n+2}\widetilde{K}_{n+6} - \widetilde{K}_{n+3}^3 = 9(-1)^n 2^{n+2}\left(5(-1)^n + 2\times 2^n\right)(1-h).$$

5 Linear Sums

In this section, we give the summation formulas of the hyperbolic generalized Jacobsthal numbers with positive and negative subscripts. Now, we present the summation formulas of the generalized Jacobsthal numbers.

Proposition 5.1. For the generalized Jacobsthal numbers, for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} V_k = \frac{1}{2}(V_{n+2} - V_1)$.

(b) $\sum_{k=0}^{n} V_{2k} = \frac{1}{3}(2V_{2n+2} - 2V_{2n+1} - V_0 + (-V_1 + 2V_0)n)$.

(c) $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{6}(-V_{2n+2} + 10V_{2n+1} - 3V_1 + 2V_0 + (2V_1 - 4V_0)n)$.

Proof . For the proof, see Soykan [34]. □

Note that we can write (c) of the above proposition as

$$\sum_{k=0}^{n} V_{2k+1} = \frac{1}{3}(2V_{2n+3} - 2V_{2n+2} - V_1 + (V_1 - 2V_0)n)$$

by using (a) and (b) of Proposition 5.1 and the identities

$$\begin{aligned}
\sum_{k=0}^{n} V_{2k+1} &= \sum_{k=0}^{2n+1} V_k - \sum_{k=0}^{n} V_{2k}, \\
V_{2n+3} - 2V_{2n+2} &= V_1 - 2V_0.
\end{aligned}$$

Next, we present the formulas which give the summation of the first n dual hyperbolic generalized Jacobsthal numbers.

Theorem 5.2. For $n \geq 0$, dual hyperbolic generalized Jacobsthal numbers have the following formulas:.

(a) $\sum_{k=0}^{n} \widetilde{V}_k = \frac{1}{2}(\widetilde{V}_{n+2} - \widetilde{V}_1)$.

(b) $\sum_{k=0}^{n} \widetilde{V}_{2k} = \frac{1}{3}(2\widetilde{V}_{2n+2} - 2\widetilde{V}_{2n+1} - \widetilde{V}_0 + (-\widetilde{V}_1 + 2\widetilde{V}_0)n)$.

(c) $\sum_{k=0}^{n} \widetilde{V}_{2k+1} = \frac{1}{3}(2\widetilde{V}_{2n+3} - 2\widetilde{V}_{2n+2} - \widetilde{V}_1 + (\widetilde{V}_1 - 2\widetilde{V}_0)n)$.

Proof . Note that using Proposition 5.1 (a) we get

$$\sum_{k=0}^{n} V_{k+1} = \frac{1}{2}(V_{n+3} - V_1 - 2V_0).$$

Then it follows that

$$\begin{aligned}
\sum_{k=0}^{n} \widetilde{V}_k &= \sum_{k=0}^{n} V_k + h\sum_{k=0}^{n} V_{k+1} \\
&= \frac{1}{2}(V_{n+2} - V_1) + h\frac{1}{2}(V_{n+3} - V_1 - 2V_0) \\
&= \frac{1}{2}(\widetilde{V}_{n+2} - (V_1 + hV_2)) \\
&= \frac{1}{2}(\widetilde{V}_{n+2} - \widetilde{V}_1).
\end{aligned}$$

This proves (a).

(b) Note that using Proposition 5.1 (b) and (c) we get

$$\begin{aligned}
\sum_{k=0}^{n} \widetilde{V}_{2k} &= \sum_{k=0}^{n} V_{2k} + h\sum_{k=0}^{n} V_{2k+1} \\
&= \frac{1}{3}(2V_{2n+2} - 2V_{2n+1} - V_0 + (-V_1 + 2V_0)n) + h\ \frac{1}{3}(2V_{2n+3} - 2V_{2n+2} - V_1 + (V_1 - 2V_0)n) \\
&= \frac{1}{3}(2(V_{2n+2} + hV_{2n+3}) - 2(V_{2n+1} + hV_{2n+2}) - (V_0 + hV_1) + ((-V_1 + 2V_0) + h(V_1 - 2V_0))n) \\
&= \frac{1}{3}(2\widetilde{V}_{2n+2} - 2\widetilde{V}_{2n+1} - \widetilde{V}_0 + ((-V_1 + 2V_0) + h(-V_2 + 2V_1))n) \\
&= \frac{1}{3}(2\widetilde{V}_{2n+2} - 2\widetilde{V}_{2n+1} - \widetilde{V}_0 + (-(V_1 + hV_2) + 2(V_0 + hV_1)n) \\
&= \frac{1}{3}(2\widetilde{V}_{2n+2} - 2\widetilde{V}_{2n+1} - \widetilde{V}_0 + (-\widetilde{V}_1 + 2\widetilde{V}_0)n).
\end{aligned}$$

(c) Note that using Proposition 5.1 (b) and (c) we get

$$\sum_{k=0}^{n} V_{2k+2} = \frac{1}{3}(2V_{2n+4} - 2V_{2n+3} - V_1 - 2V_0 + (-V_1 + 2V_0)n).$$

Then it follows that

$$
\begin{aligned}
\sum_{k=0}^{n}\widetilde{V}_{2k+1} &= \sum_{k=0}^{n}V_{2k+1}+h\sum_{k=0}^{n}V_{2k+2}\\
&= \frac{1}{3}(2V_{2n+3}-2V_{2n+2}-V_1+(V_1-2V_0)n)+h\frac{1}{3}(2V_{2n+4}-2V_{2n+3}-V_1-2V_0+(-V_1+2V_0)n)\\
&= \frac{1}{3}(2(V_{2n+3}+hV_{2n+4})-2(V_{2n+2}+hV_{2n+3})-(V_1+h(V_1+2V_0))+((V_1-2V_0)\\
&\quad +h(-V_1+2V_0))n)\\
&= \frac{1}{3}(2\widetilde{V}_{2n+3}-2\widetilde{V}_{2n+2}-(V_1+hV_2)+((2V_1-4V_0)+h(2V_2-4V_1))n)\\
&= \frac{1}{3}(2\widetilde{V}_{2n+3}-2\widetilde{V}_{2n+2}-\widetilde{V}_1+(\widetilde{V}_1-2\widetilde{V}_0)n).
\end{aligned}
$$

□

As a first special case of the above theorem, we have the following summation formulas for dual hyperbolic Jacobsthal numbers:

Corollary 5.3. For $n \geq 0$, dual hyperbolic Jacobsthal numbers have the following properties:

(a) $\sum_{k=0}^{n}\widetilde{J}_k=\frac{1}{2}(\widetilde{J}_{n+2}-\widetilde{J}_1)$.

(b) $\sum_{k=0}^{n}\widetilde{J}_{2k}=\frac{1}{3}(2\widetilde{J}_{2n+2}-2\widetilde{J}_{2n+1}-\widetilde{J}_0+(-\widetilde{J}_1+2\widetilde{J}_0)n)$.

(c) $-\sum_{k=0}^{n}\widetilde{J}_{2k+1}=\frac{1}{3}(2\widetilde{J}_{2n+3}-2\widetilde{J}_{2n+2}-\widetilde{J}_1+(\widetilde{J}_1-2\widetilde{J}_0)n)$.

As a second special case of the above theorem, we have the following summation formulas for dual hyperbolic Jacobsthal-Lucas numbers:

Corollary 5.4. For $n \geq 0$, dual hyperbolic Jacobsthal-Lucas numbers have the following properties.

(a) $\sum_{k=0}^{n}\widetilde{K}_k=\frac{1}{2}(\widetilde{K}_{n+2}-\widetilde{K}_1)$.

(b) $\sum_{k=0}^{n}\widetilde{K}_{2k}=\frac{1}{3}(2\widetilde{K}_{2n+2}-2\widetilde{K}_{2n+1}-\widetilde{K}_0+(-\widetilde{K}_1+2\widetilde{K}_0)n)$.

(c) $\sum_{k=0}^{n}\widetilde{K}_{2k+1}=\frac{1}{3}(2\widetilde{K}_{2n+3}-2\widetilde{K}_{2n+2}-\widetilde{K}_1+(\widetilde{K}_1-2\widetilde{K}_0)n)$.

Now, we present the formula which give the summation formulas of the generalized Jacobsthal numbers with negative subscripts.

Proposition 5.5. For $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n}V_{-k}=\frac{1}{2}(-3V_{-n-1}-2V_{-n-2}+V_1)$.

(b) $\sum_{k=1}^{n}V_{-2k}=\frac{1}{3}(2V_{-2n}-6V_{-2n-1}+(3V_1-5V_0)+(-V_1+2V_0)n)$.

(c) $\sum_{k=1}^{n}V_{-2k+1}=\frac{1}{3}(2V_{-2n+1}-6V_{-2n}+(-2V_1+6V_0)+(V_1-2V_0)n)$.

Proof . This is given in Soykan [35]. See also Soykan [34]. □

Next, we present the formulas which give the summation of the first n dual hyperbolic generalized Jacobsthal numbers with negative subscripts

Theorem 5.6. For $n \geq 1$, dual hyperbolic generalized Jacobsthal numbers have the following formulas:

(a) $\sum_{k=1}^{n}\widetilde{V}_{-k}=\frac{1}{2}(-3\widetilde{V}_{-n-1}-2\widetilde{V}_{-n-2}+\widetilde{V}_1)$.

(b) $\sum_{k=1}^{n}\widetilde{V}_{-2k}=\frac{1}{3}(2\widetilde{V}_{-2n}-6\widetilde{V}_{-2n-1}+(3\widetilde{V}_{1}-5\widetilde{V}_{0})+(-\widetilde{V}_{1}+2\widetilde{V}_{0})n)$.

(c) $\sum_{k=1}^{n}\widetilde{V}_{-2k+1}=\frac{1}{3}(2\widetilde{V}_{-2n+1}-6\widetilde{V}_{-2n}+(-2\widetilde{V}_{1}+6\widetilde{V}_{0})+(\widetilde{V}_{1}-2\widetilde{V}_{0})n)$.

Proof . We prove (a). (b) and (c) can be proved similarly. Note that using Proposition 5.1 (a) we get

$$\sum_{k=1}^{n}V_{-k+1}=\frac{1}{2}(-3V_{-n}-2V_{-n-1}+V_{1}+2V_{0}).$$

Then it follows that

$$\begin{aligned}
\sum_{k=1}^{n}\widetilde{V}_{-k} &= \sum_{k=1}^{n}V_{-k}+h\sum_{k=1}^{n}V_{-k+1}\\
&= \frac{1}{2}(-3V_{-n-1}-2V_{-n-2}+V_{1})+h\frac{1}{2}(-3V_{-n}-2V_{-n-1}+V_{1}+2V_{0})\\
&= \frac{1}{2}(3(V_{-n-1}+hV_{-n})-(V_{-n-2}+hV_{-n-1})+V_{1}+h(V_{1}+2V_{0}))\\
&= \frac{1}{2}(-3\widetilde{V}_{-n-1}-2\widetilde{V}_{-n-2}+(V_{1}+hV_{2})\\
&= \frac{1}{2}(-3\widetilde{V}_{-n-1}-2\widetilde{V}_{-n-2}+\widetilde{V}_{1})
\end{aligned}$$

This proves (a).

(b) Note that using Proposition 5.1 (b) and (c) we get

$$\begin{aligned}
\sum_{k=1}^{n}\widetilde{V}_{-2k} &= \sum_{k=1}^{n}V_{-2k}+h\sum_{k=1}^{n}V_{-2k+1}\\
&= \frac{1}{3}(2V_{-2n}-6V_{-2n-1}+(3V_{1}-5V_{0})+(-V_{1}+2V_{0})n)\\
&\quad +h\frac{1}{3}(2V_{-2n+1}-6V_{-2n}+(-2V_{1}+6V_{0})+(V_{1}-2V_{0})n)\\
&= \frac{1}{3}(2(V_{-2n}+hV_{-2n+1})-6(V_{-2n-1}+hV_{-2n})+((3V_{1}-5V_{0})\\
&\quad +h(-2V_{1}+6V_{0}))+((-V_{1}+2V_{0})+h(V_{1}-2V_{0}))n)\\
&= \frac{1}{3}(2\widetilde{V}_{-2n}-6\widetilde{V}_{-2n-1}+(3\widetilde{V}_{1}-5\widetilde{V}_{0})+(-\widetilde{V}_{1}+2\widetilde{V}_{0})n)
\end{aligned}$$

(c) Note that using Proposition 5.1 (b) and (c) we get

$$\sum_{k=1}^{n}V_{-2k+2}=\frac{1}{3}(2V_{-2n+2}-6V_{-2n+1}+(4V_{1}-4V_{0})+(-V_{1}+2V_{0})n).$$

Then it follows that

$$\begin{aligned}
\sum_{k=0}^{n}\widetilde{V}_{-2k+1} &= \sum_{k=0}^{n}V_{-2k+1}+h\sum_{k=0}^{n}V_{-2k+2}\\
&= \frac{1}{3}(2V_{-2n+1}-6V_{-2n}+(-2V_{1}+6V_{0})+(V_{1}-2V_{0})n)\\
&\quad +h\frac{1}{3}(2V_{-2n+2}-6V_{-2n+1}+(4V_{1}-4V_{0})+(-V_{1}+2V_{0})n))\\
&= \frac{1}{3}(2(V_{-2n+1}+hV_{-2n+2})-6(V_{-2n}+hV_{-2n+1})+((6V_{0}-2V_{1})\\
&\quad +h(4V_{1}-4V_{0}))+((V_{1}-2V_{0})+h(-V_{1}+2V_{0}))n)\\
&= \frac{1}{3}(2\widetilde{V}_{-2n+1}-6\widetilde{V}_{-2n}+(-2\widetilde{V}_{1}+6\widetilde{V}_{0})+(\widetilde{V}_{1}-2\widetilde{V}_{0})n).
\end{aligned}$$

□

As a first special case of above theorem, we have the following summation formulas for dual hyperbolic Jacobsthal numbers:

Corollary 5.7. For $n \geq 1$, dual hyperbolic Jacobsthal numbers have the following properties:

(a) $\sum_{k=1}^{n} \widetilde{J}_{-k} = \frac{1}{2}(-3\widetilde{J}_{-n-1} - 2\widetilde{J}_{-n-2} + \widetilde{J}_1)$.

(b) $\sum_{k=1}^{n} \widetilde{J}_{-2k} = \frac{1}{3}(2\widetilde{J}_{-2n} - 6\widetilde{J}_{-2n-1} + (3\widetilde{J}_1 - 5\widetilde{J}_0) + (-\widetilde{J}_1 + 2\widetilde{J}_0)n)$.

(c) $\sum_{k=1}^{n} \widetilde{J}_{-2k+1} = \frac{1}{3}(2\widetilde{J}_{-2n+1} - 6\widetilde{J}_{-2n} + (-2\widetilde{J}_1 + 6\widetilde{J}_0) + (\widetilde{J}_1 - 2\widetilde{J}_0)n)$.

As a second special case of above theorem, we have the following summation formulas for dual hyperbolic Jacobsthal-Lucas numbers:

Corollary 5.8. For $n \geq 1$, dual hyperbolic Jacobsthal-Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n} \widetilde{K}_{-k} = \frac{1}{2}(-3\widetilde{K}_{-n-1} - 2\widetilde{K}_{-n-2} + \widetilde{K}_1)$.

(b) $\sum_{k=1}^{n} \widetilde{K}_{-2k} = \frac{1}{3}(2\widetilde{K}_{-2n} - 6\widetilde{K}_{-2n-1} + (3\widetilde{K}_1 - 5\widetilde{K}_0) + (-\widetilde{K}_1 + 2\widetilde{K}_0)n)$.

(c) $\sum_{k=1}^{n} \widetilde{K}_{-2k+1} = \frac{1}{3}(2\widetilde{K}_{-2n+1} - 6\widetilde{K}_{-2n} + (-2\widetilde{K}_1 + 6\widetilde{K}_0) + (\widetilde{K}_1 - 2\widetilde{K}_0)n)$.

6 Matrices related with Hyperbolic Generalized Jacobsthal Numbers

We define the square matrix D of order 2 as:

$$D = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

such that $\det D = -2$. Induction proof may be used to establish

$$D^n = \begin{pmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{pmatrix} \tag{6.1}$$

and (the matrix formulation of V_n)

$$\begin{pmatrix} V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_1 \\ V_0 \end{pmatrix}. \tag{6.2}$$

Now, we define the matrix D_V as

$$D_V = \begin{pmatrix} \widetilde{V}_3 & 2\widetilde{V}_2 \\ \widetilde{V}_2 & 2\widetilde{V}_1 \end{pmatrix}.$$

This matrice D_V is called hyperbolic generalized Jacobsthal matrix. As special cases, hyperbolic Jacobsthal matrix and hyperbolic Jacobsthal-Lucas matrix are

$$D_J = \begin{pmatrix} \widetilde{J}_3 & 2\widetilde{J}_2 \\ \widetilde{J}_2 & 2\widetilde{J}_1 \end{pmatrix}$$

and

$$D_K = \begin{pmatrix} \widetilde{K}_3 & 2\widetilde{K}_2 \\ \widetilde{K}_2 & 2\widetilde{K}_1 \end{pmatrix}$$

respectively.

Theorem 6.1. For $n \geq 0$, the following is valid:

$$D_V^n \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{V}_{n+3} & 2\widetilde{V}_{n+2} \\ \widetilde{V}_{n+2} & 2\widetilde{V}_{n+1} \end{pmatrix}. \tag{6.3}$$

Proof . We prove by mathematical induction on n. If $n = 0$, then the result is clear. Now, we assume it is true for $n = k$, that is

$$D_V D^k = \begin{pmatrix} \widetilde{V}_{k+3} & 2\widetilde{V}_{k+2} \\ \widetilde{V}_{k+2} & 2\widetilde{V}_{k+1} \end{pmatrix}.$$

If we use (2.1), then we have $\widetilde{V}_{k+2} = \widetilde{V}_{k+1} + 2\widetilde{V}_k$. Then, by induction hypothesis, we obtain

$$\begin{aligned} D_V D^{k+1} &= (D_V D^k)D = \begin{pmatrix} \widetilde{V}_{k+3} & \widetilde{V}_{k+2} \\ \widetilde{V}_{k+2} & \widetilde{V}_{k+1} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{V}_{k+3} + 2\widetilde{V}_{k+2} & 2\widetilde{V}_{k+3} \\ \widetilde{V}_{k+2} + 2\widetilde{V}_{k+1} & 2\widetilde{V}_{k+2} \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{V}_{k+4} & 2\widetilde{V}_{k+3} \\ \widetilde{V}_{k+3} & 2\widetilde{V}_{k+2} \end{pmatrix}. \end{aligned}$$

Thus, (6.3) holds for all non-negative integers n.□

Remark 6.2. The above theorem is true for $n \leq -1$. It can also be proved by induction.

Corollary 6.3. For all integers n, the following holds:

$$\widetilde{V}_{n+2} = \widetilde{V}_2 J_{n+1} + 2\widetilde{V}_1 J_n.$$

Proof . The proof can be seen by the coefficient of the matrix D_V and (6.1). □

Taking $V_n = J_n$ and $V_n = K_n$, respectively, in the above corollary, we obtain the following results.

Corollary 6.4. For all integers n, the followings are true:

(a) $\widetilde{J}_{n+2} = \widetilde{J}_2 J_{n+1} + 2\widetilde{J}_1 J_n$.

(b) $\widetilde{K}_{n+2} = \widetilde{K}_2 J_{n+1} + 2\widetilde{K}_1 J_n$.

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