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Geometry of submanifolds of all classes of third-order ODEs as a Riemannian manifold

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Abstract

In this paper, we prove that any surface corresponding to linear second-order ODEs as a submanifold is minimal in all classes of third-order ODEs $y''' = f(x, y, p, q)$ as a Riemannian manifold where $y' = p$ and $y'' = q$, if and only if $q_{yy} = 0$. Moreover, we will see the linear second-order ODE with general form $y'' = \pm y + \beta(x)$ is the only case that is defined a minimal surface and is also totally geodesic.

Keywords: Levi-Civita connection, minimal surface, moving frame, Riemannian manifold, Riemann curvature tensor, totally geodesic
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1 Introduction

Riemannian geometry is characterized, and research is oriented towards and shaped by concepts for examples geodesics connections, curvature [12]. Originally, the geometry of submanifolds was only a part of Riemannian geometry but today it is one of several independent aspects of multi-dimensional generalizations of the classical theory of surfaces. While Riemannian geometry is the development of Gauss' idea on intrinsic geometry, the geometry of submanifolds starts from the idea of the extrinsic geometry of a surface. This theory is devoted to the study of the position and properties of a submanifold in ambient space, both in their local and global aspects [2, 3, 19].

On the other hand, the general equivalence problem is about studding when two geometrical objects are mapped on each other by a certain class of diffeomorphisms. Élie Cartan developed the general equivalence problem and provided a systematic procedure for determining the necessary and sufficient conditions. There are many papers as applications of Cartan method for third-order ODEs [11, 13, 18] and fourth-order differential operators [4]. The theory of moving frames is most closely associated with the name of Cartan, that is a powerful and algorithmic tool for studying the geometric properties of submanifolds and their invariants under the action of a transformation group [16]. Now if we apply the moving frame method for Riemannian manifold and geometry of submanifolds, we can obtain very interesting results. Recently, a lot of research has been carried out about the minimal surfaces in a three-dimensional Riemannian manifold [9, 10, 17].

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The graph of the function $u(x,y)$ is minimal if and only if u satisfies:

$$(1+u_x^2)\,u_{yy} - 2u_xu_yu_{xy} + (1+u_y^2)\,u_{xx} = 0. \tag{1.1}$$

The PDE (1.1) which is non-linear but depends linearly on the second derivatives is called quasi-linear. We will not discuss the general theory of existence of solutions to this equation, but rather describe a couple of special solutions. Of course, a function $u(x,y)$ with vanishing second derivatives is solution of Equation (1.1). Although these are not very exciting solutions, the resulting surfaces $\mathcal{S}$ being planes, they are important because of the Calabi theorem [3]. According to this theorem, if u be a solution of Equation (1.1), on $U=\mathbb{R}^2$ such that $\|\nabla u\|_0 < 1$, (hence $\mathcal{S}$ is called spacelike) then u is an affine function that means u is a function with vanishing second derivatives. There is an analogous theorem (which actually predates Calabi's) in the case of the Riemannian metric $\langle .,.\rangle_0 = dx_1^2+dx_2^2+dx_3^2$ and is known as the Bernstein theorem.

It is a classical result that any simply connected minimal surface in Euclidean space $\mathbb{R}^3$ admits a one-parameter family of minimal isometric deformations, called the *associate family.* Conversely, two minimal isometric immersions of the same Riemannian surface into $\mathbb{R}^3$ are associate. These are easy consequences of the Gauss and Codazzi equations in $\mathbb{R}^3$. More generally, analogous results hold for constant mean curvature (CMC) surfaces in 3-dimensional space forms.

Benoit Daniel in [9] investigated extensions of these results and related questions for minimal surfaces in the product manifolds $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$, where $\mathbb{S}^2$ is the 2-sphere of curvature 1 and $\mathbb{H}^2$ is the hyperbolic plane of curvature -1.

The systematic study of minimal surfaces in $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$ was initiated by H. Rosenberg and W. Meeks [14, 17] and has been very active since then. The existence of an associate family for simply connected minimal surfaces in $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$ was proved in [8].

It has been shown in Bayrakdar et al. [6], the Gaussian curvature of a surface corresponding to a first-order ODE is given by certain Burgers' equations and it is possible to obtain two-dimensional spaces of constant curvature from some integrable PDEs.

The main idea for writing present paper is raised from the papers [5], written by T. Bayrakdar and A. A. Ergin. They proved that a surface corresponding to a first-order ODE is minimal in three-dimensional Riemannian manifold which is determined by the first prolongation of $y' = p(x,y)$, if and only if $p_{yy} = 0$. Hence any linear first-order ODE characterizes a minimal surface which is not necessarily totally geodesic. Z. O. Bayrakdar and T. Bayrakdar used the same idea and got good results [7] .

In present paper, we prove that any surface corresponding to linear second-order ODEs $y'' = \alpha(x)y+\beta(x)$ where α and β are two smooth functions in term of x, as a submanifold is minimal in the class of third-order ODEs corresponding to the third-order equation $y''' = \alpha(x)y' + \alpha'(x)y + \beta'(x)$. Furthermore, we will show the linear second-order ODE $y'' = \pm y + \beta(x)$ is the only case that is defined a minimal surface and is also totally geodesic.

2 Geometry of submanifolds via moving frame method

In this section we consider the method of moving frames to investigation of geometry of submanifolds. For more details we refer to [19].

Assume N be a n-dimensional Riemannian manifold equipped with metric g. Let M is a m-dimensional submanifold of N, locally imbedded in N. The submanifold of the orthogonal frame bundle over N, denoted by $F(N,M)$, which includes of adapted frames $\{\mathbf{e}_A\},\ A=1,\ldots,n$ of which $\{\mathbf{e}_i\},\ i=1,\ldots,m$ are tangent to M and $\{\mathbf{e}_\alpha\},\ \alpha=m+1,\ldots,n$ are normal to M. We have the matrix equation $\widetilde{\mathbf{E}} = \mathbf{E}K$ between two adapted frames $\widetilde{\mathbf{E}}$ and $\mathbf{E}$ where K is the matrix

$$K = \begin{bmatrix} A & \mathrm{O} \\ \mathrm{O} & B \end{bmatrix},$$

where $A \in \mathrm{O}(m)$ and $B \in \mathrm{O}(n-m)$.

The matrix of 1-forms (ω_B^A) is the Levi-Civita connection for N where we have

$$d\omega^A + \omega_B^A \wedge \omega^B = 0, \tag{2.1}$$

$$\omega_B^A + \omega_A^B = 0. \tag{2.2}$$

We can find a connection over M by a restriction ω^A_B to ω^i_j over M. Namely, when we restrict the equation

$$d\omega^A = -\sum_B \omega^A_B \wedge \omega^B, \tag{2.3}$$

to M gives

$$d\omega^i = -\sum_B \omega^i_B \wedge \omega^B = -\sum_j \omega^i_j \wedge \omega^j, \tag{2.4}$$

because $\omega^\alpha = 0$ for tangent vectors on M. Furthermore $\omega^i_j = -\omega^j_i$, so the matrix (ω^i_j) is the uniquely defined Levi-Civita connection corresponding to the metric on M induced from that on N. Takeing the exterior derivative of $\omega^\alpha = 0$, leads to

$$0 = d\omega^\alpha = -\sum_A \omega^\alpha_A \wedge \omega^A = -\sum_i \omega^\alpha_i \wedge \omega^i. \tag{2.5}$$

From Cartan's lemma it follows that

$$\omega^\alpha_i = \sum_j h^\alpha_{ij}\ \omega^i, \tag{2.6}$$

for smooth functions $h^\alpha_{ij} = h^\alpha_{ji}$.

3 Submanifolds in Riemannian manifold

Let (S, g) be a Riemannian manifold and $U \subset S$ be an open subset. For each arbitrary $p \in U$, we can define an orthonormal frame $\mathbf{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ where $\mathbf{e}_i \in T_pU$ and its corresponding dual, the coframe $\omega = \{\omega^1, \omega^2, \omega^3, \omega^4\}$ where $\omega^i \in T^*_pU$ and we have $\omega^i(\mathbf{e}_j) = \delta_{ij}$. Therefore the metric tensor is defined by

$$g = \sum_{i,j=1}^{4} \delta_{ij}\ \omega^i \otimes \omega^j. \tag{3.1}$$

By applying the exterior derivatives on these 1-forms we obtain the first structural equations

$$d\omega^i = -\sum_{j=1}^{4}\ \theta^i_j \wedge \omega^j, \qquad 1 \le i \le 4, \tag{3.2}$$

where the skew-symmetric matrix of 1-forms $\theta = (\theta^i_j)$ is called the $\mathfrak{o}(3, \mathbb{R})$-valued torsion-free connection. Now computing the exterior derivative of θ^i_j gets the second structural equations as following

$$\Omega^i_j = d\theta^i_j + \sum_{k=1}^{4}\ \theta^i_k \wedge \theta^k_j, \tag{3.3}$$

where the skew-symmetric matrix $\Omega = (\Omega^i_j)$ is called the Riemannian curvature tensor. We can rewrite the Ω^i_j with respect to coframe ω as follows

$$\Omega^i_j = \sum_{k<l} R^i_{jkl}\ \omega^k \wedge \omega^l. \tag{3.4}$$

Let $(\widetilde{S}, \widetilde{g})$ be an isometrically immersed submanifold of the surface of (S, g) by inclusion map $\sigma : \widetilde{S} \longrightarrow S$ satisfies $\sigma^* g = \widetilde{g}$ and $\widetilde{\omega}^4 = 0$. Suppose that $\widetilde{\mathbf{E}} = (\widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2, \widetilde{\mathbf{e}}_2, \mathbf{n})$ is an adapted frame on S with corresponding coframe

$\widetilde{\omega} = (\widetilde{\omega}^1, \widetilde{\omega}^2, \widetilde{\omega}^3, \widetilde{\omega}^4)$, such that $(\widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2, \widetilde{\mathbf{e}}_3)$ is tangent to $\widetilde{S}$ and $\mathbf{n}$ is normal to $\widetilde{S}$ and $\mathbf{E}$ and $\widetilde{\mathbf{E}}$ have the same orientation. As a result, the surface $\widetilde{S}$ is equipped with metric tensor

$$\widetilde{g} = \sum_{i=1}^{4} \widetilde{\omega}^i \otimes \widetilde{\omega}^i. \tag{3.5}$$

The orthogonal frames $\widetilde{\mathbf{E}}$ and $\mathbf{E}$ are related by the equation $\widetilde{\mathbf{E}} = \mathbf{E}A$, where $A \in SO(4, \mathbb{R})$, thus we can write the associated connection 1-forms as follows

$$\widetilde{\theta} = A^{-1}dA + A^{-1}\theta A. \tag{3.6}$$

Using (2.4), we can deduce the first structure equations for coframe $\widetilde{\omega}$ by following formula

$$d\widetilde{\omega}^i = -\sum_{j=1}^{4} \widetilde{\theta}^i_j \wedge \widetilde{\omega}^j. \qquad 1 \leq i \leq 4 \tag{3.7}$$

Take a look at the formula (2.5), we derive the second structure equations

$$d\widetilde{\omega}^i = -\sum_j \widetilde{\theta}^i_j \wedge \widetilde{\omega}^j, \qquad i = 1, 2, 3 \tag{3.8}$$

$$0 = \widetilde{\theta}^4_1 \wedge \widetilde{\omega}^1 + \widetilde{\theta}^4_2 \wedge \widetilde{\omega}^2 + \widetilde{\theta}^4_3 \wedge \widetilde{\omega}^3, \tag{3.9}$$

on the $\widetilde{S}$. It is necessary to mention the (3.9) concludes by $\widetilde{\omega}^4 = 0$. An immediate consequences of the equation (3.9) are

$$\begin{aligned} \widetilde{\theta}^4_1 &= h_{11}\widetilde{\omega}^1 + h_{12}\widetilde{\omega}^2 + h_{13}\widetilde{\omega}^3, \\ \widetilde{\theta}^4_2 &= h_{21}\widetilde{\omega}^1 + h_{22}\widetilde{\omega}^2 + h_{23}\widetilde{\omega}^3, \\ \widetilde{\theta}^4_3 &= h_{31}\widetilde{\omega}^1 + h_{32}\widetilde{\omega}^2 + h_{33}\widetilde{\omega}^3, \end{aligned} \tag{3.10}$$

where $h_{ij} = h_{ji}$. These functions are the coefficients of the second fundamental form

$$\mathrm{II} = \widetilde{\theta}^4_1 \otimes \widetilde{\omega}^1 + \widetilde{\theta}^4_2 \otimes \widetilde{\omega}^2 + \widetilde{\theta}^4_3 \otimes \widetilde{\omega}^3, \tag{3.11}$$

Now one can evaluate the $\widetilde{\Omega}^i_j$ by following formula

$$\widetilde{\Omega}^i_j = \sum_{k<l} \widetilde{R}^i_{jkl}\, \widetilde{\omega}^k \wedge \widetilde{\omega}^l, \tag{3.12}$$

for metric $\widetilde{g}$ on $\widetilde{S}$. According to Gauss formula we have

$$R_{ijrs} - \widetilde{R}_{ijrs} = h_{is}h_{jr} - h_{ir}h_{js}. \tag{3.13}$$

Also in an orthonormal frame we have

$$\sum_k \delta_{ik} R^k_{jrs} = R_{ijrs}. \tag{3.14}$$

We shall define the Weingarten operator $\mathcal{A} : TM \longrightarrow TM$, for each $X, Y \in T_pM$, by

$$\langle \mathcal{A}X, Y\rangle = \langle \mathrm{II}(X, Y), \mathbf{n}\rangle_N, \tag{3.15}$$

componentwise this means that

$$\mathcal{A} = \sum_{i,j} h_{ij}\widetilde{\omega}^i \otimes \widetilde{\mathbf{e}}_j. \tag{3.16}$$

In fact, often we shall not distinguish between $\mathcal{A}$ and the second fundamental tensor in the direction of $\mathbf{n}$, that is, the map $\langle \mathrm{II}(\,,\,),\ \mathbf{n}\rangle_N : TM \times TM \longrightarrow \mathbb{R}$. The k-th mean curvatures of the hypersurface in the direction of $\mathbf{n}$ are given by

$$H_k = \binom{m}{k}^{-1} S_k, \tag{3.17}$$

where $S_0 = 1$ and, for $1 \le k \le m$, S_k is the k-th elementary symmetric function of the eigenvalues of $\mathcal{A} = (h_{ij})$. In particular $H_1 = H$ is the mean curvature that

$$H = \frac{1}{3}\mathrm{tr}\ \mathcal{A}, \tag{3.18}$$

and H_m is the Gauss-Kronecker curvature that equals to

$$H_m = \det\ \mathcal{A}, \tag{3.19}$$

and H_2 is strictly related to the scalar curvature of M, [1].

We remind the reader that the surface $\widetilde{S}$ is said to be *totally geodesic* if the second fundamental form identically vanishes on $\widetilde{S}$ and is said to be *minimal* if $H = 0$.

4 The class of third-order ODEs as a Riemannian geometry

Geometrically, one can consider the third-order ODE with the following form

$$\frac{d^3y}{dx^3} = f(x, y, y', y''), \tag{4.1}$$

as a submanifold $\mathcal{S}$ in the third-order jet bundle J^3, which has local coordinates

$$\Upsilon = \{(x, y, p, q, r) \in \mathrm{J}^3 : p = y', q = y'', r = y'''\},$$

and it is denoted by the zero set of the function $F(x, y, p, q, r) = r - f(x, y, p, q)$, that means $\mathcal{S} = F^{-1}(0)$ is presented on J^2 as the graph of the function $r = f(x, y, p, q)$.

From the geometric theory of such equations, it follows that there exists a collection of independent 1-forms as a coframe

$$\begin{aligned} \omega^1 &= dx, \\ \omega^2 &= dy - p\,dx, \\ \omega^3 &= dp - q\,dx, \\ \omega^4 &= dq - f\,dx, \end{aligned} \tag{4.2}$$

on $\mathbb{R}^5$ with coordinates (x, y, p, q, r). The third prolongation of a solution curve of differential equation (4.1) is a curve on $\mathcal{S}$ represented by 3-jet of a smooth section $\sigma(x, y, p) = (x, y, p, q, r)$ of the trivial bundle $\pi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ on which the contact forms ω^2, ω^3 and ω^4 vanish. Since the local coframe $\omega = \{\omega^1, \omega^2, \omega^3, \omega^4\}$ is dual to the frame of the vector fields $\Omega = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ that means $\omega^i(\mathbf{e}_j) = \delta_{ij}$ therefore we have

$$\mathbf{e}_1 = \partial_x + p\partial_y + q\partial_p + f\partial_q, \qquad \mathbf{e}_2 = \partial_y, \qquad \mathbf{e}_3 = \partial_p, \qquad \mathbf{e}_4 = \partial_q. \tag{4.3}$$

The Riemannian metric

$$g = \sum_{i,j=1}^{4} \delta_{ij}\ \omega^i \otimes \omega^j, \tag{4.4}$$

on $\mathcal{S}$ is given in coordinates (x, y, p, q) as

$$ds^2 = (1 + p^2 + q^2 + f^2)dx^2 - 2pdxdy - 2qdxdp - 2fdxdq + dy^2 + dp^2 + dq^2. \tag{4.5}$$

Differentiating the coframe (4.2), we have

$$\begin{aligned} d\omega^1 &= 0, \\ d\omega^2 &= \omega^1 \wedge \omega^3, \\ d\omega^3 &= \omega^1 \wedge \omega^4, \\ d\omega^4 &= f_y\ \omega^1 \wedge \omega^2 + f_p\ \omega^1 \wedge \omega^3 + f_q\ \omega^1 \wedge \omega^4. \end{aligned} \tag{4.6}$$

Now using the (3.2) formula, we can write

$$\begin{bmatrix} d\omega^1 \\ d\omega^2 \\ d\omega^3 \\ d\omega^4 \end{bmatrix} = - \begin{bmatrix} 0 & \theta^1_2 & \theta^1_3 & \theta^1_4 \\ \theta^2_1 & 0 & \theta^2_3 & \theta^2_4 \\ \theta^3_1 & \theta^3_2 & 0 & \theta^3_4 \\ \theta^4_1 & \theta^4_2 & \theta^4_3 & 0 \end{bmatrix} \wedge \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix},$$

where $\Theta = (\theta^i_j)$ is a $\mathfrak{o}(4,\mathbb{R})$-valued torsion-free connection is a antisymmetric matrix, where

$$\begin{aligned} \theta^1_2 &= -\theta^2_1 = -\frac{1}{2}\ \omega^3 - \frac{1}{2} f_y\ \omega^4, \\ \theta^1_3 &= -\theta^3_1 = -\frac{1}{2}\ \omega^2 - \frac{1}{2}(1+f_p)\ \omega^4, \\ \theta^1_4 &= -\theta^4_1 = -\frac{1}{2} f_y\ \omega^2 - \frac{1}{2}(1+f_p)\ \omega^3 - f_q\ \omega^4, \\ \theta^2_3 &= -\theta^3_2 = -\frac{1}{2}\ \omega^1, \\ \theta^2_4 &= -\theta^4_2 = \frac{1}{2} f_y\ \omega^1, \\ \theta^3_4 &= -\theta^4_3 = -\frac{1}{2}(1-f_p)\ \omega^1. \end{aligned} \tag{4.7}$$

Now according to (3.4), the components of the curvature 2-form are

$$\begin{aligned} \Omega^1_2 &= -\frac{1}{2}\Big[\frac{1}{2}(-1+3f_y^2)\ \omega^1\wedge\omega^2 + \frac{1}{2} f_y(1+3f_p)\ \omega^1\wedge\omega^3 + f_{yy}\ \omega^2\wedge\omega^4 \\ &\quad + f_{yp}\ \omega^3\wedge\omega^4 + \Big(\frac{1}{2} + 2f_yf_q - \frac{f_p}{2} + pf_{yy} + f_{xy} + ff_{yq} + qf_{yp}\Big)\ \omega^1\wedge\omega^4\Big], \\ \Omega^1_3 &= \frac{1}{2}\Big[-\frac{1}{2} f_y(1+3f_p)\ \omega^1\wedge\omega^2 - (1+f_p+\frac{3}{2}f_p^2)\ \omega^1\wedge\omega^3 \\ &\quad - (2f_pf_q + pf_{yp} + ff_{pq} + \frac{1}{2}f_y + f_{xp})\ \omega^1\wedge\omega^4 - f_{yp}\ \omega^2\wedge\omega^4 - f_{pp}\ \omega^3\wedge\omega^4\Big], \\ \Omega^1_4 &= -\frac{1}{2}\Big(2f_yf_q + \frac{1}{2}(1-f_p) + pf_{yy} + f_{xy} + ff_{yq} + qf_{yp}\Big)\ \omega^1\wedge\omega^2 - \frac{1}{2}\Big(2f_pf_q + pf_{yp} + ff_{pq} \\ &\quad + qf_{pp} + \frac{1}{2}f_y + f_{xp}\Big)\ \omega^1\wedge\omega^3 - \frac{1}{2}\Big(\frac{3}{2} - \frac{1}{2}f_p^2 - \frac{1}{2}f_y^2 + 2f_q^2 + f_p + 2qf_{pq} + 2ff_{qq} + 2pf_{yq} \\ &\quad + 2f_{xq}\Big)\ \omega^1\wedge\omega^4 - \frac{1}{2}f_{yq}\ \omega^2\wedge\omega^4 - \frac{1}{2}f_{pq}\ \omega^3\wedge\omega^4, \\ \Omega^2_3 &= \frac{1}{2}\Big[\frac{1}{2}\ \omega^2\wedge\omega^3 + \frac{1}{2}f_y\ \omega^2\wedge\omega^4 - \frac{1}{2}(1+f_p)\ \omega^3\wedge\omega^4\Big], \\ \Omega^2_4 &= \frac{1}{2}\Big[- f_{yy}\ \omega^1\wedge\omega^2 - f_{yp}\ \omega^1\wedge\omega^3 - f_{yq}\ \omega^1\wedge\omega^4 + \frac{1}{2}f_y\ \omega^2\wedge\omega^3 + \frac{1}{2}f_y^2\ \omega^2\wedge\omega^4 \\ &\quad - \Big(f_q - \frac{1}{2}f_y - \frac{1}{2}f_yf_q\Big)\ \omega^3\wedge\omega^4\Big], \\ \Omega^3_4 &= \frac{1}{2}\Big[- f_{yp}\ \omega^1\wedge\omega^2 - f_{pp}\ \omega^1\wedge\omega^3 - f_{pq}\ \omega^1\wedge\omega^4 - \frac{1}{2}(1+f_p)\ \omega^2\wedge\omega^3 + \frac{1}{2}\Big(f_y(1+f_p) \\ &\quad - 2f_q\Big)\ \omega^2\wedge\omega^4 + \frac{1}{2}(1+f_p)^2\ \omega^3\wedge\omega^4\Big]. \end{aligned}$$

Thus independent components of the Riemann curvature tensor are given below

$$
\begin{aligned}
R^1_{212} &= \frac{1}{4}\Big(1-3f_y^2\Big), \quad R^1_{213} = -\frac{1}{4}f_y(1+3f_p),\\
R^1_{214} &= -\frac{1}{2}\Big(\frac{1}{2}+2f_yf_q-\frac{f_p}{2}+pf_{yy}+f_{xy}+ff_{yq}+qf_{yp}\Big),\\
R^1_{224} &= -\frac{1}{2}f_{yy}, \qquad R^1_{234} = -\frac{1}{2}f_{yp},\\
R^1_{312} &= -\frac{1}{4}f_y(1+3f_p), \quad R^1_{313} = -\frac{1}{2}(1+f_p+\frac{3}{2}f_p^2),\\
R^1_{314} &= -\frac{1}{2}(2f_pf_q+pf_{yp}+ff_{pq}+\frac{1}{2}f_y+f_{xp}), \quad R^1_{324} = -\frac{1}{2}f_{yp}, \quad R^1_{334} = -\frac{1}{2}f_{pp},\\
R^1_{412} &= -\frac{1}{2}\Big(2f_yf_q+\frac{1}{2}(1-f_p)+pf_{yy}+f_{xy}+ff_{yq}+qf_{yp}\Big),\\
R^1_{413} &= \frac{1}{2}\Big(2f_pf_q+pf_{yp}+ff_{pq}+qf_{pp}+\frac{1}{2}f_y+f_{xp}\Big),\\
R^1_{414} &= -\frac{1}{2}\Big(\frac{3}{2}-\frac{1}{2}f_p^2-\frac{1}{2}f_y^2+2f_q^2+f_p+2qf_{pq}+2ff_{qq}+2pf_{yq}+2f_{xq}\Big),\\
R^1_{424} &= -\frac{1}{2}f_{yq}, \qquad R^1_{434} = -\frac{1}{2}f_{pq},\\
R^2_{323} &= \frac{1}{4}, \qquad R^2_{324} = \frac{1}{4}f_y, \qquad R^2_{334} = -\frac{1}{4}(1+f_p),\\
R^2_{412} &= -\frac{1}{2}f_{yy}, \quad R^2_{413} = -\frac{1}{2}f_{yp}, \quad R^2_{414} = -\frac{1}{2}f_{yq}, \quad R^2_{423} = \frac{1}{4}f_y, \quad R^2_{424} = \frac{1}{4}f_y^2,\\
R^2_{434} &= -\frac{1}{4}\Big(f_q-\frac{1}{2}f_y-\frac{1}{2}f_yf_q\Big), \quad R^3_{412} = -\frac{1}{2}f_{yp}, \quad R^3_{413} = -\frac{1}{2}f_{pp}, \quad R^3_{414} = -\frac{1}{2}f_{pq},\\
R^3_{423} &= -\frac{1}{4}(1+f_p), \quad R^3_{424} = \frac{1}{4}\Big(f_y(1+f_p)-2f_q\Big), \quad R^3_{434} = \frac{1}{4}(1+f_p)^2.
\end{aligned}
$$

5 Geometry of submanifolds

Let $\widetilde{\mathcal{S}}$ is a submanifold in $\mathcal{S}$ determined by the smooth section

$$\sigma : (x,y,p) \longrightarrow (x,y,p,q(x,y,p)), \tag{5.1}$$

corresponding to the equation $y'' = q(x,y,p)$ such that $dp - qdx$ vanishes on the first prolongation of an integral curve of (4.1). In fact, 3-graph of an integral curve of (4.1) is specified by

$$y''' = q_x + q_y y' + q_p p' = q_x + pq_y + qq_p,$$

lies on $\widetilde{\mathcal{S}}$. We can compute the pullbacks of ω^i by σ as follow

$$
\begin{aligned}
\sigma^*\omega^1 &= \omega^1,\\
\sigma^*\omega^2 &= \omega^2,\\
\sigma^*\omega^3 &= \omega^3,\\
\sigma^*\omega^4 &= q_y\,\omega^2 + q_p\,\omega^3.
\end{aligned}
$$

With a simple calculation we find

$$
\begin{aligned}
\sigma^*(d\omega^4) &= (q_{xy}+pq_{yy}+qq_{yp}+q_yq_p+q_y\sigma^*(f_q))\,\omega^1\wedge\omega^2 + (q_{xp}+q_y+pq_{yp}+qq_{pp}\\
&\quad + q_p^2 + q_p\sigma^*(f_q))\,\omega^1\wedge\omega^3,\\
d(\sigma^*\omega^4) &= (q_{xy}+pq_{yy}+qq_{yp})\,\omega^1\wedge\omega^2 + (q_y+q_{xp}+pq_{yp}+qq_{pp})\,\omega^1\wedge\omega^3 + q_p\,\omega^1\wedge\omega^4.
\end{aligned}
$$

The equality $\sigma^*(d\omega^4) = d(\sigma^*\omega^4)$, leads to $q_p = 0$ and $q_y\sigma^*(f_q) = 0$. Now we can consider two different cases:

5.1 Case of $q_y \neq 0$

Since $\sigma^*(f_q) = 0$, that is, $f_q = 0$ on the submanifold $\widetilde{\mathcal{S}}$, the induced metric on $\widetilde{\mathcal{S}}$ is readily found as

$$\sigma^* ds^2 = \omega^1 \otimes \omega^1 + (1 + q_y^2)\, \omega^2 \otimes \omega^2 + \ \omega^3 \otimes \omega^3. \tag{5.2}$$

sur In local coordinates (x, y, p), we can rewrite the metric tensor by $\widetilde{g}$ with following form

$$\widetilde{g} = [1 + p^2(1 + q_y)^2 + q^2]\, dx^2 + (1 + q_y^2)\, dy^2 + \ dp^2 - 2p(1 + q_y^2)\, dxdy - 2q\, dxdp.$$

For each point in a coordinate neighborhood $(\widetilde{U}; x, y, p)$ of $T\widetilde{\mathcal{S}}$, the tangent space $T\widetilde{\mathcal{S}}$ is spanned by the vector fields $\widehat{\mathbf{e}}_1 = \sigma_*\partial_x$, $\widehat{\mathbf{e}}_2 = \sigma_*\partial_y$ and $\widehat{\mathbf{e}}_3 = \sigma_*\partial_p$,

$$\widehat{\mathbf{e}}_1 = \partial_x + q_x\partial_q, \qquad \widehat{\mathbf{e}}_2 = \partial_y + q_y\partial_q, \qquad \widehat{\mathbf{e}}_3 = \partial_p. \tag{5.3}$$

Now we can rewrite the vector fields $\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2$ and $\widehat{\mathbf{e}}_3$ in terms of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ with following form

$$\widehat{\mathbf{e}}_1 = \mathbf{e}_1 - p\mathbf{e}_2 - q\mathbf{e}_3 + (q_x - f)\mathbf{e}_4, \quad \widehat{\mathbf{e}}_2 = \mathbf{e}_2 + q_y\mathbf{e}_4, \quad \widehat{\mathbf{e}}_3 = \mathbf{e}_3. \tag{5.4}$$

To find unit normal vector field to $\widetilde{\mathcal{S}}$, we can consider the cross product of $X, Y, Z \in T\mathcal{S}$ at a given point of a four-dimensional Riemannian manifold. The cross product on $\mathcal{S}$ is defined in terms of the Riemannian metric (4.5) and the volume form by

$$g(X \times Y \times Z, W) := \mathrm{vol}_g(X, Y, Z, W), \quad \forall\ W \in T\mathcal{S}, \tag{5.5}$$

where the positive definite matrix (g_{ij}) is defined by

$$(g_{ij}) = \begin{bmatrix} 1 + p^2 + q^2 + f^2 & -p & -q & -f \\ -p & 1 & 0 & 0 \\ -q & 0 & 1 & 0 \\ -f & 0 & 0 & 1 \end{bmatrix}.$$

Let $X = \sum_{i=1}^{4} x_i\, \mathbf{e}_i$, $Y = \sum_{i=1}^{4} y_i\, \mathbf{e}_i$ and $Z = \sum_{i=1}^{4} z_i\, \mathbf{e}_i$ in $T\mathcal{S}$, using [15] we can define the cross product ,

$$\begin{aligned} X \times Y \times Z &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} \\ &= \begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix} \mathbf{e}_1 + \begin{vmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ z_1 & z_3 & z_4 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{vmatrix} \mathbf{e}_3 + \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \mathbf{e}_4. \end{aligned} \tag{5.6}$$

According to (5.6), we can compute

$$\widehat{\mathbf{e}}_1 \times \widehat{\mathbf{e}}_2 \times \widehat{\mathbf{e}}_3 = q_y\mathbf{e}_2 - \mathbf{e}_4,$$

and therefore we obtain the unit normal vector field on $\widetilde{\mathcal{S}}$ as follow

$$\mathbf{n} = \frac{1}{\sqrt{1 + q_y^2}} \left(-q_y\mathbf{e}_2 + \mathbf{e}_4\right). \tag{5.7}$$

Thus the adapted orthogonal frame $\widetilde{\mathbf{E}} = (\widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2, \widetilde{\mathbf{e}}_3, \mathbf{n})$ on $\widetilde{\mathcal{S}}$ is

$$\widetilde{\mathbf{e}}_1 = \mathbf{e}_1, \quad \widetilde{\mathbf{e}}_2 = \frac{1}{\sqrt{1 + q_y^2}} \left(\mathbf{e}_2 + q_y\mathbf{e}_4\right), \quad \widetilde{\mathbf{e}}_3 = \mathbf{e}_3, \qquad \mathbf{n} = \frac{1}{\sqrt{1 + q_y^2}} \left(-q_y\mathbf{e}_2 + \mathbf{e}_4\right), \tag{5.8}$$

that frame $\widetilde{\mathbf{\Omega}} = (\widetilde{\omega}^1, \widetilde{\omega}^2, \widetilde{\omega}^3, \widetilde{\omega}^4)$ on $\widetilde{\mathcal{S}}$ is

$$\widetilde{\omega}^1 = \omega^1, \quad \widetilde{\omega}^2 = \frac{1}{\sqrt{1+q_y^2}}\left(\omega^2 + q_y\omega^4\right), \qquad \widetilde{\omega}^3 = \omega^3, \quad \widetilde{\omega}^4 = \frac{1}{\sqrt{1+q_y^2}}\left(-q_y\omega^2 + \omega^4\right). \tag{5.9}$$

Now since we have $\sigma^*\omega^4 = q_y\,\omega^2$ on the submanifold $\widetilde{\mathcal{S}}$, we will have $\widetilde{\omega}^4 = 0$ on it. Therefore the Riemannian metric (5.2) can be rewritten based on the members of coframe $\widetilde{\mathbf{\Omega}}$ as follows

$$\widetilde{g} = \widetilde{\omega}^1 \otimes \widetilde{\omega}^1 + \widetilde{\omega}^2 \otimes \widetilde{\omega}^2 + \widetilde{\omega}^3 \otimes \widetilde{\omega}^3, \tag{5.10}$$

on the submanifold $\widetilde{\mathcal{S}}$, here $\widetilde{\omega}^2$ is equal to $\sqrt{1+q_y^2}\,\omega^2$.

We have the relation $\widetilde{\mathbf{E}} = \mathbf{E}A$ between the frames $\mathbf{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ and $\widetilde{\mathbf{E}} = (\widetilde{\mathbf{e}}_1, \widetilde{\mathbf{e}}_2, \widetilde{\mathbf{e}}_3, \mathbf{n})$ where $A \in \mathrm{SO}(4,\mathbb{R})$, with following representation

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varepsilon) & 0 & -\sin(\varepsilon) \\ 0 & 0 & 1 & 0 \\ 0 & \sin(\varepsilon) & 0 & \cos(\varepsilon) \end{bmatrix}, \tag{5.11}$$

is defined by putting $q_y = \tan(\varepsilon)$ for the sufficiently small values of ε. Using the formula (3.6), the connection matrix $\widetilde{\theta}$ is equal to

$$\widetilde{\theta} = \begin{bmatrix} 0 & \cos(\varepsilon)\ \theta_2^1 + \sin(\varepsilon)\ \theta_4^1 & \theta_3^1 & -\sin(\varepsilon)\ \theta_2^1 + \cos(\varepsilon)\ \theta_4^1 \\ -\cos(\varepsilon)\ \theta_2^1 - \sin(\varepsilon)\ \theta_4^1 & 0 & \cos(\varepsilon)\ \theta_3^2 - \sin(\varepsilon)\ \theta_4^3 & -d\varepsilon + \theta_4^2 \\ -\theta_3^1 & -\cos(\varepsilon)\ \theta_3^2 + \sin(\varepsilon)\ \theta_4^3 & 0 & \sin(\varepsilon)\ \theta_3^2 + \cos(\varepsilon)\ \theta_4^3 \\ \sin(\varepsilon)\ \theta_2^1 - \cos(\varepsilon)\ \theta_4^1 & d\varepsilon - \theta_4^2 & -\sin(\varepsilon)\ \theta_3^2 - \cos(\varepsilon)\ \theta_4^3 & 0 \end{bmatrix},$$

where the 1-forms θ_i^js are defined in (4.7). Since $q_y = \tan(\varepsilon)$, thus

$$d\varepsilon = \frac{(q_{xy} + p q_{yy})\widetilde{\omega}^1 + q_{yy}(1+q_y^2)^{-\frac{1}{2}}\ \widetilde{\omega}^2}{1+q_y^2}, \tag{5.12}$$

and by putting $\sin(\varepsilon) = q_y(1+q_y^2)^{-\frac{1}{2}}$ and $\cos(\varepsilon) = (1+q_y^2)^{-\frac{1}{2}}$, we can rewrite the arrays of the matrice $\widetilde{\theta}$ as follow

$$\begin{aligned}
\widetilde{\theta}_2^1 &= -\frac{1}{2}\Big[f_y \sin(\varepsilon)\cos(\varepsilon)\ \widetilde{\omega}^2 + (\cos(\varepsilon) + (1+f_p)\sin(\varepsilon))\ \widetilde{\omega}^3\Big], \\
\widetilde{\theta}_3^1 &= -\frac{1}{2}\Big[(\cos(\varepsilon) + (1+f_p)\sin(\varepsilon))\ \widetilde{\omega}^2\Big], \\
\widetilde{\theta}_4^1 &= -\frac{1}{2}\Big[f_y \cos(2\varepsilon)\ \widetilde{\omega}^2 + (\sin(\varepsilon) - (1+f_p)\cos(\varepsilon))\ \widetilde{\omega}^3\Big], \\
\widetilde{\theta}_3^2 &= -\frac{1}{2}\Big[\cos(\varepsilon) - (1-f_p)\sin(\varepsilon)\Big]\widetilde{\omega}^1, \\
\widetilde{\theta}_4^2 &= -d\varepsilon + \frac{1}{2} f_y\ \widetilde{\omega}^1, \\
\widetilde{\theta}_4^3 &= -\frac{1}{2}\Big[\sin(\varepsilon) + (1-f_p)\cos(\varepsilon)\Big]\widetilde{\omega}^1.
\end{aligned} \tag{5.13}$$

Looking at (3.10), the coefficients of the second fundamental form are determined by

$$
\begin{aligned}
h_{11} &= 0,\\
h_{12} = h_{21} &= \frac{1-q_y^2}{1+q_y^2}\,(q_x+pq_y)_y,\\
h_{13} = h_{31} &= -\frac{1}{2\sqrt{1+q_y^2}},\\
h_{22} &= -\frac{q_{yy}}{(1+q_y^2)^{\frac{3}{2}}},\\
h_{23} = h_{32} &= 0,\\
h_{33} &= 0,
\end{aligned}
$$

here $\mathcal{A} = (h_{ij})$. Now Using the formula (3.11), the second fundamental form is

$$
\mathrm{II} = -\left(\frac{1-q_y^2}{1+q_y^2}\right)(q_x+pq_y)_y\,\widetilde{\omega}^1\otimes\widetilde{\omega}^2 - \left(\frac{q_y-(q_x+pq_y)_p}{\sqrt{1+q_y^2}}\right)\widetilde{\omega}^1\otimes\widetilde{\omega}^3 - \frac{q_{yy}}{(1+q_y^2)^{3/2}}\,\widetilde{\omega}^2\otimes\widetilde{\omega}^2 \tag{5.14}
$$

Then using (3.19) and (3.18), we have

$$
\mathcal{K}_e = \frac{q_{yy}}{4(1+q_y^2)^{\frac{5}{2}}}, \tag{5.15}
$$

$$
H = -\frac{q_{yy}}{3(1+q_y^2)^{\frac{3}{2}}}. \tag{5.16}
$$

Using above findings we can summarize following theorems:

Theorem 5.1. A submanifold $\widetilde{\mathcal{S}} \subset \mathcal{S}$ which is determined by the section (5.1) is minimal if and only if

$$
q_{yy} = 0. \tag{5.17}
$$

In this case, the Gauss-Kronecker curvature of a minimal surface equals to zero.

Theorem 5.2. Let $\widetilde{\mathcal{S}}$ be a submanifold corresponding to linear second-order differential equation

$$
y'' = \alpha(x)y + \beta(x) \tag{5.18}
$$

where α and β are two smooth functions in term of x, then $\widetilde{\mathcal{S}}$ determines a minimal surface in four-dimensional manifold corresponding to the following third-order equation

$$
y''' = \alpha(x)y' + \alpha'(x)y + \beta'(x), \tag{5.19}
$$

Theorem 5.3. A submanifold $\widetilde{\mathcal{S}} \subset \mathcal{S}$, which is defined by the section

$$
(x, y, p) \mapsto (x, y, p, \alpha(x)y + \beta(x)),
$$

is totally geodesic if and only if $\alpha(x) = \pm 1$.

5.2 Case of $q_y = 0$

In this case, we want to consider the submanifold determined with $\sigma^*\omega^4 = 0$, on the submanifold $\widetilde{\mathcal{S}}$. Since $\sigma^*\omega^4 = q_y\,\omega^2$, this case concludes to $q_y = 0$. Thus the matrix (5.11) reduces to the identity and then we have $\widetilde{\theta} = \theta$.

According to above discussion ω^4 equals to zero on the submanifold $\widetilde{\mathcal{S}}$.

$$\begin{aligned}
\theta^1_2 &= -\theta^2_1 = -\frac{1}{2}\,\omega^3,\\
\theta^1_3 &= -\theta^3_1 = -\frac{1}{2}\,\omega^2,\\
\theta^1_4 &= -\theta^4_1 = -\frac{1}{2}f_y\,\omega^2 - \frac{1}{2}(1+f_p)\,\omega^3,\\
\theta^2_3 &= -\theta^3_2 = -\frac{1}{2}\,\omega^1,\\
\theta^2_4 &= -\theta^4_2 = \frac{1}{2}f_y\,\omega^1,\\
\theta^3_4 &= -\theta^4_3 = -\frac{1}{2}(1-f_p)\,\omega^1,
\end{aligned} \tag{5.20}$$

Therefore

$$\begin{aligned}
\widetilde{\theta}^1_2 &= \frac{1}{2}\Big[f_y\,\widetilde{\omega}^2 + (1+f_p)\,\widetilde{\omega}^3\Big],\\
\widetilde{\theta}^1_3 &= \frac{1}{2}(1+f_p)\,\widetilde{\omega}^2,\\
\widetilde{\theta}^1_4 &= \frac{1}{2}\Big[f_y\,\widetilde{\omega}^2 + \widetilde{\omega}^3\Big],\\
\widetilde{\theta}^2_3 &= -\frac{1}{2}(1-f_p)\widetilde{\omega}^1,\\
\widetilde{\theta}^2_4 &= -\frac{1}{2}f_y\,\widetilde{\omega}^1,\\
\widetilde{\theta}^3_4 &= -\frac{1}{2}\widetilde{\omega}^1.
\end{aligned} \tag{5.21}$$

That means $h_{11} = h_{22} = h_{33} = h_{23} = 0$ and $h_{12} = \frac{1}{2}f_y$, $h_{13} = \frac{1}{2}$, and therefore $H = \mathcal{K}_e = 0$ and the second fundamental form is

$$\mathrm{II} = \frac{1}{2}(q_x + pq_y)_y\,\widetilde{\omega}^1 \otimes \widetilde{\omega}^2 + \frac{1}{2}\,\widetilde{\omega}^1 \otimes \widetilde{\omega}^3, \tag{5.22}$$

Theorem 5.4. A submanifold $\widetilde{\mathcal{S}}$ of $\mathcal{S}$ which is determined by the section (5.1) is a minimal but not totally geodesic. Its Gauss-Kronecker curvature equals to zero.

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