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On the food chain model with prey refuge and fear effect

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Abstract

Of concern the present study deals with an updated food chain model in a natural environment with the inclusion of fear effect in the prey population through Holling type II functional response in presence of prey refuge effect. The present model is affluent with intra-specific competition among the hunter species having specific mortality. The model system emphasizes its characteristics in the proximity of the probable equilibrium position in the realm of biological dynamics. The response of the system is explored further for its stability analysis based on prerequisites and Hopf-bifurcation phenomena as well with respect to some significant model parameters. Extensive numerical simulation reveals the validity of the proposed model so as to indicate the ecological implications.

Keywords: (Food chain model, Fear effect, Intra-specific competition, Prey refuge, Coexistence state, Stability, Hopf-bifurcations)
2020 MSC: 34K18, 34K20, 37B25, 37G15

1 Introduction

Mathematical model formulations including prey-predator, food chain and other ecological models and their analyses have established a new trend of research in theoretical ecology (cf. [21], [23], [18], [22]). The predator-prey interaction mechanism plays an important role in the behaviour of the proposed model.

In the ecosystem, there are enormous numbers of species of different kinds, of which prey and predator species play crucial role in various types of interactions among the species in the system. Quite often the interaction between the prey and the predator becomes the center of attraction in an ecosystem [24], [35] for complete understanding their dynamical behavior. For this purpose, various mathematical models [16], [27] have been made use of so that the outcomes of the models depending on model parameters often yielded new dimension in the domain of research. With the evolution of mathematical models in ecosystem, Rosenzweig and MacArthur [31] developed a model by combining Lotka- Volterra modified model having logistic growth rate for prey with predation rate of predator by using Holling type II [15] in order to establish their findings closer to the real situation. Many more researchers [11], [14] put forward three-dimensional models and explored various changes with the motivation of looking at different domains of interest. Such modifications include many factors in the realm of the dynamics of the ecosystem like fear effect, prey harvesting, delay effect, intra-specific competition and hunting cooperation as well [25], [8], [9], [28], [32]. A nonautonomous predator-prey model with fear, prey refuge and additional food together has been studied in [36]. The influence of Allee [33] with defense mechanism in the dynamical complexity is however, nor ruled out from the investigation. Evolutionary process in nature has been studied to illustrate the diversity of living animals [6], [38].

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The choice of various functional responses depending on the nature of interacting species [2], [4], [29], [34], [41] has been made to uphold the research worthwhile. So, one must realize that present-day research in the relevant domain of interest is going to be more and more challenging. The ecosystem envelopes quite a good number of interacting species and one may classify them as diverse levels in a food pyramid. In a food pyramid arrangement, prey species exist in the lowest trophic level while predators reside in the level above that of prey to their feeding habit. Predator species are solely depended on the prey population for their existence in the living kingdom.

On the other hand, prey species always encounter the risk of predation resulting in sense of fear in the mind of prey population which largely perturbs their process of evolution. Here fear is an indirect effect [20] that manipulates the mind set of prey psychologically [29]. Obviously, predators have the advantage to predate prey for food and thus affect the growth rate of the prey population. Predation is an example of direct purport of predators on prey [20], [39] while fear is an example of indirect effect. Several studies on the effect of fear in a tri-trophic food chain model have been carried out by many researchers of which the first work of its kind was introduced by [40] to show how the fear effect influences the growth rate of prey species. Attention has also been focused on the effects of fear of large carnivores on herbivores and meso-carnivores by eminent researchers. Three dimensional food chain models are of great use to exemplify the fear effect in the complex dynamics of such interacting species while two dimensional models have got obvious limitations.

In 1969, Pielou [30] modified the Lotka-Volterra preator-prey model by incorporating intra-specific competition in predator population. Bazykin [3] studied the predator-prey model with Holling type II functional response for predation process with inra-specific competition among predator. A ratio-dependent predator-prey model with intra-specific competition among predator population is studied by Haque [12] and showed that intra-specific competition has stabilizing potential on the system behaviour. Three species Hasting Powell food chain model is studied by Haque et. al. [13] by incorporating intra-specific competition among both the predator populations and conclude that intra-specific competition has stabilizing potential when all the species coexist. Ali and Chakravarty [1] studied the impact of prey refuge in the three species Hasting Powell food chain model in presence of intra-specific competition among predator populations.

In 1991, Hastings & Powell [14] studied the following food chain model

$$\frac{dx}{dt} = rx - d_1x - h_1x^2 - \frac{a_1xy}{b_1+x}, \quad x(0) > 0, \tag{1.1a}$$

$$\frac{dy}{dt} = \frac{e_1a_1xy}{b_1+x} - d_2y - \frac{a_2yz}{b_2+x}, \quad y(0) > 0, \tag{1.1b}$$

$$\frac{dz}{dt} = \frac{e_2a_2yz}{b_2+x} - d_3z, \quad z(0) > 0, \tag{1.1c}$$

where x, y, z stands for population density of prey, predator and top-predator respectively. The parameters r and d_1 are the birth rate and death rate of the prey, h_1 is the intra specific competition among prey population, a_i are the respective predation rates of predator and top-predator $(i = 1, 2)$, b_i are the respective half saturation constants $(i = 1, 2)$, e_i are the respective conversion factors $(i = 1, 2)$, d_i are the respective death rates of predator and top-predator $(i = 2, 3)$.

The model (1.1) is studied by many researchers ([17], [10], [26]). Haque et. al. [13] studied the food chain model (1.1) incorporating intra-specific competition among both the predator populations and conclude that intra-specific competition has the potential to control chaotic dynamics. Ali and Chakravarty [1] studied the influence of prey refuge in the model (1.1) with intra-specific competition among predator populations. Kumar and Kumari [19] incorporate fear effect in the model (1.1) and conclude that fear can control chaotic dynamics. Cong et. al. [7] formulate a three-species food chain model e by using the classical Holling's time budget argument where the cost and benefit of anti-predator behaviours are included. But no one study the impact of fear effect in presence of prey refuge on the dynamics of (1.1). Keeping in mind all the studies as mentioned above, at the moment, the interest is centered on the consequences of fear in prey species and prey refuge effect in both the prey and predator species in a tri-trophic food chain model (1.1) incorporating intra-specific competition among both the predator populations.

The present investigation is organized as follows. In Section 2, the formulation of the team model under consideration and its assumptions are stated. Section 3 contains some preliminary results. Subsequently in Section 4 the model with intra-specific competition is analyzed, identifying its equilibria, providing conditions for their feasibility, stability and bifurcation. Numerical simulation has finally been carried out in Section 5. The investigation concludes with a discussion of the results obtained.

2 Mathematical model formulation

We consider fear factor following Wang et. al. [40] in the prey population due to predator's hunting in the model (1.1) in presence of prey refuge effect. We also consider the intra-specific competition among both the predator populations. The dynamics of the model described above can be represented by the following set of differential equations:

$$\frac{dx}{dt} = \frac{rx}{1+ky} - d_1x - h_1x^2 - \frac{a_1(1-m_1)xy}{b_1+(1-m_1)x}, \quad x(0)>0, \tag{2.1a}$$

$$\frac{dy}{dt} = \frac{e_1a_1(1-m_1)xy}{b_1+(1-m_1)x} - d_2y - \frac{a_2(1-m_2)yz}{b_2+(1-m_2)y} - h_2y^2, \quad y(0)>0, \tag{2.1b}$$

$$\frac{dz}{dt} = \frac{e_2a_2(1-m_2)yz}{b_2+(1-m_2)y} - d_3z - h_3z^2, \quad z(0)>0, \tag{2.1c}$$

where all the variables and parameters are defined earlier except k, m_1, m_2, h_2, h_3. k is the fear parameter of prey population. The parameters m_i $(i=1,2)$ represents the constant proportion of prey refuge during predation process and h_i $(i=1,2)$ are the intra specific competition among individuals of predator and top-predator respectively.

3 Preliminary results

3.1 Equilibria and their feasibility

System (2.1) has the following four positive equilibria $E_i(x_i,y_i,z_i)$, $i=0,1,2,3$. E_0 is the origin, $E_1 \equiv (\frac{r-d_1}{h_1},0,0)$, $E_2 \equiv (x_2,y_2,0)$, $E_3 \equiv (x_3,y_3,z_3)$. For E_2, we have x_2, y_2 are positive roots of the equations

$$0 = \frac{r}{1+ky} - d_1 - h_1x - \frac{a_1(1-m_1)y}{b_1+(1-m_1)x}, \tag{3.1}$$

$$0 = \frac{e_1a_1(1-m_1)x}{b_1+(1-m_1)x} - d_2 - h_2y. \tag{3.2}$$

By numerical calculations, for the set of parameter values $r=2.0$, $k=0.5$, $a_1=2.0$, $a_2=0.3$, $b_1=3.2$, $b_2=1.0$, $e_1=0.9$, $e_2=0.8$, $m_1=0.1$, $m_2=0.1$, $d_1=0.3$, $d_2=0.3$, $d_3=0.2$, $h_1=0.1$, $h_2=0.1$, $h_3=0.1$; E_2 becomes $(1.230148922, 1.626838264, 0)$.

For the coexistence equilibrium $E_3(x_3,y_3,z_3)$; x_3, y_3 and z_3 are positive roots of the equations

$$0 = \frac{r}{1+ky} - d_1 - h_1x - \frac{a_1(1-m_1)y}{b_1+(1-m_1)x},$$

$$0 = \frac{e_1a_1(1-m_1)x}{b_1+(1-m_1)x} - d_2 - \frac{a_2(1-m_2)z}{b_2+(1-m_2)x} - h_2y,$$

$$0 = \frac{e_2a_2(1-m_2)y}{b_2+(1-m_2)x} - d_3 - h_3z.$$

By numerical calculations, for the set of parameter values $r=1.4$, $k=0.9$, $a_1=2.0$, $a_2=1.0$, $b_1=3.2$, $b_2=1.0$, $e_1=0.9$, $e_2=0.8$, $m_1=0.1$, $m_2=0.1$, $d_1=0.3$, $d_2=0.3$, $d_3=0.2$, $h_1=0.1$, $h_2=0.1$, $h_3=0.1$; E_3 becomes $(3.894309020, 0.6753642011, 1.024342722)$.

3.2 Boundedness

Proposition 1. The solutions of system (2.1) which initiate in R^3_+ are uniformly bounded.

Proof. Define a positive definite function

$$\Omega(t) = x(t) + \frac{y(t)}{e_1} + \frac{e_1z(t)}{e_2}. \tag{3.3}$$

From definition, $\Omega(t)$ is differentiable in some maximal interval $(0, t_b)$. For an arbitrary $\eta > 0$, the time derivative of (3.3) along the solution of the system (2.1) is

$$\begin{aligned}\frac{d\Omega}{dt} + \eta\Omega &= x\left(\eta + \frac{r}{1+ky} - d_1 - h_1 x\right) + y\left(\eta - \frac{d_2}{e_1} - \frac{h_2 y}{e_1}\right) + z\left(\eta - \frac{e_1 d_3}{e_2} - \frac{e_1 h_3 z}{e_2}\right)\\ &\leq \left(\eta + r - d_1 - h_1 x\right) + y\left(\eta - \frac{d_2}{e_1} - \frac{h_2 y}{e_1}\right) + z\left(\eta - \frac{e_1 d_3}{e_2} - \frac{e_1 h_3 z}{e_2}\right)\\ &\leq \frac{(\eta + r - d_1)^2}{4h_1} + \frac{(\eta - \frac{d_2}{e_1})^2}{4\frac{h_2}{e_1}} + \frac{(\eta - \frac{e_1 d_3}{e_2})^2}{4\frac{e_1 h_3}{e_2}}.\end{aligned}$$

Hence, we can find $\mu > 0$ such that

$$\frac{d\Omega}{dt} + \eta\Omega \leq \mu \quad \forall \quad t \in (0, t_b).$$

Applying the theory of differential equation [5], we get

$$0 < \Omega(x, y, z) < \frac{\mu}{\eta}(1 - e^{-\eta t)} + \Omega(x(0), y(0), z(0))e^{-\eta t} \quad \forall \quad t \in (0, t_b)$$

and for $t_b \to \infty$, $0 < \Omega(x, y, z) < \frac{\mu}{\eta}$. Hence all the solutions of system (2.1) that initiate at $(x(0), y(0), z(0))$ lie in R^3_+ and are confined in the compact region

$$\Gamma = \{(x, y, z) \in R^3_+; x(t) + \frac{1}{e}y(t) + z(t) = \frac{\mu}{\eta} + \varepsilon, \quad \forall \quad \varepsilon > 0\}. \tag{3.4}$$

4 Stability and bifurcation analysis

In order to investigate the dynamics of the proposed model (2.1) around the above equilibrium points, the Jacobian matrix of the system (2.1) at any arbitrary point (x, y, z) is given by

$$J(x, y, z) = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix},$$

where $J_{11} = \frac{r}{1+ky} - d_1 - 2h_1 x - \frac{a_1(1-m_1)y}{b_1+(1-m_1)x} + \frac{a_1(1-m_1)^2 xy}{\{b_1+(1-m_1)x\}^2}$, $J_{12} = -\frac{rkx}{(1+ky)^2} - \frac{a_1(1-m_1)x}{b_1+(1-m_1)x}$, $J_{13} = 0$, $J_{21} = \frac{e_1 a_1 b_1(1-m_1)zy}{\{b_1+(1-m_1)x\}^2}$, $J_{22} = \frac{e_1 a_1(1-m_1)x}{b_1+(1-m_1)x} - d_2 - \frac{a_2(1-m_2)z}{b_2+(1-m_2)y} + \frac{a_2(1-m_2)^2 yz}{(\{b_2+(1-m_2)y\}^2} - 2h_2 y$, $J_{23} = -\frac{a_2(1-m_2)y}{b_2+(1-m_2)y}$, $J_{31} = 0$, $J_{32} = \frac{e_2 a_2 b_2(1-m_2)z}{\{b_2+(1-m_2)y\}^2}$, $J_{33} = \frac{e_2 a_2(1-m_2)y}{b_2+(1-m_2)y} - d_3 - 2h_3 z$.

4.1 Dynamics of the system around $E_0(0,0,0)$

The eigenvalues of the Jacobian matrix J_0 at E_0 are $r - d_1 > 0$, $-d_2 < 0$, $-d_3 < 0$. Hence E_0 is unstable manifold in x direction.

4.2 Dynamics of the system around $E_1(x_1, 0, 0)$

The eigenvalues of the Jacobian matrix J_1 at E_1 are $-(r - d_1)$, $\frac{e_1 a_1(1-m_1)x_1}{b_1+(1-m_1)x_1} - d_2$, $-d_3$. Hence E_1 will be locally assymptotically stable if $\frac{e_1 a_1(1-m_1)x_1}{b_1+(1-m_1)x_1} < d_2$.

4.3 Dynamics of the system around $E_2(x_2, y_2, 0)$

(i) E_2 will be locally assymptotically stable if $a_1(1 - m_1)^2 x_2 y_2 < h_1 x_2\{b_1 + (1 - m_1)x_2\}^2$, $e_2 a_2(1 - m_2)y_2 < d_3\{b_2 + (1 - m_2)y_2\}$, $b_1 > m_1 x_2$.

(ii) E_2 experiences hopf-bifurcation at $a_1 = a_1^{[1HB]}$ where $a_1^{[1HB]} = \frac{(h_1 x_2 + h_2 y_2)\{b_1+(1-m_1)x_2\}^2}{(1-m_1)^2 x_2 y_2}$.

Proof: (i) The Jacobian matrix J_2 evaluated at E_2 is given by $J_2 = (c_{ij})_{3\times 3}$, where $c_{11} = -h_1x_2 + \frac{a_1(1-m_1)^2x_2y_2}{\{b_1+(1-m_1)x_2\}^2}$, $c_{12} = -\frac{rkx_2}{(1+ky_2)^2} - \frac{a_1(1-m_1)x_2}{b_1+(1-m_1)x_2} < 0$, $c_{13} = 0$, $c_{21} = \frac{e_1a_1y_2(1-m_1)(b_1-m_1x_2)}{\{b_1+(1-m_1)x_2\}^2}$, $c_{22} = -h_2y_2 < 0$, $c_{23} = -\frac{a_2(1-m_2)y_2}{b_2+(1-m_2)y_2} < 0$, $c_{31} = 0$, $c_{32} = 0$, $c_{33} = \frac{e_2a_2(1-m_2)y_2}{b_2+(1-m_2)y_2} - d_3$. Its eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left[c_{11} + c_{22} \pm \sqrt{(c_{11}+c_{22})^2 - 4(c_{11}c_{22} - c_{12}c_{21})}\right] \quad \lambda_3 = c_{33}. \tag{4.1}$$

If we assume $c_{11} < 0$, $c_{33} < 0$, $c_{21} > 0$ then $\lambda_3 < 0$ and $\lambda_{1,2}$ both are either negative or complex numbers with negative real parts. Hence, E_3 will be locally assymptotically stable if $c_{11} < 0$, $c_{33} < 0$, $c_{21} > 0$ that is, $a_1(1-m_1)^2x_2y_2 < h_1x_2\{b_1 + (1-m_1)x_2\}^2$, $e_2a_2(1-m_2)y_2 < d_3\{b_2 + (1-m_2)y_2\}$, $b_1 > m_1x_2$.

Proof: (ii) From (4.1), we see that λ_3 is real, λ_1 and λ_2 will be purely imaginary if and only if there is a $a_1 = a_1^{[1HB]}$ such that $a_1^{[1HB]} = \frac{(h_1x_2+h_2y_2)\{b_1+(1-m_1)x_2\}^2}{(1-m_1)^2x_2y_2}$.
But for $i = 1, 2$,

$$Re\left(\frac{d\lambda_i}{da_1}\right)\Big|_{a_1=a_1^{[1HB]}} = \frac{(1-m_1)^2x_2y_2}{\{b_1+(1-m_1)x_2\}^2} \neq 0.$$

Therefore, the system enters into hopf-bifurcation at $a_1 = a_1^{[1HB]}$ around E_2.

4.4 System behaviour near the coexistence equilibrium $E_3(x_3, y_3, z_3)$

The Jacobian matrix J_3 evaluated at E_3 has the components

$$\begin{aligned} J_{11} &= -h_1x_3 + \frac{a_1(1-m_1)x_3y_3}{\{b_1+(1-m_1)x_3\}^2}, \quad J_{12} = -\frac{rkx_3}{1+ky_3} - \frac{a_1(1-m_1)x_3}{b_1+(1-m_1)x_3} < 0, \\ J_{13} &= 0, \quad J_{21} = \frac{e_1a_1y_3(1-m_1)(b_1-m_1x_3)}{\{b_1+(1-m_1)x_3\}^2}, \quad J_{22} = -h_2y_3 + \frac{a_2(1-m_2)y_3z_3}{\{b_2+(1-m_2)y_3\}^2}, \\ J_{23} &= -\frac{a_2(1-m_2)y_3}{b_2+(1-m_2)y_3} < 0, \quad J_{31} = 0, \quad J_{32} = \frac{e_2a_2z_3(1-m_2)(b_2-m_2y_3)}{\{b_2+(1-m_2)y_3\}^2}, \\ J_{33} &= -h_3z_3. \end{aligned} \tag{4.2}$$

The characteristic equation of J_3 is $\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$ where $A_1 = -J_{11} - J_{22} - J_{33}$, $A_2 = J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} - J_{12}J_{21} - J_{23}J_{32}$, $A_3 = J_{11}J_{23}J_{32} + J_{12}J_{21}J_{33} - J_{11}J_{22}J_{33}$.
4.4.1: The system (2.1) is locally asymptotically stable if $A_1 > 0$, $A_3 > 0$ and $A_1A_2 - A_3 > 0$.
Proof: From the Routh-Hurwitz criterion, the equilibrium point E_3 is locally asymptotically stable if $A_1 > 0$, $A_3 > 0$ and $A_1A_2 > A_3$. Here,

$$\begin{aligned} A_1A_2 - A_3 &= -{J_{11}}^2J_{22} - {J_{11}}^2J_{33} - J_{11}{J_{22}}^2 - {J_{22}}^2J_{33} - J_{11}{J_{33}}^2 - J_{22}{J_{33}}^2 \\ &\quad + J_{11}J_{12}J_{21} + J_{22}J_{23}J_{32} + J_{22}J_{12}J_{21} + J_{33}J_{23}J_{32} - 2J_{11}J_{22}J_{33}. \end{aligned}$$

Assuming $J_{11} < 0$, $J_{22} < 0$, $J_{21} > 0$ and $J_{32} > 0$, $A_1A_2 - A_3 > 0$. Hence $A_1 > 0$, $A_3 > 0$ and $A_1A_2 - A_3 > 0$ i.e. E_3 is locally asymptotically stable if $J_{11} < 0$, $J_{22} < 0$, $J_{21} > 0$ and $J_{32} > 0$ which implies the conditions are as follows: $a_1(1-m_1)x_3y_3 < h_1x_3\{b_1 + (1-m_1)x_3\}^2$, $a_2(1-m_2)y_3z_3 < h_2y_3\{b_2 + (1-m_2)y_3\}^2$, $b_1 > m_1x_3$, $b_2 > m_2y_3$.

4.4.2: The system (2.1) is globally asymptotically stable if $h_1b_1\{b_1 + (1-m_1)x_3\} > a_1y_3(1-m_1)^2$, $h_2b_2\{b_2 + (1-m_2)y_3\} > a_2z_3(1-m_2)^2$.
Proof: Let $R_*^3 = \{(x,y,z) \in R_+^3, x > 0, y > 0, z > 0\}$ and consider the scalar function $L : R_*^3 \to R$ defined by

$$L = k_1\left[x - x_3 - x_3 ln\frac{x}{x_3}\right] + k_2\left[y - y_3 - y_3 ln\frac{y}{y_3}\right] + k_3\left[z - z_3 - z_3 ln\frac{z}{z_3}\right] \tag{4.3}$$

where k_1, k_2, k_3 are positive constants determined latter. The derivative of the above equation (4.3) along the solution of the system (2.1) is given by

$$
\begin{aligned}
\frac{dL}{dt} &= k_1\left[1-\frac{x_3}{x}\right]\dot{x}+k_2\left[1-\frac{y_3}{y}\right]\dot{y}+k_3\left[1-\frac{z_3}{z}\right]\dot{z}\\
&= k_1(x-x_3)\left[\frac{r}{1+ky}-d_1-h_1x-\frac{a_1(1-m_1)y}{b_1+(1-m_1)x}\right]\\
&\quad +k_2(y-y_3)\left[\frac{e_1a_1(1-m_1)x}{b_1+(1-m_1)x}-d_2-\frac{a_2(1-m_2)z}{b_2+(1-m_2)y}-h_2y\right]\\
&\quad +k_3(z-z_3)\left[\frac{e_2a_2(1-m_2)y}{b_2+(1-m_2)y}-d_3-h_3z\right]
\end{aligned}
$$

At the equilibrium point E_3 of the system (2.1), we have

$$
\begin{aligned}
d_1 &= \frac{r}{1+ky_3}-h_1x_3-\frac{a_1(1-m_1)y_3}{b_1+(1-m_1)x_3},\\
d_2 &= \frac{e_1a_1(1-m_1)x_3}{b_1+(1-m_1)x_3}-\frac{a_2(1-m_2)z_3}{b_2+(1-m_2)y_3}-h_2y_3,\\
d_3 &= \frac{e_2a_2(1-m_2)y_3}{b_2+(1-m_2)y_3}-h_3z_3.
\end{aligned}
\tag{4.4}
$$

Using (4.4), the time derivative of L becomes

$$
\begin{aligned}
\frac{dL}{dt} &= k_1(x-x_3)\Bigg[-h_1(x-x_3)+\frac{r}{1+ky}-+\frac{r}{1+ky_3}-\frac{a_1(1-m_1)y}{b_1+(1-m_1)x}\\
&\quad +\frac{a_1(1-m_1)y_3}{b_1+(1-m_1)x_3}\Bigg]+k_2(y-y_3)\Bigg[\frac{e_1a_1(1-m_1)x}{b_1+(1-m_1)x}-\frac{e_1a_1(1-m_1)x_3}{b_1+(1-m_1)x_3}\\
&\quad -\frac{a_2(1-m_2)z}{b_2+(1-m_2)y}+\frac{a_2(1-m_2)z_3}{b_2+(1-m_2)y_3}-h_2(y-y_3)\Bigg]+k_3(z-z_3)\\
&\quad \left[\frac{e_2a_2(1-m_2)y}{b_2+(1-m_2)y}-\frac{e_2a_2(1-m_2)y_3}{b_2+(1-m_2)y_3}-h_3(z-z_3)\right],\\
&= k_1(x-x_3)\Bigg[-h_1(x-x_3)-\frac{rk(y-y_3)}{(1+ky)(1+ky_3)}+\frac{a_1y_3(1-m_1)^2(x-x_3)}{\{b_1+(1-m_1)x\}\{b_1+(1-m_1)x_3\}}\\
&\quad -\frac{a_1(1-m_1)(y-y_3)}{b_1+(1-m_1)x}\Bigg]+k_2(y-y_3)\Bigg[\frac{e_1a_1b_1(1-m_1)(x-x_3)}{\{b_1+(1-m_1)x\}\{b_1+(1-m_1)x_3\}}\\
&\quad +\frac{a_2z_3(1-m_2)^2(y-y_3)}{\{b_2+(1-m_2)y\}\{b_2+(1-m_2)y_3\}}-\frac{a_2(1-m_2)(z-z_3)}{b_2+(1-m_2)y}-h_2(y-y_3)\Bigg]\\
&\quad +k_3(z-z_3)\left[\frac{e_2a_2b_2(1-m_2)(y-y_3)}{\{b_2+(1-m_2)y\}\{b_2+(1-m_2)y_3\}}-h_3(z-z_3)\right]\\
&\le k_1(x-x_3)\Bigg[-h_1(x-x_3)-\frac{rk(y-y_3)}{(1+ky_3)}+\frac{a_1y_3(1-m_1)^2(x-x_3)}{b_1\{b_1+(1-m_1)x_3\}}\\
&\quad -\frac{a_1(1-m_1)(y-y_3)}{b_1}\Bigg]+k_2(y-y_3)\Bigg[\frac{e_1a_1b_1(1-m_1)(x-x_3)}{b_1\{b_1+(1-m_1)x_3\}}\\
&\quad +\frac{a_2z_3(1-m_2)^2(y-y_3)}{b_2\{b_2+(1-m_2)y_3\}}-\frac{a_2(1-m_2)(z-z_3)}{b_2}-h_2(y-y_3)\Bigg]\\
&\quad +k_3(z-z_3)\left[\frac{e_2a_2b_2(1-m_2)(y-y_3)}{b_2\{b_2+(1-m_2)y_3\}}-h_3(z-z_3)\right]
\end{aligned}
$$

Now we choose k_1, k_2, k_3 in such a way that the coefficients of $(x-x_3)(y-y_3)$, $(y-y_3)(z-z_3)$ will be zero i.e. $\frac{e_1a_1b_1(1-m_1)k_2}{b_1\{b_1+(1-m_1)x_3\}}-\frac{a_1(1-m_1)k_1}{b_1}-\frac{rkk_1}{(1+ky_3)}=0$,
$\frac{e_2a_2b_2(1-m_2)k_3}{b_2\{b_2+(1-m_2)y_3\}}-\frac{a_2(1-m_2)k_2}{b_2}=0$. Taking $k_3=1$, we get $k_2=\frac{e_2b_2}{b_2+(1-m_2)y_3}$, $k_1=\frac{e_1a_1b_1(1-m_1)k_2}{b_1\{b_1+(1-m_1)x_3\}}\frac{b_1(1+ky_3)}{a_1(1-m_1)(1+ky_3)+rkb_1}$.

Now,

$$\frac{dL}{dt} \leq -[Ak_1(x-x_3)^2 + Bk_2(y-y_3)^2 + h_3k_3(z-z_3)^2],$$

where $A = h_1 - \frac{a_1 y_3(1-m_1)^2}{b_1\{b_1+(1-m_1)x_3\}}$, $B = h_2 - \frac{a_2 z_3(1-m_2)^2}{b_2\{b_2+(1-m_2)y_3\}}$.
Hence L will be the lyapunov function and the system (2.1) will be globally asymptotically stable around the coexistence equilibrium point if $A>0$ and $B>0$ i.e. $h_1b_1\{b_1+(1-m_1)x_3\} > a_1y_3(1-m_1)^2$, $h_2b_2\{b_2+(1-m_2)y_3\} > a_2z_3(1-m_2)^2$.

4.4.3: The system enters into a Hopf-bifurcation at E_3 for $\lambda = \lambda_i$, for a suitable value $k = k^{[2HB]}$ if $A_1A_2 - A_3 = 0$ hold.
Proof: The Routh-Hurwitz conditions are satisfied, as mentioned above, if we assume $J_{11} < 0$, $J_{22} < 0$, $J_{21} > 0$ and $J_{32} > 0$. To have a Hopf bifurcation, we need however $A_1A_2 = A_3$ for some value of k, say $k = k^{[2HB]}$. Since $A_2 > 0$ at $k = k^{[2HB]}$, for some $k > \epsilon > 0$ there is an interval $(k^{[2HB]} - \epsilon, k^{[2HB]} + \epsilon)$ in which $A_2 > 0$. Thus in this interval the characteristic equation cannot have real positive roots.
Now, for $k = k^{[2HB]}$, the characteristic equation factorizes $(\lambda^2 + A_2)(\lambda + A_1) = 0$ to give the three roots $\lambda_1 = i\sqrt{A_2}$, $\lambda_2 = -i\sqrt{A_2}$, $\lambda_3 = -A_1$. These roots are functions of $k \in (k^{[2HB]} - \epsilon, k^{[2HB]} + \epsilon)$ and can therefore be written as $\lambda_1 = \alpha(k) + i\beta(k)$, $\lambda_2 = \alpha(k) - i\beta(k)$, $\lambda_3 = -A_1(k)$.

Now we verify the transversality condition

$$Re\left(\frac{d\lambda_i}{dk}\right)|_{k=k^{[2HB]}} \neq 0, \quad i = 1,2.$$

Substituting $\lambda_j = \alpha(k) + i\beta(k)$, $j = 1,2$, into the characteristic equation and differentiating w.r.t k, we have

$$\omega(k)\alpha'(k) - \phi(k)\beta'(k) + \eta(k) = 0,$$
$$\phi(k)\alpha'(k) + \omega(k)\beta'(k) + \mu(k) = 0,$$

where

$$\omega(k) = 3\alpha^2(k) + 2A_1(k)\alpha(k) + A_2(k) - 3\beta^2(k), \quad (4.5)$$
$$\phi(k) = 6\alpha(k)\beta(k) + A_1(k)\beta(k), \quad (4.6)$$
$$\eta(k) = \alpha^2(k)A_1'(k) + A_2'(k)\alpha(k) + A_3'(k) - A_1'(k)\beta^2(k), \quad (4.7)$$
$$\mu(k) = 2\alpha(k)\beta(k)A_1'(k) + A_2'(k)\beta(k). \quad (4.8)$$

Since $\phi(k)\mu(k) + \omega(k)\eta(k) \neq 0$, we have

$$Re\left(\frac{d\lambda_j}{dk}\right)|_{k=k^{[2HB]}} = -\frac{\phi\mu + \omega\eta}{\phi^2 + \omega^2} \neq 0, \quad j = 1,2, \quad \lambda_3(k) = -A_1(k) \neq 0.$$

Hence, the claim.

5 Numerical simulation

In this section, the dynamical behaviour of the food chain model (2.1) are studied numerically to support our analytical results. The numerical simulation about the co-existence equilibrium point E_3 based on various model parameter values satisfying the criteria mentioned above is undertaken for the purpose of present model (2.1) validation and complete understanding the dynamical behavior. For numerical computations, we solve the system of ordinary differential equations by Runge-Kutta fourth order method and made use of software packages like Maple 16 and MATLAB R2010a with due attention on developing necessary codes for the purpose. Several diagrams are exhibited in order to illustrate the complex dynamics of the proposed model keeping in mind the entire analytical findings. The consequences of various diagrams relating to phase portraits and bifurcations behavior are presented for the purpose of understanding the nature of the system under consideration.

Figure 1 exhibits the time series and phase portrait representations of stable steady state behaviour of the system around the predator free equilibrium E_1. Figure 2 shows the hopf-bifurcation behaviour around E_2 representing in the time series and phase portrait for the set of following parameter values $r = 2.0$, $k = 0.5$, $a_1 = 2.0$, $a_2 = 0.3$, $b_1 = 1.9$, $b_2 = 1.0$, $e_1 = 0.9$, $e_2 = 0.8$, $m_1 = 0.1$, $m_2 = 0.1$, $d_1 = 0.3$, $d_2 = 0.3$, $d_3 = 0.2$, $h_1 = 0.1$, $h_2 = 0.1$, $h_3 = 0.1$. Keeping all the parameters same, if we increase the value of b_1 from 1.9 to 3.2, periodic behaviour of the system changes into stable steady state behaviour which is represented in the time series and phase portrait in Figure 3. Figure 5 exhibits the time series and phase portrait representations around E_3 for prey, predator and top-predator species over a large span of time. All the species appear to follow an undulating trend towards the onset which continues for a short period of time and eventually they all become invariant for rest of the time. One may note that the system becomes globally stable around the interior equilibrium E_3 as shown in Figure 5 for the following set of model parameter values: $r = 1.4$, $k = 0.9$, $a_1 = 2.0$, $a_2 = 1.0$, $b_1 = 3.2$, $b_2 = 1.0$, $e_1 = 0.9$, $e_2 = 0.8$, $m_1 = 0.1$, $m_2 = 0.1$, $d_1 = 0.3$, $d_2 = 0.3$, $d_3 = 0.2$, $h_1 = 0.1$, $h_2 = 0.1$, $h_3 = 0.1$. Subsequently, we focus our attention to the occurrence of Hopf-bifurcation phenomena because of its importance in the present dynamical system which is responsible for both switching and periodic solutions of the system. In the event of tumbling the equilibrium position of the model from its local stability, Hopf-bifurcation sets in with respect to the model parameter k. This situation may be illuminated mathematically that a conjugate pair of purely imaginary eigenvalues crosses the imaginary axis in the complex plane. Figure 4(a) exhibits the limit cycle stability of the model system for a specific value of $k = 0.1$ in three dimensional spaces while the corresponding time series delineations for all the species involved are included in Figure 4(b). All these trajectories are found to be periodic with respective fixed amplitudes over the entire period of time. Global stability behaviour of the system around coexistence equilibrium E_3 is presented in Figure 6. Hopf bifurcation situatuin around E_3 is depicted in Figures 7, 8, 9, 10 for the system parameters r, k, m_1, $m2$ respectively. Stabilizing potential of intra-specific competition among predator populations are shown in Figure 11.

In Figure 12, the infuence of fear on the system behaviour in absence of prey refuge effect are recorded. Influence of prey refuge in absence of fear effect are represented in Figure 13. Despite the response of the present scenario, the concluding Figure 14 depict the influence of fear in presence of prey refuge effect on the population of prey, predator and top-predator species in the system under consideration. Thus the growth rate of the population experiences substantial impact due to the presence of fear factor in the system under study.

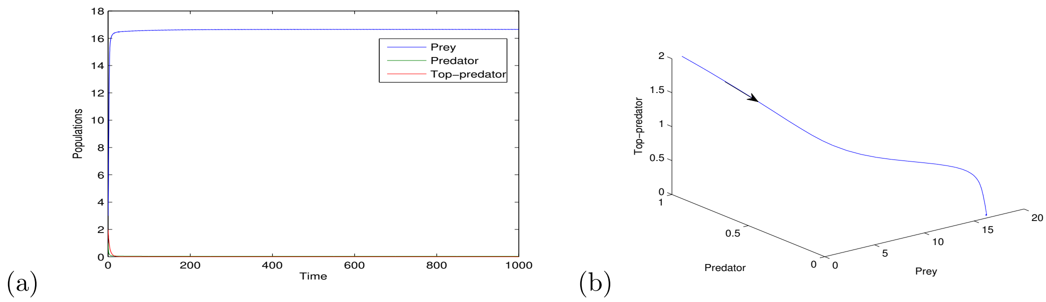

Figure 1: Solution trajectories tends to the equilibrium E_1. Here $r = 2.0$, $k = 0.5$, $a_1 = 2.0$, $a_2 = 0.3$, $b_1 = 3.2$, $b_2 = 1.0$, $e_1 = 0.9$, $e_2 = 0.8$, $m_1 = 0.1$, $m_2 = 0.1$, $d_1 = 0.3$, $d_2 = 1.48$, $d_3 = 0.2$, $h_1 = 0.1$, $h_2 = 0.1$, $h_3 = 0.1$. (a) Time series, (b) Phase portrait.

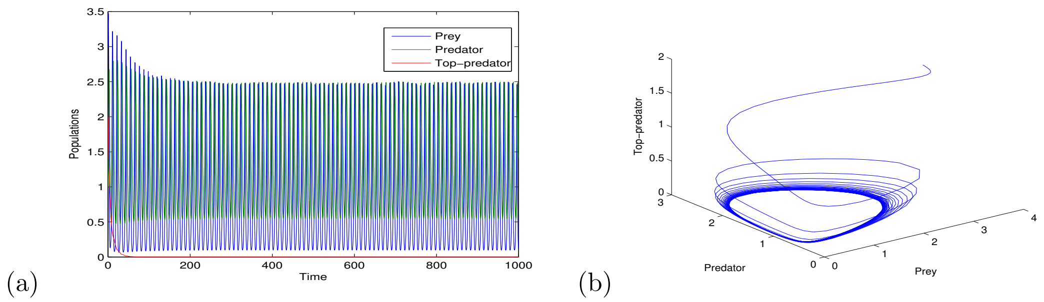

Figure 2: Hopf-bifurcation behaviour of model the system (2.1) around the equilibrium position E_2 for the set of parameter values $r = 2.0$, $k = 0.5$, $a_1 = 2.0$, $a_2 = 0.3$, $b_1 = 1.9$, $b_2 = 1.0$, $e_1 = 0.9$, $e_2 = 0.8$, $m_1 = 0.1$, $m_2 = 0.1$, $d_1 = 0.3$, $d_2 = 0.3$, $d_3 = 0.2$, $h_1 = 0.1$, $h_2 = 0.1$, $h_3 = 0.1$. (a) Time series, (b) Phase portrait.

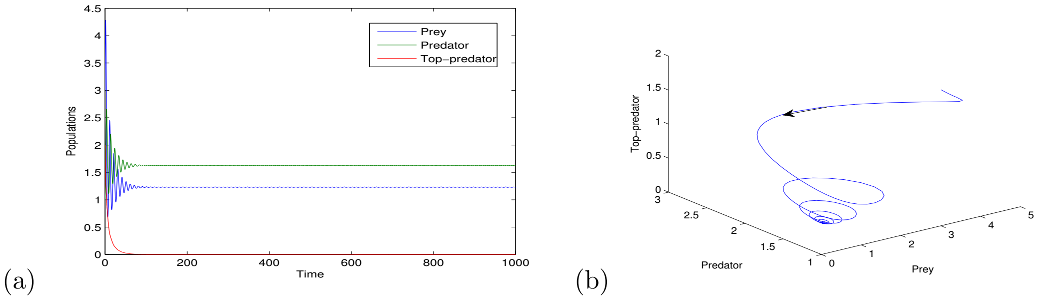

Figure 3: Solution trajectories tends to the equilibrium E_2. Here $r = 2.0$, $k = 0.5$, $a_1 = 2.0$, $a_2 = 0.3$, $b_1 = 3.2$, $b_2 = 1.0$, $e_1 = 0.9$, $e_2 = 0.8$, $m_1 = 0.1$, $m_2 = 0.1$, $d_1 = 0.3$, $d_2 = 0.3$, $d_3 = 0.2$, $h_1 = 0.1$, $h_2 = 0.1$, $h_3 = 0.1$. (a) Time series, (b) Phase portrait.

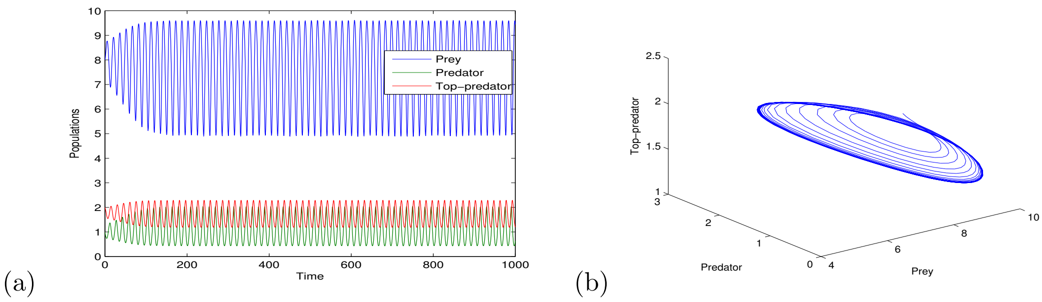

Figure 4: Hopf-bifurcation behaviour of model the system (2.1) around the equilibrium position E_3 for the set of parameter values $r = 1.4$, $k = 0.1$, $a_1 = 2.0$, $a_2 = 1.0$, $b_1 = 3.2$, $b_2 = 1.0$, $e_1 = 0.9$, $e_2 = 0.8$, $m_1 = 0.1$, $m_2 = 0.1$, $d_1 = 0.3$, $d_2 = 0.3$, $d_3 = 0.2$, $h_1 = 0.1$, $h_2 = 0.1$, $h_3 = 0.1$. (a) Time series, (b) Phase portrait.

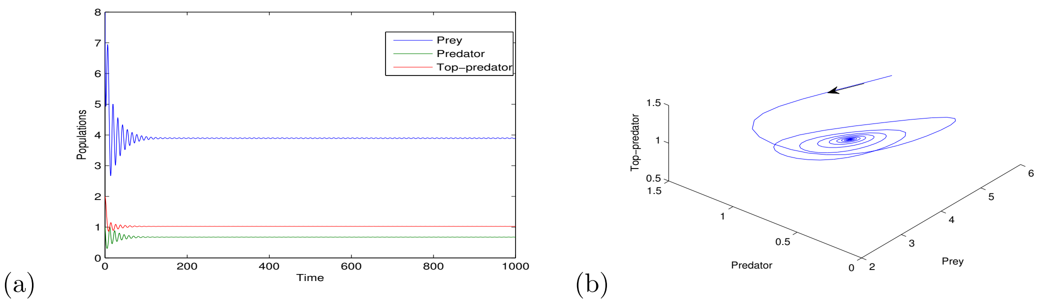

Figure 5: Solution trajectories tends to the equilibrium E_3. Here $r = 1.4$, $k = 0.9$, $a_1 = 2.0$, $a_2 = 1.0$, $b_1 = 3.2$, $b_2 = 1.0$, $e_1 = 0.9$, $e_2 = 0.8$, $m_1 = 0.1$, $m_2 = 0.1$, $d_1 = 0.3$, $d_2 = 0.3$, $d_3 = 0.2$, $h_1 = 0.1$, $h_2 = 0.1$, $h_3 = 0.1$. (a) Time series, (b) Phase portrait.

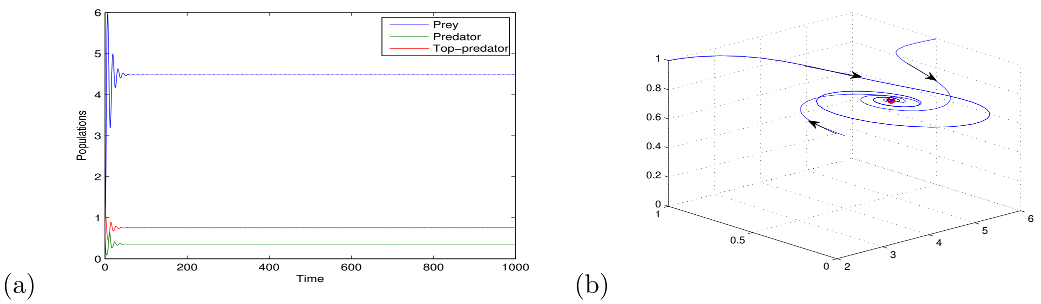

Figure 6: Global stability behaviour around the equilibrium E_3. Global stability conditions $h_1 b_1\{b_1 + (1 - m_1)x_3\} = 25.12970221 > a_1 y_3(1 - m_1)^2 = 0.5752640511$, $h_2 b_2\{b_2 + (1 - m_2)y_3\} = 2.152201223 > a_2 z_3(1 - m_2)^2 = 1.530052850$ are satisfied for the set of parameter values $r = 1.4$, $k = 0.9$, $a_1 = 2.0$, $a_2 = 1.0$, $b_1 = 3.2$, $b_2 = 1.0$, $e_1 = 0.9$, $e_2 = 0.8$, $m_1 = 0.1$, $m_2 = 0.1$, $d_1 = 0.3$, $d_2 = 0.3$, $d_3 = 0.2$, $h_1 = 0.1$, $h_2 = 0.1$, $h_3 = 0.1$. (a) Time series, (b) Phase portrait.

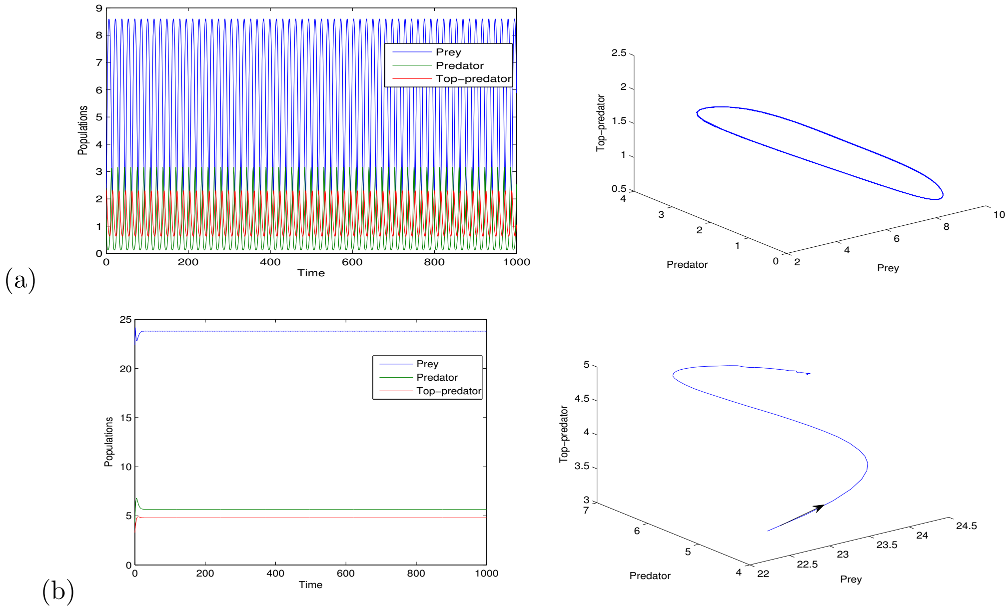

Figure 7: (a) Hopf-bifurcation around the equilibrium E_3 for $r = 1.2$ and (b) stable behaviour for $r = 3.1$. The other parameter values are $k = 0.0$, $a_1 = 2.0$, $a_2 = 1.0$, $b_1 = 3.2$, $b_2 = 1.0$, $e_1 = 0.9$, $e_2 = 0.8$, $m_1 = 0.0$, $m_2 = 0.0$, $d_1 = 0.3$, $d_2 = 0.3$, $d_3 = 0.2$, $h_1 = 0.1$, $h_2 = 0.1$, $h_3 = 0.1$.

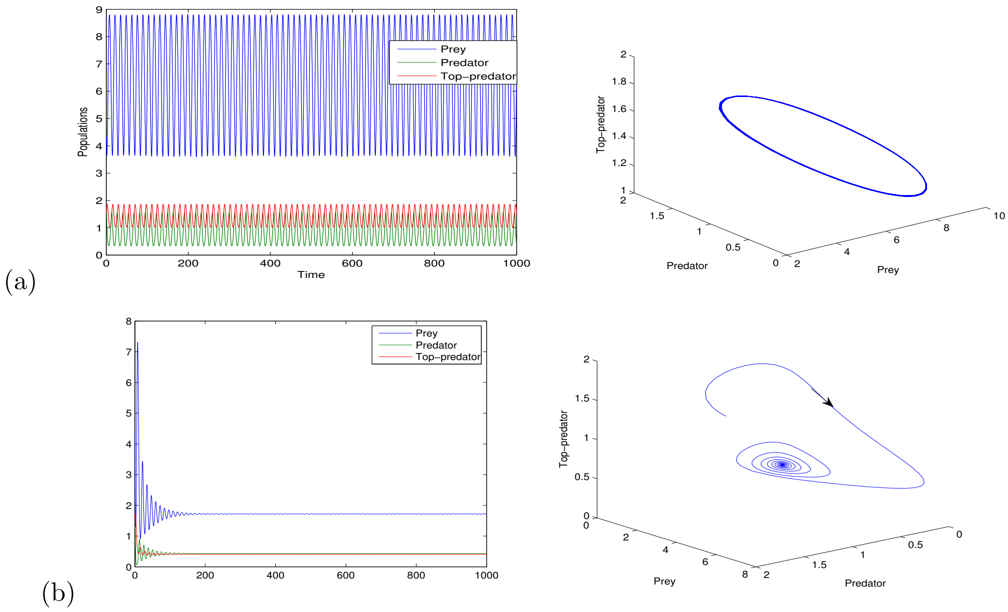

Figure 8: (a) Hopf-bifurcation around the equilibrium E_3 for $k = 0.3$ and (b) stable behaviour for $k = 2.7$. The other parameter values are $r = 1.4$, $a_1 = 2.0$, $a_2 = 1.0$, $b_1 = 3.2$, $b_2 = 1.0$, $e_1 = 0.9$, $e_2 = 0.8$, $m_1 = 0.0$, $m_2 = 0.0$, $d_1 = 0.3$, $d_2 = 0.3$, $d_3 = 0.2$, $h_1 = 0.1$, $h_2 = 0.1$, $h_3 = 0.1$.

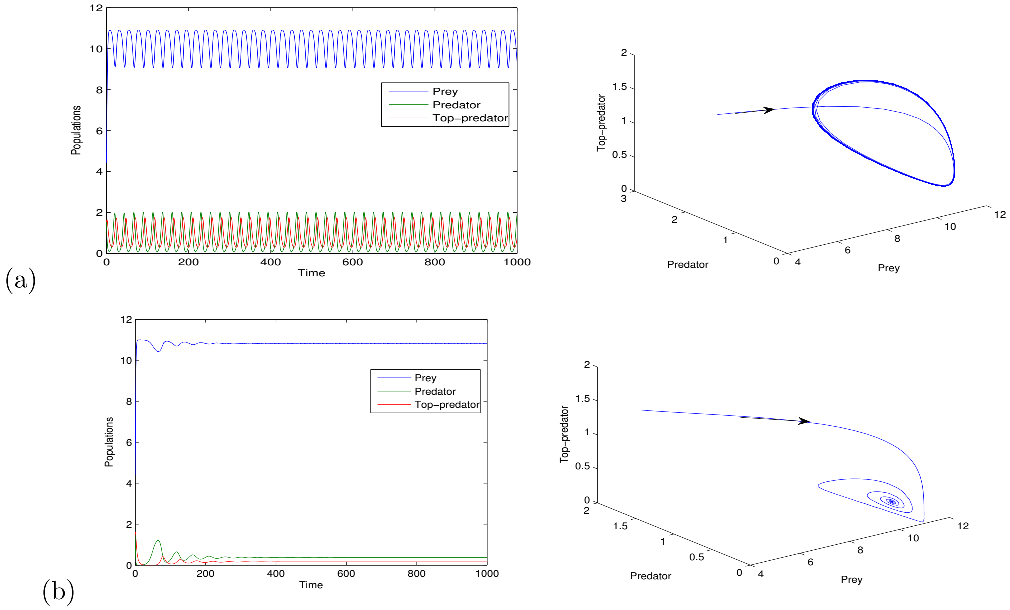

Figure 9: (a) Hopf-bifurcation around the equilibrium E_3 for $m_1 = 0.7$ and (b) stable behaviour for $m_1 = 0.9$. The other parameter values are $r = 1.4$, $k = 0.0$, $a_1 = 2.0$, $a_2 = 1.0$, $b_1 = 3.2$, $b_2 = 1.0$, $e_1 = 0.9$, $e_2 = 0.8$, $m_2 = 0.0$, $d_1 = 0.3$, $d_2 = 0.3$, $d_3 = 0.2$, $h_1 = 0.1$, $h_2 = 0.1$, $h_3 = 0.1$.

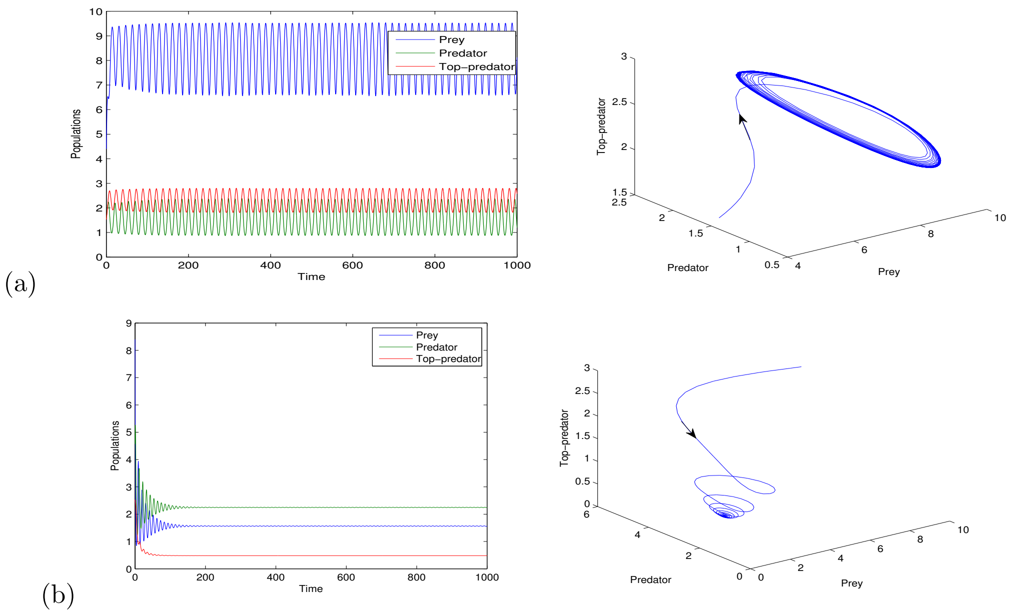

Figure 10: (a) Hopf-bifurcation around the equilibrium E_3 for $m_2 = 0.2$ and (b) stable behaviour for $m_2 = 0.8$. The other parameter values are $r = 1.4$, $k = 0.0$, $a_1 = 2.0$, $a_2 = 1.0$, $b_1 = 3.2$, $b_2 = 1.0$, $e_1 = 0.9$, $e_2 = 0.8$, $m_1 = 0.0$, $d_1 = 0.3$, $d_2 = 0.3$, $d_3 = 0.2$, $h_1 = 0.1$, $h_2 = 0.1$, $h_3 = 0.1$.

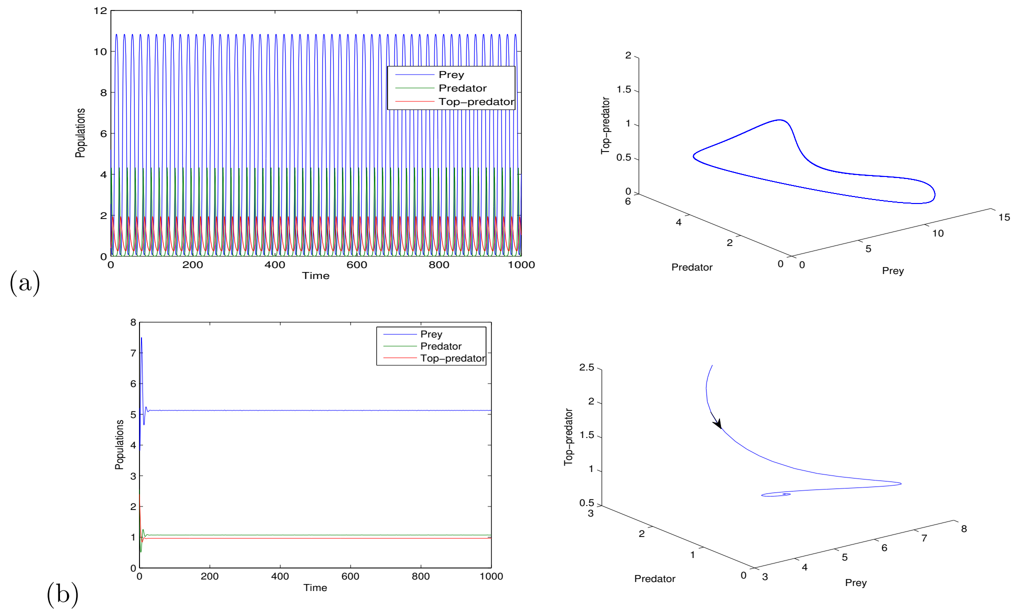

Figure 11: Stabilizing potential of intra-specific competetion among predator populations. (a) Limit cycle behaviour for $h_2 = 0.0$, $h_3 = 0.0$ (b) stable behaviour for $h_2 = 0.3$, $h_3 = 0.2$. The other parameter values are $r = 1.4$, $k = 0.3$, $a_1 = 2.0$, $a_2 = 1.0$, $b_1 = 3.2$, $b_2 = 1.0$, $e_1 = 0.9$, $e_2 = 0.8$, $m_1 = 0.1$, $m_2 = 0.1$, $d_1 = 0.3$, $d_2 = 0.3$, $d_3 = 0.2$, $h_1 = 0.1$.

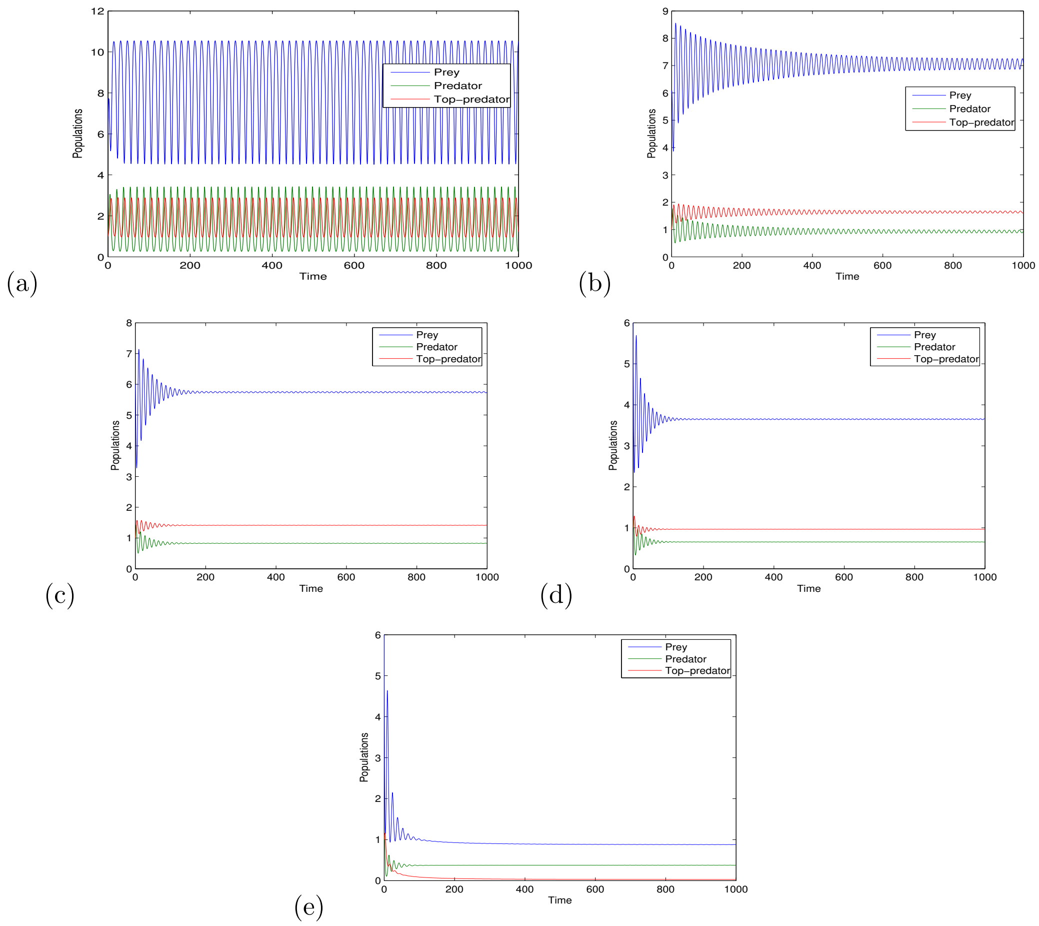

Figure 12: Influence of fear effect on the system behaviour around the equilibrium E_3. (a) $k = 0$, (b) $k = 0.2$, (c) $k = 0.4$, (d) $k = 1.0$, (e) $k = 4.0$. The other parameter values are same as mentioned in Figure 5.

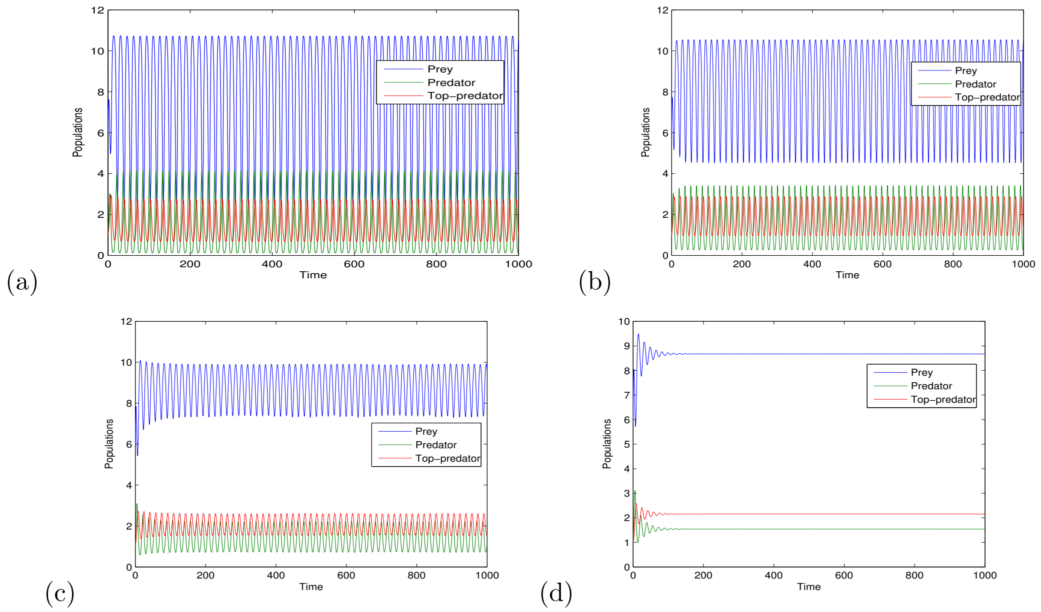

Figure 13: Influence of prey refuge effect on the system behaviour around the equilibrium E_3. (a) $m_1 = 0.0, m_2 = 0.0$ (b) $m_1 = 0.1, m_2 = 0.1$, (c) $m_1 = 0.2, m_2 = 0.2$, (d) $m_1 = 0.3, m_2 = 0.3$. The other parameter values are same as mentioned in Figure 5.

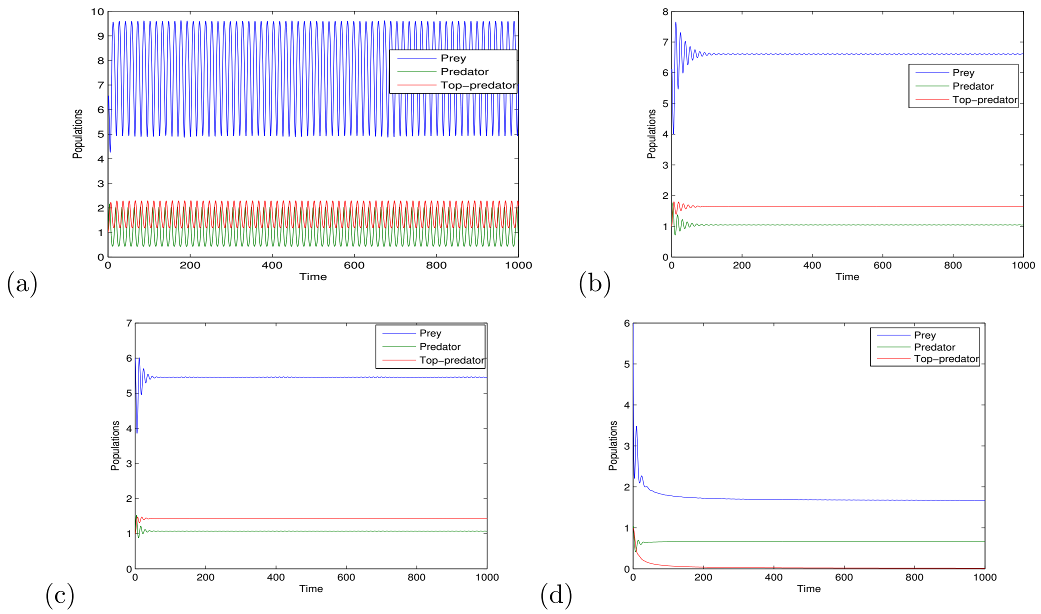

Figure 14: Influence of prey refuge and fear effect on the system behaviour around the equilibrium E_3. (a) $k = 0.1, m_1 = 0.1, m_2 = 0.1$, (b) $k = 0.2, m_1 = 0.2, m_2 = 0.2$, (c) $k = 0.3, m_1 = 0.3, m_2 = 0.3$, (d) $k = 1.8, m_1 = 0.5, m_2 = 0.5$. The other parameter values are same as mentioned in Figure 5.

6 Conclusions and future directions

The model system under consideration is dealt with three interacting species designated by the prey, the predator and the top-predator. This is basically a food web model where predation is executed following a sequence in which the predator captures prey and the top predator takes the middle predator into its custody resulting in setting up an interesting food chain model. Inclusion of logistic growth for prey species together with Holling type II functional response for both the predator and the top predator put up the model more interesting especially for the predators role in food collection. Special emphasis is paid on the impact of fear in presence of constant proportion of prey refuge in the system to examine how the growth of the interacting species gets substantially perturbed. The incorporation of the consequences of mortality together with the intra-specific competition among predator species in the present pursuit is not ruled out however.

The present article bears the potential for its merit on several counts. First, the rightful analytical proof concerning boundedness of the model system as well provided in the text affirm good foundation. Secondly, both the local and the global stability analysis for the ecologically feasible equilibrium positions are carried out theoretically and duly validated numerically in order to uphold the merit of the investigation. Thirdly, the model switches off from limit cycle to stable phase with respect to key model parameters b_1 and k. This feature adds another novelty of the work undertaken. Fourth merit counts the influence of fear factor in presence of prey refuge on the growth rate of the entire population under investigation. Finally, the fear and prey refuge effect deserve their need for rectification of the system dynamics as it may control instability of the model system. One may add further that appropriate level of intra-specific competition accomplishes the predation pressure on both the prey and the predator resulting in the reduction of mortality risk of both the predator species.

Table 1: Comparison between the model considered by Sk. et. al. [37] and the revised model (2.1). LAS≡ Locally asymptotically stable & GAS≡ Globally asymptotically stable.

Sl.No.	**Results of the model ([37]**	**Results of the model (2.1)**
1	E_0 is unstable.	E_0 is unstable.
2	E_1 is LAS.	E_1 is LAS.
3	E_3 is LAS.	E_3 is LAS.
4	E_4 is LAS.	E_4 is LAS.
5	Hopf-bifurcation	Hopf-bifurcation
6	-	E_4 is GAS.
7	-	Stabilizing potential of k (Fig. 12).
8	-	Stabilizing potential of m_1, m_2 (Fig. 13).
9	-	Stabilizing potential of h_1, h_2 (Fig. 11).

One can again study the following food chain model to get the richer dynamics incorporating hunting cooperation effect in model 2.1 as follows:

$$\frac{dx}{dt} = \frac{rx}{1+ky} - d_1x - h_1x^2 - \frac{a_1(1+\alpha_1 y)(1-m_1)xy}{b_1+(1+\alpha_1 y)(1-m_1)x}, \quad x(0)>0, \tag{6.1a}$$

$$\frac{dy}{dt} = \frac{e_1a_1(1+\alpha_1 y)(1-m_1)xy}{b_1+(1+\alpha_1 y)(1-m_1)x} - d_2y - \frac{a_2(1+\alpha_2 z)(1-m_2)yz}{b_2+(1+\alpha_2 z)(1-m_2)y} - h_2y^2, \quad y(0)>0, \tag{6.1b}$$

$$\frac{dz}{dt} = \frac{e_2a_2(1+\alpha_2 z)(1-m_2)yz}{b_2+(1+\alpha_2 z)(1-m_2)y} - d_3z - h_3z^2, \quad z(0)>0, \tag{6.1c}$$

where all the variables and parameters are defined earlier except α_1, α_2. The parameters α_i $(i=1,2)$ represents the coefficients of hunting cooperation of predator and top-predator respectively. The model 6.1 is different from the model considered by Sk et. al [37] where intra-specific competition among predator populations are not considered. One can again incorporate time delay in the model 6.1 to get more richer behaviour of food chain model.

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