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Weak Galerkin finite element method for the nonlinear Schrodinger equation

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Abstract

The numerical technique for a two-dimensional time-dependent nonlinear Schrodinger equation is the subject of this work. The approximations are produced using the weak Galerkin finite element technique with semi-discrete and fully discrete finite element methods, respectively, using the backward Euler method and the crank-Nicolson method in time. Using the elliptic projection operator, we provide optimum L^2 error estimates for semi and fully discrete weak Galerkin finite elements. Finally, we present numerical examples provided to verify our theoretical results.

Keywords: WGFEM, nonlinear Schrodinger equation, semi-discrete, Fully discrete (backward Euler scheme, Crank-Nicolson scheme), error estimates
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1 Introduction

In this section, we consider the following initial boundary value problem for the two dimensional time dependent nonlinear Schrodinger equation. We seek complex-valued function $u = u(x, y, t)$ defined on $\Omega \times [0, T]$ satisfying [10]

$$iu_t = -\Delta u + gu + |u|^2 u + f(x, y, t) \in \Omega \times [0, T] \tag{1.1}$$

$$\begin{array}{ll} u(x, y, t) = 0 & , (x, y, t) \in \partial\Omega \times [0, T] \\ (x, y, 0) = u_0(x, y), & (x, y) \in \Omega, \end{array}$$

where $i = \sqrt{-1}$ is the complex unit, $\Omega \in \mathbb{R}^2$ is a convex polygonal domain, Δ is the usual Laplace operator and $u_0(x, y)$ is a given complex valued initial data. Function $g(x, y)$ is a given real -valued external potential and non-negative bounded for all $(x, y) \in \Omega$. Function $f(x, y, t)$ is complex-valued .Schrodinger equation is the fundamental equation of physics for describing quantum mechanical behavior. Especially, the nonlinear Schrodinger equation can be met in many different areas of physics and chemistry. There are numerous works in the literature to solve the Schrodinger equations. At present the common numerical methods are finite difference method [1, 3, 5], spectral method [2, 4],two grid method [6, 12],mixed finite method [8, 13]. Bao et al [2] studied the performance of time -splitting spectral approximations for general nonlinear Schrodinger equation in the semi classical regimes. Jin et al [12],solved the time-dependent Schrodinger equation by the finite element two-grid method and analyzed the convergence. The semidiscrete

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schemes are proved to be convergent with an optimal convergence order and the full-discrete schemes are verified by a numerical example . Rest of the paper is organized as follows .In section 2 we introduce the definition of discrete weak gradient and weak function space . In section 3 , we define weak Galerkin space, some lemmas which are necessary in error estimate, and introduce the semi-discrete formulation,. In section 4 , the error analysis of semi-discrete WGFEM. In section 5 ,the error analysis of fully-discrete WGFEM (backward WGFEM, Crank-Nicolson WGFEM). In section 6, numerical experiments are given.

2 Preliminaries

In this section, we shall introduce some weak differential operators, such as weak gradient, and then we introduce some important weak function spaces which are useful in the error analysis of WGFEM. Let $K \subset \Omega$ be any polygonal domain with boundary ∂K.For any triangle $K \in T_h$,such that T_h be the quasi-uniform triangular partition of Ω with mesh size $0 < h < 1$. A weak function $v = \{v_0, v_b\}$ on K has two parts, $v_0 \in L^2(K)$ and $v_b \in L^2(\partial K)$, the first part represents the values of v in the interior K and the second part on triangle boundary ∂K.The space of weak functions and corresponding vector space defined on K is given by

$$W(K) = \left\{v = \{v_0, v_b\}, v_0 \in L^2(K), v_b \in L^2(\partial K)\right\}. \tag{2.1}$$

Define a space

$$H(\mathrm{div}, K) = \left\{v, v \in \left(L^2(K)\right)^2, \nabla \cdot v \in L^2(K)\right\}. \tag{2.2}$$

Then the weak form of problem (1.1) is defined as follows for all $t \in [0, T]$, find $u(x, y, t) \in H_0^1(\Omega)$ such that

$$(iu_t, v) = (\nabla u, \nabla v) + (gu, v) + \left(|u|^2 u, v\right) + (f, v), \forall v \in H_0^1(\Omega), t > 0 \tag{2.3}$$

$$\begin{aligned} &u(x, y, t) \in \Omega \times [0, T] \\ &u(x, y, 0) = u_0(x, y), (x, y) \in \Omega \\ &u(x, y, t) = 0 (x, y, t) \in \partial\Omega \times [0, T] \end{aligned}$$

We define the normas follow

$$\|u\|_w = \left(\|\nabla u\|^2 + \|u\|^2\right)^{\frac{1}{2}}. \tag{2.4}$$

The space of discrete weak function denote by $W(K, j, l)$ is

$$W(K, j, l) = \left\{v = \left\{v^0, v^b\right\}, v^0 \in P_j\left(K^0\right) \text{ and } v^b \in P_l(\partial K)\right\}. \tag{2.5}$$

Definition 2.1. Let $v \in W(K)$, the weak gradient operator of v is defined as a linear functional $\nabla_{d,r} v \in H(\text{ div,}K)$ for each $K \in T_h$, as following:

$$\int_K \nabla_{d,r} v \cdot q dk = -\int_K v_0 (\nabla \cdot q) dK + \int_{\partial K} v_b (q.n) ds, \forall q \in H(div, K) \tag{2.6}$$

Where n is the outward normal direction of ∂K.

Definition 2.2. Let $v \in W(K)$,the discrete weak gradient operator of v is defined as unique polynomial $\nabla_{d,r} v \in [P_{K-1}(K)]^2$ on each element K, by the following equation:

$$\int_K \nabla_{d,r} v \cdot q dK = -\int_K v_0 (\nabla \cdot q) dK + \int_{\partial K} v_b (q \cdot n) ds, \forall q \in [P_{k-1}(K)]^2 \tag{2.7}$$

3 Weak Galerkin finite element space

We define the weak Galerkin space

$$\emptyset_h = \left\{ v = \left\{ v^0, v^b \right\} \in W(K, j, l), \forall K \in T_h \right\} \tag{3.1}$$

and

$$\emptyset_h^0 = \left\{ v = \left\{ v^0, v^b \right\} \in \emptyset_h, \, v^b \big|_{\partial K \cap \partial \Omega} = 0, \forall K \in T_h \right\} \tag{3.2}$$

Denoted by $P_r(K)$ the set of polynomials on K with degree no more than r. For any non-negative integer $r \geq 0$, let $V(K, r) \subset [P_r(K)]^2$ be a subspace of the space of vector-valued polynomials of degree r. Let T_h be a quasi-uniform rectangular partition of Ω with mesh size $0 < h < 1$, and $\emptyset_h$ be the corresponding piecewise bilinear finite element subspace in $H_0^1(\Omega)$. In general ,for any fixed $t \in [0, T]$ and given $u(x, y, t) \in H_0^1(\Omega)$, we can define its elliptic projection $\pi_h u(x, y, t) \in \emptyset_h$ such that

$$a\left(\pi_h u, v_h\right) = a\left(u, v_h\right), \quad \forall v \in \emptyset_h \tag{3.3}$$

And the projection π_h is assumed to exist and satisfy the following property: $\forall q \in H(\text{div}, \Omega)$ with mildly regularity, $\pi_h q \in H(\text{ div}, K)$ such that $\pi_h q \in V(K, r)$ on each $K \in T_h$ and satisfying

$$\left(\nabla \cdot q, v^0\right)_K = \left(\nabla \cdot \pi_h q, v^0\right)_K, \forall v^0 \in P_j\left(K^0\right) \tag{3.4}$$

$$\pi_h u(x) = u(x), x = (x_i, y_j)\, i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m \tag{3.5}$$

$$\|u - \pi_h u\| \leq C h^{\mu+1} \|u\|_{\mu+1} \quad 0 \leq \mu \leq r + 1 \tag{3.6}$$

Also there exist two projections are defined for each element $K \in T_h$,one is $Q_h u = \left\{ Q_h^0 u, Q_h^b u \right\}$, the L^2 projection of $H^1(K)$ on to $P_j\left(K^0\right) \times P_l(\partial K)$ and R_h the L^2 projection of $\left[L^2(K)\right]^2$ on to $V(K, r)$.
It is easy to see the following useful identity [9]:

$$\nabla_{d,r}\left(Q_h u\right) = R_h(\nabla u), \forall u \in H^1(K), \tag{3.7}$$

Such that $P_j\left(K^0\right)$ the set of polynomials on K^0 with degree no more than j, and $P_l(\partial K)$ the set of polynomials on ∂K with degree no more than l.

Definition 3.1. If $u \in H_0^1(\Omega) \cap H^{r+1}(\Omega), Q_h u \in \emptyset_h^0$.Then

$$\|Q_0 u - u\| \leq C h^{\mu} \|u\|_{\mu}, \quad 0 \leq \mu \leq r + 1 \tag{3.8}$$

$$\|\nabla_{d,r} Q_h u - \nabla u\| \leq C h^{\mu} \|u\|_{\mu+1}, \quad 0 \leq \mu \leq r + 1 \tag{3.9}$$

Definition 3.1 show that $Q_h u \in \emptyset_h^0$ is a very good approximation for function $u \in H_0^1(\Omega) \cap H^{r+1}(\Omega)$.

Lemma 3.2. [12] For any $q \in H(\text{div}, k)$, we have

$$\sum_{K \in T_h} \left(-\nabla \cdot q, v^0\right)_K = \sum_{K \in T_h} \left(\pi_h q, \nabla_{d,r} v\right)_K, \forall v = \left\{ v^0, v^b \right\} \in \emptyset_h^0(j, j+1)$$

Lemma 3.3. [7] For $u^n \in \left(L^2(0, t), L^{1+r}(\Omega)\right)$ with $r > 0$, we have

$$\left\|\pi_h\left(\nabla u^n\right) - \nabla_d\left(Q_h u^n\right)\right\| \leq C h^{1+r} \left\|u^n\right\|_{1+r}$$

The WG FEM based on (1.1) Which is to find $u_h = \left\{ u_h^0, u_h^b \right\} \in \emptyset_h(j, l)$ satisfying $u^b = Q^b$ on $\partial \Omega$ such that

$$\left(i u_t, v^0\right) = a_w\left(u_h, v\right) + \left(|u|^2 u, v^0\right) + \left(f, v^0\right), \forall v = \left\{ v^0, v^b \right\} \in \emptyset_h^0(j, l) \tag{3.10}$$

Where

$$a_w\left(u_h, v\right) = \left(\nabla_{d,r} u_h, \nabla_{d,r} v\right) + \left(g u_h^0, v^0\right) \tag{3.11}$$

Then the semi -discrete WG FEM is find $u_h = \left\{ u_h^0, u_h^b \right\} \in \emptyset_h$ such that

$$\begin{aligned} &\left(i u_{h,t}, v^0\right) - a_w\left(u_h, v\right) - \left(\left|u_h^0\right|^2 u_h^0, v^0\right) = \left(f, v^0\right), \forall v = \left\{ v^0, v^b \right\} \in \emptyset_h^0 \\ &u_h(x, y, 0) = \pi_h u_0(x, y) \end{aligned} \tag{3.12}$$

4 The Error Analysis of semi -discrete WGFE M

In this section, we drive error estimates for the semi- discrete WGFEM in the L^2- norm

Lemma 4.1. Let $u \in H^1\left(0, T, H^2(\Omega)\right)$ be the solution of problem (1.1) .Then

$$\left(iu_t, v^0\right)=\left(\pi_h\left(\nabla_{d,r} u\right), \nabla_{d,r} v\right)+\left(gu, v^0\right)+\left(|u|^2 u, v^0\right)+\left(f, v^0\right), \forall v \in \emptyset_h^0 \tag{4.1}$$

Proof . Multiply Equation (1.1) by v^0 and integration, we get

$$\left(iu_t, v^0\right)=-\left(\Delta u, v^0\right)+\left(gu, v^0\right)+\left(|u|^2 u, v^0\right)+\left(f, v^0\right) \tag{4.2}$$

For the term-($\Delta u, v^0\big)$ in equation (4.2) and Lemma (3.1), we have

$$-\left(\Delta u, v^0\right)=-\left(\nabla .(\nabla u), v^0\right)=\left(\pi_h \nabla_{d,r} u, \nabla_{d,r} v\right). \tag{4.3}$$

Substituting (4.3) in (4.2) which complete the proof. □

Lemma 4.2. Let u $\in H^{1+r}(\Omega)$ with $r>0$ and u_h be the solution of (1.1), $Q_h u=\left\{Q_h^0 u, Q_h^b u\right\}$ the L^2- projection of the exact solution $u(x,t)$.The error equation for the WGFEM is

$$\begin{aligned}&\left(-i\left(u_h-Q_h u\right)_t, v^0\right)+a_w\left(u_h-Q_h u, v\right)+\left(\left|u_h\right|^2\left(u_h-Q_h^0 u\right), v^0\right)= \\ &\left(\pi_h\left(\nabla_{d,r} u\right)-R_h(\nabla u), \nabla_{d,r} v\right)+\left(g\left(u-Q_h^0 u\right), v^0\right)+\left[\left(|u|^2\left(u-Q_h u\right), v^0\right)+\right. \\ &\left.\left(Q_h\left(\left(|u|+\left|u_h\right|\right)\left(u-Q_h u\right), v^0\right)+\left(Q_h\left(\left(|u|+\left|u_h\right|\right)\left(Q_h u-u_h\right), v^0\right)\right)\right]\end{aligned} \tag{4.4}$$

Where

$$a_w\left(u_h-Q_h u, v\right)=\left(\pi_h\left(\nabla_{d,r} u-R_h(\nabla u), \nabla_{d,r} v\right)+\left(g\left(u-Q_h^0 u\right), v^0\right)\right.$$

Proof . Let $v=\left\{v^0, v^b\right\} \in \emptyset_h$ be the testing function, Adding and subtracting the term

$$a_w\left(Q_h u, v\right)+\left(\left|u_h\right|^2 Q_h u, v^0\right). \tag{4.5}$$

To the left (4.1) and using $\left(iQ_h u_t, v^0\right)=\left(iu_t, v^0\right)$, we obtain

$$\begin{aligned}&\left(iQ_h u_t, v^0\right)=\left(\pi_h\left(\nabla_{d,r} u\right)-\nabla_{d,r}\left(Q_h u\right), \nabla_{d,r} v\right)+\left(g\left(u-Q_h u\right), v^0\right)+\left(|u|^2 u-\right. \\ &\left.\left|u_h\right|^2 Q_h u, v^0\right)+\left(f, v^0\right)+a_w\left(Q_h^0 u, v\right)+\left(\left|u_h\right|^2 Q_h u, v^0\right).\end{aligned} \tag{4.6}$$

From (3.12), we obtain

$$\begin{aligned}&\left(iu_{h,t}, v^0\right)-a_w\left(u_h, v\right)-\left(\left|u_h\right|^2 u_h, v^0\right)=\left(iQ_h u_t, v^0\right)-\left(\pi_h\left(\nabla_{d,r} u\right)-\right. \\ &\left.R_h(\nabla u), \nabla_{d,r} v\right)-\left(g\left(u-Q_h u\right), v^0\right)-\left(|u|^2 u-\left|u_h\right|^2 Q_h^0 u, v^0\right)-a_w\left(Q_h u, v\right)- \\ &\left(\left|u_h\right|^2 Q_h u, v^0\right).\end{aligned} \tag{4.7}$$

It can rewritten as

$$\begin{aligned}&\left(-i\left(u_h-Q_h u\right)_t, v^0\right)+a_w\left(u_h-Q_h u, v\right)+\left(\left|u_h\right|^2\left(u_h-Q_h^0 u\right), v^0\right)= \\ &\left(\pi_h\left(\nabla_{d,r} u\right)-R_h(\nabla u), \nabla_{d,r} v\right)+\left(g\left(u-Q_h^0 u\right), v^0\right)+\left(|u|^2 u-\left|u_h\right|^2 Q_h^0 u, v^0\right)\end{aligned} \tag{4.8}$$

$$\begin{aligned}&\left(-i\left(u_h-Q_h u\right)_t, v^0\right)+a_w\left(u_h-Q_h u, v\right)+\left(\left|u_h\right|^2\left(u_h-Q_h^0 u\right), v^0\right)= \\ &\left(\pi_h\left(\nabla_{d,r} u\right)-R_h(\nabla u), \nabla_{d,r} v\right)+\left(g\left(u-Q_h^0 u\right), v^0\right)+\left[\left(|u|^2\left(u-Q_h u\right)+\right.\right. \\ &\left.Q_h u\left(|u|^2-\left|u_h\right|^2\right), v^0\right)\end{aligned}$$

$$\left(-i\left(u_h-Q_hu\right)_t, v^0\right)+a_w\left(u_h-Q_hu, v\right)+\left(\left|u_h\right|^2\left(u_h-Q_h^0u\right), v^0\right)=$$
$$\left(\pi_h\left(\nabla_{d,r}u\right)-R_h(\nabla u), \nabla_{d,r}v\right)+\left(g\left(u-Q_h^0u\right), v^0\right)+\left[\left(\left(|u|^2\left(u-Q_hu\right), v^0\right)+\right.\right.$$
$$\left(Q_hu\left(|u|+\left|u_h\right|\right)\left(|u|-\left|u_h\right|\right), v^0\right)$$

$$\left(-i\left(u_h-Q_hu\right)_t, v^0\right)+a_w\left(u_h-Q_hu, v\right)+\left(\left|u_h\right|^2\left(u_h-Q_h^0u\right), v^0\right)=$$
$$\left(\pi_h\left(\nabla_{d,r}u\right)-R_h(\nabla u), \nabla_{d,r}v\right)+\left(g\left(u-Q_h^0u\right), v^0\right)+\left[\left(|u|^2\left(u-Q_hu\right), v^0\right)+\right.$$
$$\left.\left(Q_hu\left(\left(|u|+\left|u_h\right|\right)\left(u-Q_hu\right), v^0\right)+\left(Q_hu\left(\left(|u|+\left|u_h\right|\right)\left(Q_hu-u_h\right), v^0\right)\right]\right. \tag{4.9}$$

The proof complete. □

Theorem 4.3. Let $u \in H^{1+r}\Omega$) with $r>0$ be the exact solution and u_h be approximation solution of equation (3.12) denoted by $e=u_h-Q_hu$ then there exists a constant C such that

$$\|e\|^2 \leq \left\|u_h(0)-u(0)\right\|^2+Ch^{2(r+1)}\left(\|u(0)\|^2+\int_0^t\|u(s)\|^2ds\right) \tag{4.10}$$

Proof . Let $e=\left\{u_h-Q_hu\right\},\left\{e^0, e^b\right\}=\left\{u^0-Q_h^0u, u^b-Q_h^bu\right\}$, and put $v^0=e$, in equation (4.9), we have

$$\left(-ie_t, e\right)+a_w(e, e)+\left(\left|u_h\right|^2e, e\right)=\left(\pi_h\left(\nabla_{d,r}u\right)-R_h(\nabla u), \nabla_{d,r}e\right)+$$
$$\left(g\left(u-Q_h^0u\right), e\right)+\left(|u|^2\left(u-Q_h^0u\right), e\right)+\left(Q_hu\left(\left(|u|+\left|u_h\right|\right)\left(u-Q_h^0u\right), e\right)-\right.$$
$$\left(Q_hu\left(\left(|u|+\left|u_h\right|\right)e, e\right)\right. \tag{4.11}$$

By comparing the imaginary parts of (4.11), we get

$$\begin{aligned} R_e\left(e_t, e\right)= & -\operatorname{Im}\left(\pi_h\left(\nabla_{d,r}u\right)-R_h(\nabla u), \nabla_{d,r}e\right)-\operatorname{Im}\left(g\left(u-Q_h^0u\right), e\right) \\ & -\operatorname{Im}\left(|u|^2\left(u-Q_h^0u\right), e\right)-\operatorname{Im}\left(Q_hu\left(\left(|u|+\left|u_h\right|\right)\left(u-Q_h^0u\right), e\right)\right. \end{aligned}$$

By Cauchy Schwarz inequality and Young inequality, we have

$$\frac{1}{2}\frac{d}{dt}\|e\|^2+\delta\|e\|_w^2+\left|u_h\right|^2\|e\|^2 \leq \frac{1}{4\delta}\left\|\pi_h\left(\nabla_{d,r}u\right)-R_h(\nabla u)\right\|^2+\delta\left\|\nabla_{d,r}\right\|^2+$$
$$\frac{3|g|}{4\delta}\left\|u-Q_h^0u\right\|^2+\frac{\delta}{3}\|e\|^2+\frac{3|u|^2}{4\delta}\left\|u-Q_h^0u\right\|^2+\frac{\delta}{3}\|e\|^2+\frac{3}{4\delta} \mid Q_hu(|u|+$$
$$\left|u_h\right|) \mid\left\|u-Q_h^0u\right\|^2+\frac{\delta}{3}\|e\|^2.$$

From definition (3.1), we obtain

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 \leq C_1h^{2(r+1)}\|u\|_{r+1}^2+C_2h^{2(r+1)}\|u\|_{r+1}^2+C_3h^{2(r+1)}\|u\|_{r+1}^2+C_4h^{r+1}\|u\|_{r+1}^2$$

where

$$\begin{gathered} C_1=\frac{1}{4\delta}, C_2=\frac{3|g|}{4\delta}, C_3=\frac{3|u|^2}{4\delta}, C_4=\frac{3}{4\delta}\left|Q_hu\left(|u|+\left|u_h\right|\right)\right| \\ \frac{1}{2}\frac{d}{dt}\|e\|^2 \leq Ch^{2(r+1)}\|u\|_{r+1}^2 \end{gathered}$$

where

$$C=\max\left(C_1, C_2, C_3, C_4\right).$$

Multiply by 2 and integral both side from 0 to t, we get

$$\|e\|^2 \leq\|e(0)\|^2+Ch^{2(r+1)}\int_0^t\|u(s)\|^2ds$$

where

$$\begin{aligned}\|e(0)\| &= \|u_h(0) - Q_h u(0)\| \\ &\le \|u_h(0) - u(0)\| + \|u(0) - Q_h u(0)\| \\ &\le \|u_h(0) - u(0)\| + Ch^{(r+1)}\|u(0)\|_{r+1}.\end{aligned}$$

$$\|e\|^2 \le \|u_h(0) - u(0)\|^2 + Ch^{2(r+1)}\left(\|u(0)\|^2 + \int_0^t \|u(s)\|^2 ds\right) \tag{4.12}$$

□

5 Error Analysis of Fully-Discrete WGFEM

In this section, we focus on the time discretization with the backward Euler scheme and Crank-Nicolson scheme.

5.1 backward Euler WGFEM

Let $\tau = \frac{T}{N}$ be the time step of the interval $[0,T]$,the time nodes $t_j = j\tau, j = 0, 1, \dots N$ and time elements

$$R_j = (t_j, t_{j+1}), j = 0, 1, \dots, N-1.$$

To simplify the notation, we denote $u(x, y, t_n)$ by u^n. Then we discretize (3.12) by backward Euler WGFEM and the fully discrete finite element solution $u_h^n(x,y) \in \emptyset_h, 0 \le n \le N$ of problem (1.1) can be defined by

$$\left(i\bar{\partial}_t (u_h^n)^0, v^0\right) = a_w(u_h^n, v) + \left(\left|(u_h^n)^0\right|^2 (u_h^n)^0, v^0\right) + (f, v^0), \forall v \in \emptyset_h \tag{5.1}$$

where

$$\bar{\partial}_t (u_h^n)^0 = \frac{1}{\tau}\left((u_h^n)^0 - (u_h^{n-1})^0\right),$$

and

$$a_w(u_h^n, v) = (\nabla_{d,r} u_h^n, \nabla_{d,r} v) + \left(g(u_h^n)^0, v^0\right).$$

Lemma 5.1. Let function series $u_h^n(x,y), 0 \le n \le N$, be the backward weak Galerkin finite element solution defined in (5.1), if $\tau < \frac{1}{2}$, then we have

$$\left\|(u_h^n)^0\right\| \le C \tag{5.2}$$

Proof . Taking $v = (u_h^n)^0$ in (5.1), we obtain

$$\left(\left(\frac{i}{\tau}(u_h^n)^0 - (u_h^{n-1})^0\right), (u_h^n)^0\right) = a_w(u_h^n, u_h^n) + \left(\left|(u_h^n)^0\right|^2 (u_h^n)^0, (u_h^n)^0\right) + \left(f, (u_h^n)^0\right). \tag{5.3}$$

By comparing the imaginary parts of (5.3), we have

$$\frac{1}{\tau}\left(\left\|(u_h^n)^0\right\|^2 - R_e\left((u_h^n)^0, (u_h^{n-1})^0\right)\right) = \operatorname{Im}\left\{\left(f^n, (u_h^n)^0\right)\right\}$$

which implies that

$$\left\|(u_h^n)^0\right\|^2 \le \left\|(u_h^n)^0\right\| \left\|(u_h^{n-1})^0\right\| + \tau\|f^n\| \left\|(u_h^n)^0\right\|.$$

That is $\left\|(u_h^n)^0\right\| \le \left\|(u_h^{n-1})^0\right\| + \tau\|f^n\|$. Summing form $j = 1$ to $j = n$

$$\begin{aligned}\left\|(u_h^n)^0\right\| &\leq \left\|(u_h^0)^0\right\| + \tau\sum_{j=1}^{n}\left\|f^j\right\| \\ &= \left\|\pi_h u_0\right\| + \tau\sum_{j=1}^{n}\left\|f^j\right\| \\ &\leq C\left\|u_0\right\| + T\max_{0\leq j\leq N}\left\|f^j\right\| \end{aligned} \tag{5.4}$$

Therefore, (5.2) follows from (5.4). □

Theorem 5.2. 5.1 Let u^n and u_h^n be the solution of (3.12) and (5.1) respectively ,then there exists a constant C, independent of h such that

$$\left\|e^n\right\| \leq \left\|e^0\right\| + C\tau h^{r+1}\left\|u^n\right\|_{1+r} + \left(C\tau^2\int_0^{t_n}\left\|u_{tt}\right\|^2\right)^{\frac{1}{2}} \tag{5.5}$$

Proof . Let $t = t^n$ and $u^n = u\left(t^n\right)$ in equation (4.1)

$$\left(iu_t\left(t^n\right), v^0\right) = \left(\pi_h\left(\nabla_{d,r}u^n\right), \nabla_{d,r}v\right) + \left(g\left(u^n\right)^0, v^0\right) + \left(|u|^2\left(u^n\right)^0, v^0\right) + \left(f, v^0\right). \tag{5.6}$$

Adding and subtracting

$$\begin{aligned}&\left(i\bar{\partial}_t\left(Q_h u^n\right), v^0\right) - \left(i\bar{\partial}_t\left(Q_h u^n\right), v^0\right) + \left(\nabla_{d,r}\left(Q_h u^n\right), \nabla_{d,r}v\right) - \\ &\left(\nabla_{d,r}\left(Q_h u^n\right), \nabla_{d,r}v\right) + \left(gQ_h u^n, v^0\right) - \left(gQ_h u^n, v^0\right) + \left(|u|^2 Q_h u^n, v^0\right) - \\ &\left(|u|^2 Q_h u^n, v^0\right)\end{aligned}$$

to equation (5.3) with $Q_h u^n = u\left(t^n\right)$, we get

$$\begin{aligned}&\left(i\bar{\partial}_t\left(Q_h u^n\right), v^0\right) - \left(i\bar{\partial}_t\left(Q_h u^n\right), v^0\right) + \left(\nabla_{d,r}\left(Q_h u^n\right), \nabla_{d,r}v\right) - \\ &\left(\nabla_{d,r}\left(Q_h u^n\right), \nabla_{d,r}v\right) + \left(gQ_h u^n, v^0\right) - \left(gQ_h u^n, v^0\right) + \left(|u|^2 Q_h u^n, v^0\right) - \\ &\left(|u|^2 Q_h u^n, v^0\right) = -\left(iu_t\left(t^n\right), v^0\right) + \left(\pi_h\left(\nabla_{d,r}u^n\right), \nabla_{d,r}v\right) + \left(g\left(u^n\right)^0, v^0\right) + \\ &\left(|u|^2\left(u^n\right)^0, v^0\right) + \left(f, v^0\right).\end{aligned} \tag{5.7}$$

Subtracting equation (5.7) from (5.1) and using $\nabla_{d,r}\left(Q_h u^n\right) = R_h \nabla u^n$, we get

$$\begin{aligned}\left(i\bar{\partial}_t\left(Q_h u^n - u_h^n\right), v^0\right) &+ \left(\nabla_{d,r}\left(u_h^n - Q_h u^n\right), \nabla_{d,r}v\right) + \left(g\left(u_h^n - Q_h u^n\right), v^0\right) \\ &+ \left(|u|^2\left(u_h^n - Q_h u^n, v^0\right)\right. \\ &= \left(i\left(\bar{\partial}_t\left(u^n\right) - u_t^n\right), v^0\right) + \left(\pi_h\left(\nabla_{d,r}u^n\right) - R_h\left(\nabla_{d,r}u^n\right), \nabla_{d,r}v\right) \\ &+ \left(g\left(\left(u^n\right)^0 - Q_h u^n\right), v^0\right) + \left(|u|^2\left(\left(u^n\right)^0 - Q_h u^n\right), v^0\right)\end{aligned}$$

i .e.

$$\begin{aligned}&\left(i\bar{\partial}_t\left(Q_h u^n - u_h^n\right), v^0\right) + a_w\left(u_h^n - Q_h u^n, v\right) + \left(|u|^2\left(u_h^n - Q_h u^n, v^0\right) = \right. \\ &\left(i\left(\bar{\partial}_t\left(u^n\right) - u_t^n\right), v^0\right) + \left(\pi_h\left(\nabla_{d,r}u^n\right) - R_h\left(\nabla_{d,r}u^n\right), \nabla_{d,r}v\right) + \left(g\left(\left(u^n\right)^0 - \right.\right. \\ &\left.\left.Q_h u^n\right), v^0\right) + \left(|u|^2\left(\left(u^n\right)^0 - Q_h u^n\right), v^0\right).\end{aligned} \tag{5.8}$$

Since $e^n = u_h^n - Q_h u^n$, we have

$$\begin{aligned}&\left(-i\overline{\partial_t}e^n, v^0\right) + a_w\left(e^n, v\right) + \left(|u|^2 e^n, v^0\right) = i\left(\overline{\partial_t}u^n - u_t^n, v^0\right) + \left(\pi_h\left(\nabla_{d,r}u^n\right) - \right. \\ &\left.R_h\left(\nabla_{d,r}u^n\right), \nabla_{d,r}v\right) + \left(g\left(\left(u^n\right)^0 - Q_h u^n\right), v^0\right) + \left(|u|^2\left(\left(u^n\right)^0 - Q_h u^n\right), v^0\right).\end{aligned} \tag{5.9}$$

Use $v^0 = e^n$ in equation (5.9), we obtain

$$\left(-i\overline{\partial_t}e^n, e^n\right) + a_w\left(e^n, e^n\right) + \left(|u|^2 e^n, e^n\right) = \left(i\left(\overline{\partial_t}u^n - u_t^n\right), e^n\right) + \left(\pi_h\left(\nabla_{d,r}u^n\right) - R_h\left(\nabla_{d,r}u^n\right), \nabla_{d,r}e^n\right) + \left(g\left(\left(u^n\right)^0 - Q_h u^n\right), e^n\right) + \left(|u|^2\left(\left(u^n\right)^0 - Q_h u^n\right), e^n\right). \tag{5.10}$$

By comparing the imaginary parts of (5.10), we get

$$-R_e\left(\overline{\partial_t}e^n, e^n\right) = R_e\left(\overline{\partial_t}u^n - u_t^n, e^n\right) + \operatorname{Im}\left(\pi_h\left(\nabla_{d,r}u^n\right) - R_h\left(\nabla_{d,r}u^n\right), \nabla_{d,r}e^n\right) + \operatorname{Im}\left(\left(g\left(\left(u^n\right)^0\right)\right) - Q_h u^n\right), e^n\right) + \operatorname{Im}\left(|u|^2\left(\left(u^n\right)^0 - Q_h u^n\right), e^n\right).$$

Let

$$\begin{aligned}
E_1^n &= R_e\left(\bar{\partial}_t u^n - u_t^n\right)\\
E_2^n &= \operatorname{Im}\left(\pi_h\left(\nabla_{d,r}u^n\right) - R_h\left(\nabla u^n\right)\right)\\
E_3^n &= \operatorname{Im}\left(g\left(\left(u^n\right)^0 - Q_h u^n\right)\right)\\
E_4^n &= \operatorname{Im}\left(|u|^2\left(\left(u^n\right)^0 - Q_h u^n\right)\right).
\end{aligned}$$

We have

$$\left(-\bar{\partial}_t e^n, e^n\right) + a_w\left(e^n, e^n\right) + \left(|u|^2 e^n, e^n\right) = \left(E_1^n, e^n\right) + \left(E_2^n, \nabla_{d,r}e^n\right) + \left(E_3^n, e^n\right) + \left(E_4^n, e^n\right). \tag{5.11}$$

Hence

$$\left(\frac{e^n - e^{n-1}}{\tau}, e^n\right) + a_w\left(e^n, e^n\right) + \left(|u|^2 e^n, e^n\right) = \left(E_1^n, e^n\right) + \left(E_2^n, \nabla_{d,r}e^n\right) + \left(E_3^n, e^n\right) + \left(E_4^n, e^n\right). \tag{5.12}$$

By (∅-elliptic) of the bilinear form and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&\|e^n\|^2 + \tau\|e^n\|_w^2 + \tau|u|^2\|e^n\|^2\\
&\quad\le \tau\|E_1^n\|\|e^n\| + \tau\|E_2^n\|\|\nabla_{d,r}e^n\| + \tau\|E_3^n\|\|e^n\| + \tau\|E_4^n\|\|e^n\|\\
&\quad+ \frac{1}{2}\left\|e^{n-1}\right\|^2 + \frac{1}{2}\|e^n\|^2.
\end{aligned}$$

Using Young's inequality ,we obtain

$$\begin{aligned}
&\frac{1}{2}\|e^n\|^2 + \tau\|e^n\|_w^2 + \tau|u|^2\|e^n\|^2\\
&\le \tau C\left(\|E_1^n\|^2 + \|E_2^n\|^2 + \|E_3^n\|^2 + \|E_4^n\|^2\right) + \tau\|e^n\|_w^2.
\end{aligned}$$

Then, by induction, we arrive to

$$\|e^n\|^2 \le \left\|e^0\right\|^2 + \tau C\left(\sum_{j=1}^n\left\|E_1^j\right\|^2 + \sum_{j=1}^n\left\|E_2^j\right\|^2 + \sum_{j=1}^n\left\|E_3^j\right\|^2 + \sum_{j=1}^n\|E_4^n\|^2\right). \tag{5.13}$$

The first term of the equation (5.13)

$$\begin{aligned}
E_1^j &= u_t^j - \frac{1}{\tau}\left(u^j - u^{j-1}\right)\\
\tau E_1^j &= \tau\left(u_t^j\right) - \int_{t_{j-1}}^{t_j} u_t dt\\
&= \left(t_j - t_{j-1}\right)\left(u_t^j\right) - \int_{t_{j-1}}^{t_j} u_t dt.
\end{aligned}$$

Adding and subtraction $t_{j-1}\left(u_t^{j-1}\right)$, we get

$$\begin{aligned}&= t_j\left(u_t^j\right)\left(-t_{j-1}\left(u_t^{j-1}\right)\right)-\int_{t_{j-1}}^{t_j} u_t dt-\left(t_{j-1}\left(u_t^j\right)-t_{j-1}\left(u_t^{j-1}\right)\right)\\ &=\int_{t_{j-1}}^{t_j} t u_{tt} dt - t_{j-1}\int_{t_{j-1}}^{t_j} u_{tt} dt\\ &\tau E_1^j \leq \tau\int_{t_{j-1}}^{t_j}\left\|u_{tt}\right\| dt.\end{aligned}$$

By Jensen's inequality

$$\left\|E_1^j\right\|^2 \leq \tau^2\left(\int_{t_{j-1}}^{t_j}\left\|u_{tt}\right\|\frac{dt}{\tau}\right)^2 \leq \tau^2\int_{t_{j-1}}^{t_j}\left\|u_{tt}\right\|^2\frac{dt}{\tau}=\tau\int_{t_{j-1}}^{t_j}\left\|u_{tt}\right\|^2 dt. \tag{5.14}$$

To approximate E_2^j by Lemma (3.3), we get,

$$\begin{aligned}E_2^j &= C\tau\left\|\pi_h\left(\nabla_{d,r}u^n\right)-R_h\left(\nabla_{d,r}u^n\right)\right\|^2\\ &\leq C\tau h^{2(r+1)}\left\|u^n\right\|_{1+r}^2.\end{aligned} \tag{5.15}$$

To approximate E_3^j by equation (3.6)

$$E_3^j \leq Ch^{2(r+1)}\left\|u^n\right\|_{1+r}^2 \tag{5.16}$$

To approximate E_4^j by definition (3.1)

$$E_4^j \leq Ch^{2(r+1)}\left\|u^n\right\|_{1+r}^2. \tag{5.17}$$

Substitution (5.14),(5.15), (5.16) and (5.17)in to (5.13), we get

$$\left\|e^n\right\|^2 \leq \left\|e^0\right\|^2 + C\tau h^{2(r+1)}\left\|u^n\right\|_{1+r}^2 + C\tau^2\int_0^{t_n}\left\|u_{tt}\right\|^2 dt.$$

Hence, we have

$$\left\|e^n\right\| \leq \left\|e^0\right\| + C\tau h^{r+1}\left\|u^n\right\|_{1+r} + \left(C\tau^2\int_0^{t_n}\left\|u_{tt}\right\|^2 dt\right)^{\frac{1}{2}}. \tag{5.18}$$

The proof is complete. □

5.2 The Crank-Nicolson WGFEM

In this sub section, we'll look at how to discretize time using the CrankNicolson technique. Let $\tau=\frac{T}{N}$ be the time step of the interval $[0,T]$, the time nodes $t_j=j\tau, j=0,1,\ldots,N$,

$$t_{j+\frac{1}{2}}=\frac{1}{2}\left(t_{j+1}+t_j\right)$$

and time element $k_j=\left(t_j,t_{j+1}\right), j=0,1,\ldots,N-1$. For function series $u_h^n(x,y), n=0,1,\ldots$, Let

$$\begin{aligned}\bar{\partial}_t u_h^{n+\frac{1}{2}} &= \frac{1}{\tau}\left[u_h^{n+1}-u_h^n\right],\\ u_h^{n+\frac{1}{2}} &= \frac{1}{2}\left[u_h^{n+1}+u_h^n\right].\end{aligned}$$

To simplify the notation, we denote $u(x,y,t_n)$ by u^n. Then we define the fully discrete finite element solution $u_h^n(x,y) \in \emptyset_h, 0 \le n \le N$ of problem (1.1) satisfying the Crank-Nicolson weak Galerkin finite element

$$\left(i\bar{\partial}_t u_h^{n+\frac{1}{2}}, v^0\right) = a_w\left(u_h^{n+\frac{1}{2}}, v\right) + \left(\left|u_h^{n+\frac{1}{2}}\right|^2 u_h^{n+\frac{1}{2}}, v^0\right) + \left(f^{n+\frac{1}{2}}, v^0\right), \forall v \in \emptyset_h \tag{5.19}$$

$$u_h^0(x,y) = \pi_h u_0(x,y),$$

where

$$a_w\left(u_h^{n+\frac{1}{2}}, v\right) = \left(\nabla_{d,r} u_h^{n+\frac{1}{2}}, \nabla_{d,r} v\right) + \left(g\left(u_h^{n+\frac{1}{2}}\right)^0, v^0\right).$$

Theorem 5.3. Let u^n and u_h^n be the solution of (3.12) and (5.19) respectively ,then there exists a constant C, independent of h such that

$$\|e^n\| \le \left\|e^0\right\| + C\tau h^{r+1}\left(\|u^n\|_{r+1} - \left\|u^0\right\|_{r+1}\right) + \left(C\tau^2 \int_0^{t_n} \|u_{tt}\|^2 dt\right)^{\frac{1}{2}} \tag{5.20}$$

Proof . Let $t = t^{n+\frac{1}{2}}$ and $u^{n+\frac{1}{2}} = u\left(t^{n+\frac{1}{2}}\right)$ in equation (4.1)

$$\begin{aligned}\left(iu_t\left(t^{n+\frac{1}{2}}\right), v^0\right) &= \left(\pi_h\left(\nabla_{d,r} u^{n+\frac{1}{2}}\right), \nabla_{d,r} v\right) + \left(g\left(u^{n+\frac{1}{2}}\right)^0, v^0\right) \\ &+ \left(\left|u_h^{n+\frac{1}{2}}\right|^2 \left(u_h^{n+\frac{1}{2}}\right)^0, v^0\right)\Big) + \left(f^{n+\frac{1}{2}}, v^0\right).\end{aligned} \tag{5.21}$$

Adding and subtracting

$$\begin{aligned}&\left(i\bar{\partial}_t\left(Q_h u^{n+\frac{1}{2}}\right), v^0\right) - \left(i\bar{\partial}_t\left(Q_h u^{n+\frac{1}{2}}\right), v^0\right) + \left(\nabla_{d,r}\left(Q_h u^{n+\frac{1}{2}}\right), \nabla_{d,r} v\right) \\ &- \left(\nabla_{d,r}(Q_h) u^{n+\frac{1}{2}}, \nabla_{d,r} v\right) + \left(gQ_h u^{n+\frac{1}{2}}, v^0\right) - \left(gQ_h u^{n+\frac{1}{2}}, v^0\right) \\ &+ \left(\left|u^{n+\frac{1}{2}}\right|^2 Q_h u^{n+\frac{1}{2}}, v^0\right) - \left(\left|u^{n+\frac{1}{2}}\right|^2 Q_h u^{n+\frac{1}{2}}, v^0\right)\end{aligned}$$

to equation (5.21) with $Q_h u^{n+\frac{1}{2}} = u\left(t^{n+\frac{1}{2}}\right)$, we get

$$\begin{aligned}&\left(i\bar{\partial}_t\left(Q_h u^{n+\frac{1}{2}}\right), v^0\right) - \left(i\bar{\partial}_t\left(Q_h u^{n+\frac{1}{2}}\right), v^0\right) + \left(\nabla_{d,r}\left(Q_h u^{n+\frac{1}{2}}\right), \nabla_{d,r} v\right) \\ &- \left(\nabla_{d,r}\left(Q_h u^{n+\frac{1}{2}}\right), \nabla_{d,r} v\right) + \left(gQ_h u^{n+\frac{1}{2}}, v^0\right) - \left(gQ_h u^{n+\frac{1}{2}}, v^0\right) \\ &+ \left(\left|u^{n+\frac{1}{2}}\right|^2 Q_h u^{n+\frac{1}{2}}, v^0\right) - \left(\left|u^{n+\frac{1}{2}}\right|^2 Q_h u^{n+\frac{1}{2}}, v^0\right) = -\left(iu_t\left(t^{n+\frac{1}{2}}\right), v^0\right) \\ &+ \left(\pi_h\left(\nabla_{d,r} u^{n+\frac{1}{2}}\right), \nabla_{d,r} v\right) + \left(g\left(u^{n+\frac{1}{2}}\right)^0, v^0\right) + \left(\left|u_h^{n+\frac{1}{2}}\right|^2 \left(u_h^{n+\frac{1}{2}}\right)^0, v^0\right) \\ &+ \left(f^{n+\frac{1}{2}}, v^0\right).\end{aligned} \tag{5.22}$$

Subtracting equation (5.22) from (5.19) and using $\nabla_{d,r}\left(Q_h u^{n+\frac{1}{2}}\right) = R_h \nabla u^{n+\frac{1}{2}}$, we get

$$\begin{aligned}&\left(i\bar{\partial}_t\left(Q_h u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}\right), v^0\right) + \left(\nabla_{d,r}\left(u_h^{n+\frac{1}{2}} - Q_h u^{n+\frac{1}{2}}\right), \nabla_{d,r} v\right) + \Big(g\Big(u_h^{n+\frac{1}{2}} - \\ &Q_h u^{n+\frac{1}{2}}\Big), v^0\Big) + \left(\left|u_h^{n+\frac{1}{2}}\right|^2 \left(u_h^{n+\frac{1}{2}} - Q_h u^{n+\frac{1}{2}}\right), v^0\right) = \Big(i\Big(\bar{\partial}_t\left(u^{n+\frac{1}{2}}\right) - \\ &u_t^{n+\frac{1}{2}}\Big), v^0\Big) + \left(\pi_h\left(\nabla_{d,r} u^{n+\frac{1}{2}}\right) - R_h\left(\nabla_{d,r} u^{n+\frac{1}{2}}\right), \nabla_{d,r} v\right) + \Big(g\Big(\left(u^{n+\frac{1}{2}}\right)^0 - \\ &Q_h u^{n+\frac{1}{2}}\Big), v^0\Big) + \left(\left|u_h^{n+\frac{1}{2}}\right|^2 \left(\left(u^{n+\frac{1}{2}}\right)^0 - Q_h u^{n+\frac{1}{2}}\right), v^0\right)\end{aligned}$$

i.e.

$$\left(i\bar{\partial}_t\left(Q_hu^{n+\frac{1}{2}}-u_h^{n+\frac{1}{2}}\right),v^0\right)+a_w\left(u_h^{n+\frac{1}{2}}-Q_hu^{n+\frac{1}{2}},v\right)+||u_h^{n+\frac{1}{2}}\Big|^2\left(u_h^{n+\frac{1}{2}}-\right.$$
$$\left.Q_hu^{n+\frac{1}{2}}\right),v^0\Big)=\left(i\left(u_t^{n+\frac{1}{2}}-\bar{\partial}_t\left(u^{n+\frac{1}{2}}\right)\right),v^0\right)+\left(\pi_h\left(\nabla_{d,r}u^{n+\frac{1}{2}}\right)-\right.$$
$$R_h\left(\nabla_{d,r}u^{n+\frac{1}{2}}\right),\nabla_{d,r}v\Big)+\left(g\left(\left(u^{n+\frac{1}{2}}\right)^0-Q_hu^{n+\frac{1}{2}}\right),v^0\right)+\left(|u_h^{n+\frac{1}{2}}|^2\left(\left(u^{n+\frac{1}{2}}\right)^0-\right.\right.$$
$$\left.\left.Q_hu^{n+\frac{1}{2}}\right),v^0\right). \tag{5.23}$$

Since $e^{n+\frac{1}{2}}=u_h^{n+\frac{1}{2}}-Q_hu^{n+\frac{1}{2}}$, we have

$$\left(-i\bar{\partial}_te^{n+\frac{1}{2}},v^0\right)+a_w\left(e^{n+\frac{1}{2}},v\right)+\left(\left|u_h^{n+\frac{1}{2}}\right|^2e^{n+\frac{1}{2}},v^0\right)=i\left(\bar{\partial}_tu^{n+\frac{1}{2}}-u_t^{n+\frac{1}{2}},v^0\right)$$
$$+\left(\pi_h\left(\nabla_{d,r}u^{n+\frac{1}{2}}\right)-R_h\left(\nabla_{d,r}u^{n+\frac{1}{2}}\right),\nabla_{d,r}v\right)+\left(g\left(\left(u^{n+\frac{1}{2}}\right)^0-Q_hu^{n+\frac{1}{2}}\right),v^0\right)$$
$$+\left(\left|u_h^{n+\frac{1}{2}}\right|^2\left(\left(u^{n+\frac{1}{2}}\right)^0-Q_hu^{n+\frac{1}{2}}\right),v^0\right). \tag{5.24}$$

Use $v^0=e^{n+\frac{1}{2}}$ in equation (5.24), we obtain

$$\left(-i\bar{\partial}_te^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)+a_w\left(e^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)+\left(\left|u_h^{n+\frac{1}{2}}\right|^2e^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)$$
$$=\left(i\left(\bar{\partial}_tu^{n+\frac{1}{2}}-u_t^{n+\frac{1}{2}}\right),e^{n+\frac{1}{2}}\right)+\left(\pi_h\left(\nabla_{d,r}u^{n+\frac{1}{2}}\right)-R_h\left(\nabla_{d,r}u^{n+\frac{1}{2}}\right),\nabla_{d,r}e^{n+\frac{1}{2}}\right)$$
$$+\left(g\left(\left(u^{n+\frac{1}{2}}\right)^0-Q_hu^{n+\frac{1}{2}}\right),e^{n+\frac{1}{2}}\right)+\left(\left|u_h^{n+\frac{1}{2}}\right|^2\left(\left(u^{n+\frac{1}{2}}\right)^0-Q_hu^{n+\frac{1}{2}}\right),e^{n+\frac{1}{2}}\right). \tag{5.25}$$

By comparing the imaginary parts of (5.25), we get

$$-R_e\left(\bar{\partial}_te^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)=R_e\left(\partial_tu^{n+\frac{1}{2}}-u_t^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)+\mathrm{Im}\left(\pi_h\left(\nabla_{d,r}u^{n+\frac{1}{2}}\right)\right.$$
$$\left.-\ R_h\left(\nabla_{d,r}u^{n+\frac{1}{2}}\right),\nabla_{d,r}e^{n+\frac{1}{2}}\right)+\mathrm{Im}\left(\left(g\left(\left(u^{n+\frac{1}{2}}\right)^0\right)\right)-Q_hu^{n+\frac{1}{2}}\right),e^{n+\frac{1}{2}}\Big)$$
$$+\mathrm{Im}\left(\left|u_h^{n+\frac{1}{2}}\right|^2\left(\left(u^{n+\frac{1}{2}}\right)^0-Q_hu^{n+\frac{1}{2}}\right),e^n\right).$$

Let

$$E_1^{n+\frac{1}{2}}=R_e\left(\bar{\partial}_tu^{n+\frac{1}{2}}-u_t^{n+\frac{1}{2}}\right)$$
$$E_2^{n+\frac{1}{2}}=\mathrm{Im}\left(\pi_h\left(\nabla_{d,r}u^{n+\frac{1}{2}}\right)-R_h\left(\nabla u^{n+\frac{1}{2}}\right)\right)$$
$$E_3^{n+\frac{1}{2}}=\mathrm{Im}\left(g\left(\left(u^{n+\frac{1}{2}}\right)^0-Q_hu^{n+\frac{1}{2}}\right)\right)$$
$$E_4^{n+\frac{1}{2}}=\mathrm{Im}\left(\left|u_h^{n+\frac{1}{2}}\right|^2\left(\left(u^{n+\frac{1}{2}}\right)^0-Q_hu^{n+\frac{1}{2}}\right)\right)$$

We have

$$\left(\overline{\partial_t}e^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)+a_w\left(e^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)+\left(\left|u_h^{n+\frac{1}{2}}\right|^2e^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)=\left(E_1^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)$$
$$+\left(E_2^{n+\frac{1}{2}},\nabla_{d,r}e^{n+\frac{1}{2}}\right)+\left(E_3^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)+\left(E_4^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right). \tag{5.26}$$

Hence

$$\left(\frac{e^{n+1}-e^{n}}{\tau},\frac{e^{n+1}+e^{n}}{2}\right)+a_{w}\left(e^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)+\left(\left|u_{h}^{n+\frac{1}{2}}\right|^{2}e^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)=\left(E_{1}^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)$$
$$+\left(E_{2}^{n+\frac{1}{2}},\nabla_{d,r}e^{n+\frac{1}{2}}\right)+\left(E_{3}^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right)+\left(E_{4}^{n+\frac{1}{2}},e^{n+\frac{1}{2}}\right). \tag{5.27}$$

By ($\varnothing$-elliptic) of the bilinear form and Cauchy-Schwartz inequality, we obtain

$$\frac{1}{2}\left\|e^{n+1}\right\|^{2}+\tau e^{n+\frac{1}{2}}\left\|_{w}^{2}+\tau\left|u_{h}^{n+\frac{1}{2}}\right|^{2}\right\|e^{n+\frac{1}{2}}\|^{2}$$
$$\leq\tau\left\|E_{1}^{n+\frac{1}{2}}\right\|\left\|e^{n+\frac{1}{2}}\right\|+\tau\left\|E_{2}^{n+\frac{1}{2}}\right\|\left\|\nabla_{d,r}e^{n+\frac{1}{2}}\right\|+\tau\left\|E_{3}^{n+\frac{1}{2}}\right\|\left\|e^{n+\frac{1}{2}}\right\|$$
$$+\tau\left\|E_{4}^{n+\frac{1}{2}}\right\|\left\|e^{n+\frac{1}{2}}\right\|+\frac{1}{2}\left\|e^{n}\right\|^{2}$$

by Young's inequality on the right hand side of the above equation, it follows

$$\frac{1}{2}\left\|e^{n+1}\right\|^{2}+\tau\left\|e^{n+\frac{1}{2}}\right\|_{w}^{2}+\tau\left|u_{h}^{n+\frac{1}{2}}\right|^{2}\left\|e^{n+\frac{1}{2}}\right\|^{2}\leq\tau C\left(\left\|E_{1}^{n+\frac{1}{2}}\right\|^{2}+\left\|E_{2}^{n+\frac{1}{2}}\right\|^{2}+\right.$$
$$\left.\left\|E_{3}^{n+\frac{1}{2}}\right\|^{2}+\left\|E_{4}^{n+\frac{1}{2}}\right\|^{2}\right)+\tau\left\|e^{n+\frac{1}{2}}\right\|_{w}^{2}+\frac{1}{2}\left\|e^{n}\right\|^{2}$$
$$\left\|e^{n+1}\right\|^{2}\leq 2\tau C\left(\left\|E_{1}^{n+\frac{1}{2}}\right\|^{2}+\left\|E_{2}^{n+\frac{1}{2}}\right\|^{2}+\left\|E_{3}^{n+\frac{1}{2}}\right\|^{2}+\left\|E_{4}^{n+\frac{1}{2}}\right\|^{2}\right)+\frac{1}{2}\left\|e^{n}\right\|^{2}$$

then by induction, we arrive to

$$\left\|e^{n}\right\|^{2}\leq\left\|e^{0}\right\|^{2}+\tau C\left(\sum_{j=1}^{n}\left\|E_{1}^{j+\frac{1}{2}}\right\|^{2}+\left\|E_{2}^{j+\frac{1}{2}}\right\|^{2}+\left\|E_{3}^{j+\frac{1}{2}}\right\|^{2}+\left\|E_{4}^{n+\frac{1}{2}}\right\|^{2}\right). \tag{5.28}$$

The first term of the equation (5.28)

$$E_{1}^{j+\frac{1}{2}}=\left(u_{t}^{j+\frac{1}{2}}-\frac{1}{\tau}\left(u^{j+1}-u^{j}\right)\right)$$
$$\tau E_{1}^{j+\frac{1}{2}}=\tau\left(u_{t}^{j+\frac{1}{2}}\right)-\int_{t_{i}}^{t_{j+1}}u_{t}dt$$

$$=\left(t_{j+1}-t_{j}\right)\left(\frac{1}{2}\left(u_{t}^{j+1}+u_{t}^{j}\right)\right)-\int_{t_{j}}^{t_{j+1}}u_{t}dt$$
$$=\frac{1}{2}\left(t_{j+1}-t_{j}\right)u_{t}^{j+1}+\frac{1}{2}\left(t_{j+1}-t_{j}\right)u_{t}^{j}-\int_{t_{j}}^{t_{j+1}}u_{t}dt$$
$$=\frac{1}{2}\left(t_{j+1}-t_{j}\right)u_{t}^{j+1}+\frac{1}{2}\left(t_{j+1}-t_{j}\right)u_{t}^{j}-\frac{1}{2}\int_{t_{j}}^{t_{j+1}}u_{t}dt-\frac{1}{2}\int_{t_{j}}^{t_{j+1}}u_{t}dt.$$

Adding and subtraction $\frac{1}{2}t_{j}u_{t}^{j},\frac{1}{2}t_{j+1}u_{t}^{j+1}$ we get

$$=\frac{1}{2}\left[t_{j+1}u_{t}^{j+1}-t_{j}u_{t}^{j}-\int_{t_{j}}^{t_{j+1}}u_{t}dt\right]-\frac{1}{2}\left(t_{j}u_{t}^{j+1}-t_{j}u_{t}^{j}\right)$$
$$+\frac{1}{2}\left[t_{j+1}u_{t}^{j+1}-t_{j}u_{t}^{j}-\int_{t_{j}}^{t_{j+1}}u_{t}dt\right]-\frac{1}{2}\left(t_{j+1}u_{t}^{j+1}-t_{j+1}u_{t}^{j}\right)$$
$$=\frac{1}{2}\int_{t_{j}}^{t_{j+1}}tu_{tt}dt-\frac{1}{2}t_{j}\int_{t_{j}}^{t_{j+1}}u_{tt}dt+\frac{1}{2}\int_{t_{j}}^{t_{j+1}}tu_{tt}dt-\frac{1}{2}t_{j+1}\int_{t_{j}}^{t_{j+1}}u_{tt}dt$$
$$\tau E_{1}^{j+\frac{1}{2}}\leq\frac{\tau}{2}\int_{t_{i}}^{t_{j+1}}\left\|u_{tt}\right\|dt.$$

By Jensen's inequality

$$\left\|E_1^{j+\frac{1}{2}}\right\|^2 \leq \left(\int_{t_j}^{t_{j+1}} \|u_{tt}\| \frac{ds}{\tau}\right)^2 = \tau^2 \left(\int_{t_j}^{t_{j+1}} \|u_{tt}\| \frac{ds}{\tau}\right)$$
$$= \tau \int_{t_j}^{t_{j+1}} \|u_{tt}\|^2 \, dt. \quad (5.29)$$

To approximate $E_2^{j+\frac{1}{2}}$ by Lemma (3.2), we get

$$E_2^{j+\frac{1}{2}} = C\tau \left\|\pi_h \left(\nabla_{d,r} u^{n+\frac{1}{2}}\right) - R_h \left(\nabla_{d,r} u^{n+\frac{1}{2}}\right)\right\|^2$$
$$\leq C\tau h^{2(r+1)} \left\|u^{n+\frac{1}{2}}\right\|_{1+r}^2. \quad (5.30)$$

To approximate $E_3^{j+\frac{1}{2}}$ by equation (3.6)

$$E_3^{j+\frac{1}{2}} \leq C h^{2(r+1)} \left\|u^{n+\frac{1}{2}}\right\|_{1+r}^2. \quad (5.31)$$

To approximate $E_4^{j+\frac{1}{2}}$ by definition (3.1)

$$E_4^{j+\frac{1}{2}} \leq C h^{2(r+1)} \left\|u^{n+\frac{1}{2}}\right\|_{1+r}^2. \quad (5.32)$$

Substitution $(5.29), (5.30), (5.31)$ and (5.32) in to (5.28), we get

$$\|e^n\|^2 \leq \left\|e^0\right\|^2 + C\tau h^{2(r+1)} \left\|u^{n+\frac{1}{2}}\right\|_{1+r}^2 + C\tau^2 \int_0^{t_n} \|u_{tt}\|^2 \, dt.$$

Hence, we have

$$\|e^n\| \leq \left\|e^0\right\| + C\tau h^{r+1} \left\|u^{n+\frac{1}{2}}\right\|_{r+1} + \left(C\tau^2 \int_0^{t_n} \|u_{tt}\|^2 \, dt\right)^{\frac{1}{2}}$$
$$\leq \left\|e^0\right\| + C\tau h^{r+1} \left(\frac{1}{2}\left(\left\|u^{n+1}\right\|_{r+1} + \|u^n\|_{r+1}\right)\right) + \left(C\tau^2 \int_0^{t_n} \|u_{tt}\|^2 \, dt\right)^{\frac{1}{2}}$$
$$\leq \left\|e^0\right\| + C\tau h^{r+1} \left(\left\|u^{n+1}\right\|_{r+1} + \|u^n\|_{r+1}\right) + \left(C\tau^2 \int_0^{t_n} \|u_{tt}\|^2 \, dt\right)^{\frac{1}{2}}.$$

By induction we arrive to

$$\|e^n\| \leq \left\|e^0\right\| + C\tau h^{r+1} \left(\|u^n\|_{r+1} - \left\|u^0\right\|_{r+1}\right) + \left(C\tau^2 \int_0^{t_n} \|u_{tt}\|^2 \, dt\right)^{\frac{1}{2}}.$$

We complete the proof. □

6 Numerical Experiments

In this section, we use a uniform triangular mesh T_h and discrete weak space $U_h\left(P_r(K), P_r(\partial K), r = 0\right.$, i. e, space consisting of piecewise polynomial of order r on the triangles and edges respectively with $V_{r+1}(K)$ to be the Raviart-Thomas element RT_r as a space of discrete weak gradient, which were used in [11] for the numerical studies of the weak Galerkin method for second order elliptic problems .to present the numerical results of the error between the L^2- projection $Q_h u$ of the exact solution and the numerical solution u_h. We use the following norms: Element-based L^2- norm,

$$\|e_0\| = \left(\sum_{K \in T_h} \int_K |e_0|^2 \, dx\right)^{\frac{1}{2}}.$$

Edge-based L^2- norm,

$$\|e_b\| = \left(\sum_{e\in\varepsilon_h} h_e \int_{\varepsilon_h} |e_b|^2 \, ds\right)^{\frac{1}{2}},$$

where ε_e the of all edges and h_e the diameter of edge e.

6.1 Test Problem

In this subsection, we present the test problem to illustrate the backward Euler WGFEM for the time dependent nonlinear Schrodinger equation (1.1) over a square domain $\Omega : [0,1] \times [0,1]$, the time interval $[0,T] = [0,1]$, and the function $f(x,y,t)$ is chosen corresponding to the exact solution and $g(x,y) = 1$ [10]

$$u = e^t(1-x)(1-y)\sin x \sin y + ie^t xy \sin(1-x)\sin(1-y)$$

$$U_h = \left\{ \begin{array}{l} u_0 \in P_0(K), \forall K \in T_h \\ u = (u_0, u_b) : u_b \in P_0(e), \forall K \in T_h \text{ and } e \subset \partial K \notin \partial\Omega, \\ u_b = \varepsilon_h, \forall K \in T_h \text{ and } \partial K \in \partial\Omega \end{array} \right\}$$

Table 1: convergence rate for real part where $\tau = 0.1, t = 1$

h	L2 Error	order
1/2	3.508×10^{-2}	0
1/4	1.7628×10^{-2}	0.99277
1/8	8.8987×10^{-3}	0.98623
1/16	4.4941×10^{-3}	0.98557
1/32	2.2697×10^{-3}	0.98551

Table 2: convergence rate for the imaginary part where $\tau = 0.1, t = 1$

h	L2 Error	order
1/2	1.3268×10^{-2}	0
1/4	6.6339×10^{-3}	1
1/8	3.3319×10^{-3}	0.99349
1/16	1.6743×10^{-3}	0.99284
1/32	8.4133×10^{-4}	0.99278

Tables 1 and 2, show the rate convergence for the WG solution in L^2 norm on triangle meshes, the triangle mesh is obtained by divided each diagonal line with a negative slope. In the test $\tau = 0.1$ is used to check the order of convergence with respect to time step size τ and mesh size $h = \frac{1}{n}, n = 2, 4, 8, 16, 32$, the numerical results show that the WG solution with constant space $(r = 0)$ equivalent to slandered finite element solutions with linear space $(r = 1)$ with convergence rate $o(h)$ in L^2-norm, this results are show in figures 1 and 2.

7 Conclusions

In this paper we introduced the weak Galerkin finite element method to solve the nonlinear Schrodinger equation, the error was determined in two ways :semi -discrete and fully - discrete .In fully discrete WGFEM that has been addressed upon (backward,Crank-Nicolson).We found that the convergence order of the error $o\left(h^{r+1}\right)$ is $r+1$, where r is the degree of the polynomial used in the research. The practical results show the theoretical side.

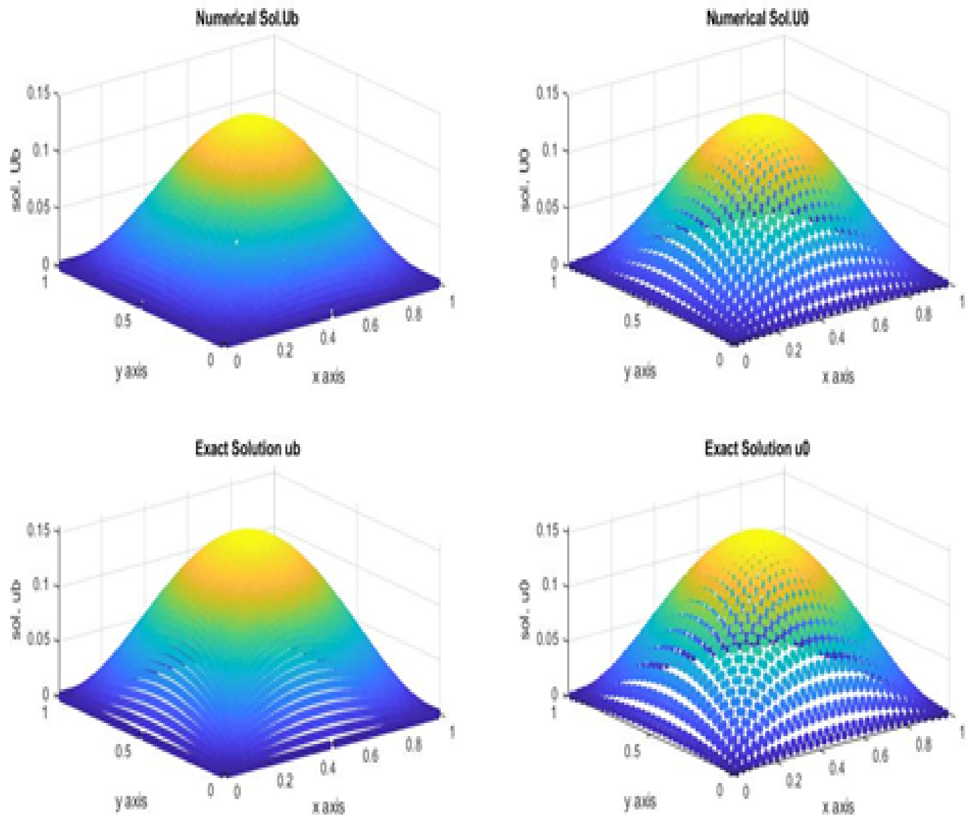

Figure 1: Numerical and exact solutiom u for real part in case $\tau = 0.1, t = 1$

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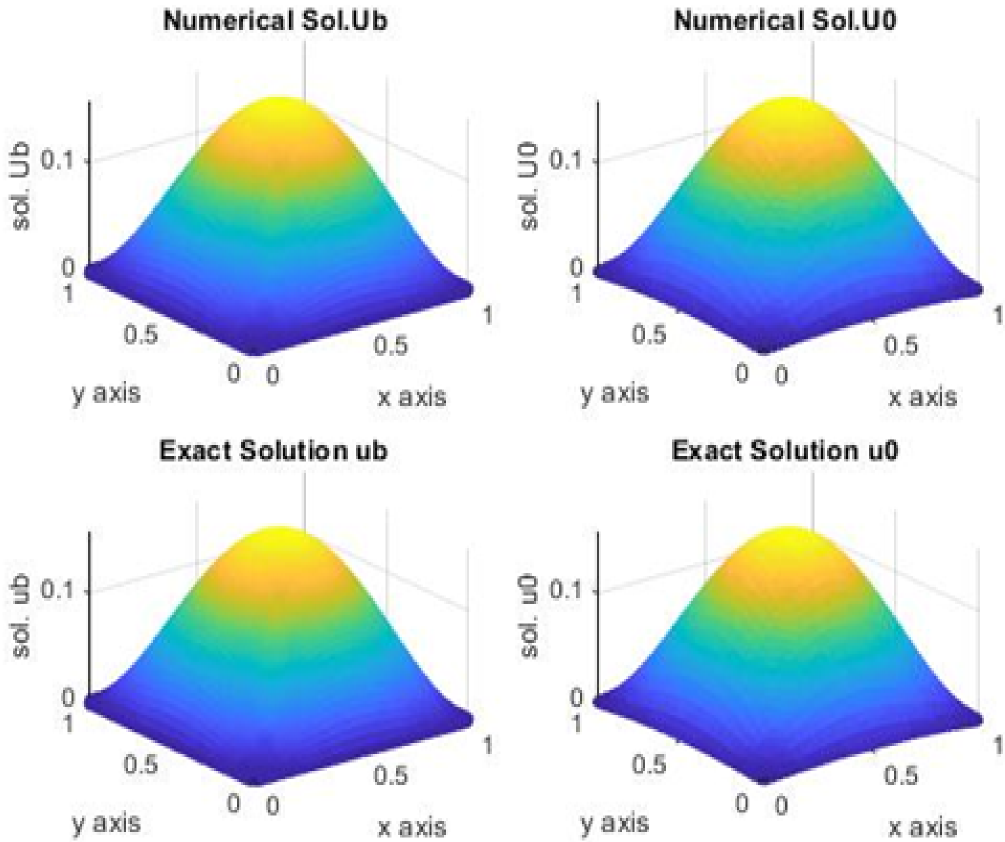

Figure 2: Numerical and exact solution u for imaginary part in case $\tau = 0.1, t = 1$

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