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Existence of solutions for time fractional order diffusion equations on weighted graphs

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Abstract

We generalize the concept of diffusion equations on weighted graphs, which is also known as ω-diffusion equations, to study fractional order diffusion equations on weighted graphs. More precisely, we replace the ordinary first order derivative in time by a fractional derivative of order α in the sense of Riemann-Liouville and Caputo fractional derivatives. We prove the existence of solutions of fractional order diffusion equations on graphs using the concept of α-exponential matrix and illustrate the solutions through numerical simulation in various examples.

Keywords: Calculus on Graphs; Diffusion Equations; Fractional Calculus
2020 MSC: Primary 34A08, Secondary 34B45, 35K57

1 Introduction

Differential equation is a very important tool in applied science, especially in modelling of dynamic processes. Diffusion equation [23], which is a partial differential equation of the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

is one of the main type of differential equation that has extensive applications. Physically, the term $u(x,t)$ represents the density of the diffusing material. The equation is used to describe the behavior of motion of particles associated with the random movement in a material. Diffusion equation can be found in various application such as the Markov process and applied to many fields, such as biology and chemistry. In particular, it is used for the diffusion of pollution in the atmosphere, cell diffusion, network traffic, the spread of contaminants in underground water, the transmission of signal through strong magnetic fields, etc. However, there are some complex systems that cannot be identified by classical differential equations. Therefore, fractional integral and derivative was developed for solving these problems [13, 15, 16, 18].

Fractional calculus has triggered research interests in modelling of dynamical systems since some behavior can be reflected and described better than the models based on integer order derivatives. Indeed, it is observed that

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fractional calculus provides more realistic models to describe systems with memory effects. There are several types of fractional derivatives and fractional integrals found in the literature based on various kernels including Riemann-Liouville, Caputo, Hilfer, Riesz, Erdelyi-Kober, Hadamard, Atangana–Baleanu (see for example, [3, 15, 16, 21]). The common definitions that have been used in the literature are Riemann-Liouville and Caputo fractional calculus. Apart from the study of new type of fractional derivative and integral, there is also a research focus on integral inequalities arising from new fractional operators [1].

Based on the development of fractional calculus, there have been extensively studied on fractional diffusion equations that generalize the diffusion equations with classical derivative. In 2007, [5] formalized the systems of linear fractional differential equations using Riemann-Liouville and Caputo derivative operators to solve homogeneous and non-homogeneous with constant coefficients. In 2011, [22] introduced a space fractional derivative with parameter β in the heat equation. They used this new fractional model to generalize properties of medium, and Fourier law. Moreover, they applied it into the second law of thermodynamics to find cases that verify and do not verify. In 2017, [2] presented maximum principle, uniqueness and stability of solutions for fractional diffusion equations with the Caputo fractional derivative of non-singular kernel. Moreover, [20] presented the well-posedness of sub-diffusion equations with can be reduced to Ginzburg-Landau equations and Burger equations. Also, in [19], authors studied the final value problem for nonlinear time fractional reaction–diffusion equation by considering the problem as discrete data from grid points of the domain.

While there have been extensive studied on fractional diffusion equations, there is also some investigation of discrete structure that could affect the diffusion based on connected graph and network. In 2007, diffusion equations was generalized on graph by Chung et al. [9]. They discussed discrete version of the heat equations which are called the ω-diffusion equations. It is assumed that the diffusion occurs on network from one vertex to another adjacent vertex through an edge with the rate of change of the energy flow proportional to the difference of the quantity of the material of two vertices and the conductivity of the adjacent edges. They solved the ω-diffusion equations under three conditions including no boundary condition, the initial condition and the Dirichlet boundary condition. Moreover, they derived some properties on the ω-diffusion equations such as the minimum and maximum principles, Huygens property and uniqueness with energy methods. Moreover, the various problems on graph, solving direct and inverse problems of an equation, called ω-Laplace equations on graphs which can be interpreted as a diffusion equation on graphs has been studied by [7, 8, 10, 11, 12, 14].

Motivated by the development of fractional calculus and the discrete structure on networks, we generalize ω-diffusion equation on graphs to consider homogeneous and non-homogeneous fractional order ω-diffusion equations using Riemann-Liouville and Caputo derivative. This generalization would explain diffusion process that reflects discrete structure through connected edges and the memory effects from fractional derivatives, which contributes a new aspect to the literature. The purpose of this paper is to find solutions of fractional order system of diffusion equation on graphs and illustrate the behavior of solution from numerical simulation.

2 Preliminaries

In this section, we give some preliminary background on fractional calculus and calculus on weighted graphs.

2.1 Fractional Order Differential Equations

Definition 2.1. [17] The *Riemann-Liouville's fractional derivative* of order α for a continuous function $f(t)$ is definded by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds \quad \text{for } \alpha \in (0,1]$$

where Γ is the Gamma function.

Definition 2.2. [6] The *Caputo's fractional derivative* of order α for a continuous function $f(t)$ is defined by

$${}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{'}(s) ds \quad \text{for } \alpha \in (0,1]$$

where Γ is the Gamma function.

Remark 2.3. [5] There is a relationship between the Caputo and the Riemann-Liouville derivatives given by

$$D^{\alpha} f(t) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{\Gamma(1+j-\alpha)} t^{j-\alpha} +^{C} D^{\alpha} f(t)$$

For fractional order linear system, we can guarantee the existence of solution from the result of [5] as stated the next theorem.

Theorem 2.4. [5] Consider the systems of linear differential equations of fractional order

$$D^{\alpha} f(t) = A(t) f(t) + B(t) \tag{2.1}$$

where D^{α} is the Riemann-Liouville fractional derivative of order α $(0 < \alpha \leq 1)$, and

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix}, B(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix}$$

are matrices of real functions. The system (2.1) has the general solution of the form

$$f(t) = e_{\alpha}^{A(t)} C + \int_0^t e_{\alpha}^{A(t-\xi)} B(\xi) d\xi, \tag{2.2}$$

where $e_{\alpha}^{A(t)} = t^{\alpha-1} \sum_{k=0}^{\infty} A^k \frac{t^{k\alpha}}{\Gamma[(k+1)\alpha]}$ is a fundamental solution and C is an arbitrary constant matrix. In addition, the function $e_{\alpha}^{A(t)}$ satisfies the following properties:

1. If $\|A\| = \max_{i,j} |a_{i,j}|$, then $\|e^{A(t)_{\alpha}}\| \leq \sum_{k=0}^{\infty} \|A\|^k \frac{x^{(k-1)\alpha-1}}{\Gamma[(k+1)\alpha]}$ where $(x > 0)$
2. $e_{\alpha}^{At} \cdot e_{\alpha}^{Ct} \neq e_{\alpha}^{(A+C)t}$ where $\alpha \neq 0$
3. $D^{\alpha} e_{\alpha}^{A(t)} = A e_{\alpha}^{A(t)}$, where $A, C \in M_n(\mathbb{R})$, and $\alpha \in (0, 1]$.

2.2 Graph Theory and Calculus on Weighted Graph

In this section, we introduce some basic background of graph theory and calculus on weighted graphs [4].

Definition 2.5. For a graph $G = G(V, E)$, we refer to a finite set V of vertices with a set E of two element-subsets of V (whose element are called edges). We denote either $x \in V$ or $x \in G$ when x is a vertex of G.

Definition 2.6. A graph G is called *simple* if has neither multiple edges nor loops.

Definition 2.7. A graph G is called *connected* if for every pair of vetices x and y there exist a sequence of vertices $x = x_0 \sim x_1 \sim x_2 \ldots \sim x_{n-1} \sim x_n = y$ such that x_{j-1} and x_j are connected by an edge for $j = 1, 2, \ldots, n$, where $x \sim y$ means that two vertices x and y are connected by an edge in E.

Definition 2.8. A graph $S = S(V', E')$ is called *subgraph* of $G = G(V, E)$ if $V' \subset V$ and $E' \subset E$.

Definition 2.9. A graph $S = S(V', E')$ is called *induced subgraph* of $G = G(V, E)$ if S is a subgraph of G and E' consists of all the edges from E which connected the vertices of V'.

Definition 2.10. *Weighted Graph* is a graph $G = G(V, E)$ is associated with a weight function $\omega : V \times V \to [0, \infty)$ satisfying

1. $\omega(x, y) = \omega(y, x)$, for $x, y \in V$,
2. $\omega(x, y) = 0$ if and only if $x, y \notin E$.

Definition 2.11. The *degree* $d_\omega x$ of a vertex x is defined by

$$d_\omega x := \sum_{y \in V} \omega(x, y). \tag{2.3}$$

Definition 2.12. [9] The *directional derivative* of a function $f : V(G) \to \mathbb{R}$, is given by

$$D_{\omega_y} f(x) := [f(y) - f(x)]\sqrt{\omega(x, y)} \qquad , x, y \in V. \tag{2.4}$$

Definition 2.13. [9] *The gradient* ∇_ω of a function f is defined by

$$\nabla_\omega f(x) := (D_{\omega,y} f(x))_{y \in V}. \tag{2.5}$$

Note: All the subgraphs in our concern are assumed to be induced, simple and connected subgraphs of a weighted graph.

Definition 2.14. [9] The *ω-Laplacian* $\triangle_\omega$ of a function $f : V(G) \to \mathbb{R}$ on a graph G is defined by

$$\begin{aligned} \triangle_\omega f(x) &:= -\sum_{y \in V} D_{(\omega,y)}(D_{(\omega,y)} f(x)) \\ &= \sum_{y \in V} [f(y) - f(x)] \cdot \omega(x, y). \end{aligned} \tag{2.6}$$

Definition 2.15. [9] (ω-Laplacian Matrix)
For a function $f : V(G) \to \mathbb{R}$, with $|V| = N$. The ω-Laplacian operator $\triangle_\omega$ also can be considered as matrix defined by

$$\triangle_\omega(x, y) = \begin{cases} -d_\omega x, & x = y, \\ \omega(x, y), & \text{otherwise.} \end{cases} \tag{2.7}$$

Definition 2.16. [9] Symmetric normalized ω-Laplacian matrix is defined as

$$L_\omega = D^{1/2} \triangle_\omega D^{-1/2}, \tag{2.8}$$

where D denotes the diagonal matrix with (x, x)-th entry having the value $d_\omega x$ for each $x \in V$.

Definition 2.17. [9] (ω-Diffusion Equation)
Let a graph $G = G(V, E)$ and a weight ω be given. Let $F : V(G) \times [0, T) \to \mathbb{R}$ with T a given positive real number or ∞. The ω-diffusion equation is defined by

$$\partial_t F(x, t) - \triangle_\omega F(x, t) = H(x, t), x \in V, t \in (0, T), \tag{2.9}$$

where $H(x, t)$ is a given function in $V(G) \times [0, T)$.

3 Existence of Solutions of Fractional Order Diffusion Equations on Weighted Graphs

The main result presented in this section is the existence of solutions of fractional order diffusion equations on weighted graphs. The existence result is proved for homogeneous and non-homogeneous linear diffusion equations which can be extended to other types such as semilinear or perturbation, etc.

Theorem 3.1. (Homogeneous ω-diffusion on graphs by using Riemann-Liouville's derivative)
Let $G(V, E)$ be a graph with a weigh ω.Then, every solution $F(x, t)$ of the equation

$$\partial_t^\alpha F(x, t) - \triangle_\omega F(x, t) = 0, \quad x \in V, t \in [0, T) \tag{3.1}$$

is represented by

$$F(x, t) = \sum_{i=1}^{N} c_i \Phi_i(x) e_\alpha^{-\lambda_i t}, \quad x \in V, t \in [0, T) \tag{3.2}$$

for some $c_1, c_2, ..., c_N \in \mathbb{R}$,where $N = |V|$.

Proof . From (3.1),we have

$$\partial_t^\alpha F = \triangle_\omega F,$$

where $\triangle_\omega$ defined by

$$\triangle_\omega(x_i,x_j) = \begin{bmatrix} -d_\omega x_1 & \omega(x_1,x_2) & \dots & \omega(x_1,x_N) \\ \omega(x_2,x_1) & -d_\omega x_2 & \dots & \omega(x_2,x_N) \\ \vdots & \vdots & \ddots & \vdots \\ \omega(x_N,x_1) & \omega(x_N,x_2) & \dots & -d_\omega x_N \end{bmatrix}$$

By applying the Theorem 2.4, we obtain that

$$F(x,t) = e_\alpha^{At} C$$

where $A = \triangle_\omega$ $C = [C(x_1), C(x_2), \dots, C(x_N)]^T$ and $x_1, x_2, \dots, x_N \in V$.
Since $-\triangle_\omega$ is a non negative definite symmetric matrix, it has eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ and the corresponding eigenfunction $\Phi_1, \Phi_2, \dots, \Phi_N$. Then, for $x \in V$ we can express

$$F(x,t) = \sum_{i=1}^{N} c_i \Phi_i(x) e_\alpha^{-\lambda_i t}$$

where c_i is a constant. □

Theorem 3.2. (Non-homogeneous ω-diffusion on graphs by using Riemann-Liouville's derivative)
Let $G(V,E)$ be a graph with a weigh ω and $H(x,t) \in C^0(V \times (0,T)) \cap L^1(V \times (0,T))$.Then, every solution $F(x,t)$ of the equation

$$\partial_t^\alpha F(x,t) - \triangle_\omega F(x,t) = H(x,t), \quad x \in V, t \in [0,T) \tag{3.3}$$

is represented by

$$F(x,t) = \sum_{i=1}^{N} \Big(c_i e_\alpha^{-\lambda_i t} + \int_0^t e_\alpha^{-(t-\tau)\lambda_i} \sum_{y \in V} H(y,\tau)\Phi_i(y) d\tau \Big) \Phi_i(x) \tag{3.4}$$

for $x \in V$ and $t \in (0,T)$, where $N = |V|$.

Proof . First, we consider the expansion

$$F(x,t) = \sum_{i=1}^{N} a_i(t)\Phi_i(x)$$

for $(x,t) \in V \times (0,T)$,where

$$a_i(t) = \sum_{y \in V} F(y,t)\Phi_i(y) \qquad , i = 1, 2, \dots, N.$$

Since $-\triangle_\omega \Phi_i = \lambda_i \Phi_i$,we have

$$\begin{aligned} -\lambda_i a_i(t) &= -\lambda_i \sum_{y \in V} F(y,t)\Phi_i(y) \\ &= \sum_{y \in V} F(y,t)(-\lambda_i \Phi_i(y)) \\ &= \sum_{y \in V} F(y,t)\triangle_\omega \Phi_i(y) \\ &= \sum_{y \in V} \big[(\partial_t^\alpha F(y,t) - H(y,t))\Phi_i(y) \big] \\ &= \sum_{y \in V} \partial_t^\alpha F(y,t)\Phi_i(y) - \sum_{y \in V} H(y,t)\Phi_i(y) \\ &= \partial_t^\alpha \Big(\sum_{y \in V} F(y,t)\Phi_i(y) \Big) - \sum_{y \in V} H(y,t)\Phi_i(y) \\ &= \partial_t^\alpha a_i(t) - \sum_{y \in V} H(y,t)\Phi_i(y). \end{aligned}$$

It follows that

$$\partial_t^\alpha a_i(t) = -\lambda_i a_i(t) + \sum_{y\in V} H(y,t)\Phi_i(y).$$

Applying Theorem 2.4 and Theorem 3.1 ,we obtain

$$a_i(t) = c_i e_\alpha^{-\lambda_i t} + \int_0^t e_\alpha^{-(t-\tau)\lambda_i} \sum_{y\in V} H(y,\tau)\Phi_i(y)d\tau$$

for some real constants c_i. Hence,

$$F(x,t) = \sum_{i=1}^{N} \Big(c_i e_\alpha^{-\lambda_i t} + \int_0^t e_\alpha^{-(t-\tau)\lambda_i} \sum_{y\in V} H(y,\tau)\Phi_i(y)d\tau\Big)\Phi_i(x).$$

□

Corollary 3.3. (Homogeneous ω-diffusion on graphs by using Caputo derivative) Let $G(V,E)$ be a graph with a weigh ω.Then, every solution $F(x,t)$ of the equation

$$^{C}\partial_t^\alpha F(x,t) - \triangle_\omega F(x,t) = 0, \quad x\in V, t\in[0,T) \tag{3.5}$$

$$F(x,0) = F_0(x), \quad x\in V \tag{3.6}$$

is represented by

$$F(x,t) = F_0(x) + \sum_{i=1}^{N}\left(\int_0^t \sum_{y\in V} \triangle_\omega F_0(y)\Phi_i(y)e_\alpha^{-(t-\tau)\lambda_i}d\tau\right)\Phi_i(x) \tag{3.7}$$

Proof . Set

$$\tilde{F}(x,t) = F(x,t) - F_0(x).$$

Then, we have

$$\begin{aligned}
^{C}\partial_t^\alpha \tilde{F}(x,t) &= \sum_{j=0}^{N-1} \frac{\frac{\partial^j}{\partial t^j}\tilde{F}(x,0)t^{j-\alpha}}{\Gamma(1+j-\alpha)} + \partial_t^\alpha \tilde{F}(x,t)\\
&= \sum_{j=0}^{N-1} \frac{\frac{\partial^j}{\partial t^j}\tilde{F}(x,0)t^{j-\alpha}}{\Gamma(1+j-\alpha)} + \partial_t^\alpha F(x,t) - \partial_t^\alpha F_0(x)\\
&= \triangle_\omega F(x,t)\\
&= \triangle_\omega \tilde{F}(x,t) + \triangle_\omega F_0(x)
\end{aligned}$$

Applying the previous theorem, we get

$$\tilde{F}(x,t) = \sum_{i=1}^{N}\left(c_i e_\alpha^{-\lambda_i t} + \int_0^t e_\alpha^{-(t-\tau)\lambda_i} \sum_{y\in V} \triangle_\omega F_0(y)\Phi_i(y)d\tau\right)\Phi_i(x)$$

Since $\tilde{F}(x,0) = 0$, we obtain

$$\sum_{i=1}^{N} c_i\Phi_i(x)e_\alpha^{-\lambda_i t} = 0.$$

It follows that

$$\begin{aligned}
F(x,t) &= \tilde{F}(x,t) + F_0(x)\\
&= F_0(x) + \sum_{i=1}^{N}\left(\int_0^t \sum_{y\in V} \triangle_\omega F_0(y)\Phi_i(y)e_\alpha^{-(t-\tau)\lambda_i}d\tau\right)\Phi_i(x)
\end{aligned}$$

□

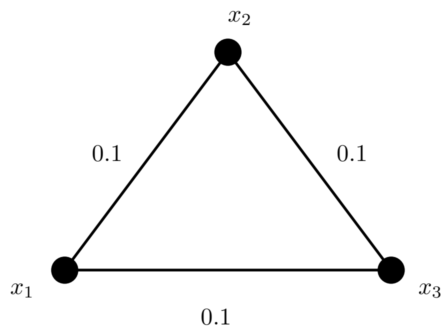

Figure 1:

4 Examples and Numerical simulation

In this section, we provide examples of homogeneous fractional order diffusion equations (3.1) on graphs in Example 1-4 and also non-homogeneous fractional order diffusion equations (3.3) in Example 5. First we provide example of graph with uniform weight.

Example 4.1. Let G be a graph with weight ω as Figure 1.

From the graph, we can write down the ω-Laplacian matrix:

$$\triangle_\omega = \begin{bmatrix} -0.2 & 0.1 & 0.1 \\ 0.1 & -0.2 & 0.1 \\ 0.1 & 0.1 & -0.2 \end{bmatrix}$$

which $-\triangle_\omega$ has the eigenvalues $\lambda_1 = 0, \lambda_2 = 0.3$ and $\lambda_3 = 0.3$, with corresponding eigenvectors:

$$\Phi_1(x) = \begin{bmatrix} -0.5774 \\ -0.5774 \\ -0.5774 \end{bmatrix}, \Phi_2(x) = \begin{bmatrix} 0.4894 \\ -0.8107 \\ 0.3214 \end{bmatrix} \text{ and } \Phi_3(x) = \begin{bmatrix} -0.6536 \\ -0.0970 \\ 0.7506 \end{bmatrix}.$$

Then the solution when the initial values at $t_0 = 0.01$ given by $F(x_1, t_0) = 0.1, F(x_2, t_0) = 0.2$ and $F(x_3, t_0) = 0.3$ are shown in Figure 2. Figure 2 shows the solution at each vertex in graph when fractional order derivative varies from $\alpha = 0.2$, 0.5, 0.9 and 1. It can be seen that the value on each vertex increase. When fractional order derivative is closed to 1, the trend becomes linear and the magnitude of the energy increases faster.
In the next example, we consider non-complete graph with equal weight.

Example 4.2. Let G be a graph with weight ω as Figure 3.
From the graph, we can write down the ω-Laplacian matrix:

$$\triangle_\omega = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0.1 & -0.2 & 0.1 \\ 0 & 0.1 & -0.1 \end{bmatrix}$$

which $-\triangle_\omega$ has the eigenvalues $\lambda_1 = 0, \lambda_2 = 1$ and $\lambda_3 = 3$, with corresponding eigenvectors:

$$\Phi_1(x) = \begin{bmatrix} -0.5774 \\ -0.5774 \\ -0.5774 \end{bmatrix}, \Phi_2(x) = \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix} \text{ and } \Phi_3(x) = \begin{bmatrix} 0.4082 \\ -0.8165 \\ 0.4082 \end{bmatrix}.$$

Then the solution when the intial values at $t_0 = 0.1181$ given by $F(x_1, t_0) = 0.1, F(x_2, t_0) = 0.2$ and $F(x_3, t_0) = 0.3$ are shown in Figure 4. The behavior of solution is similar to example 1. Figure 4 shows the solution at each vertex in graph when fractional order derivative varies from $\alpha = 0.2$, 0.5, 0.9 and 1. It can be seen that the value on each vertex increase. When fractional order derivative is closed to 1, the trend becomes linear and the magnitude of the energy increases faster.

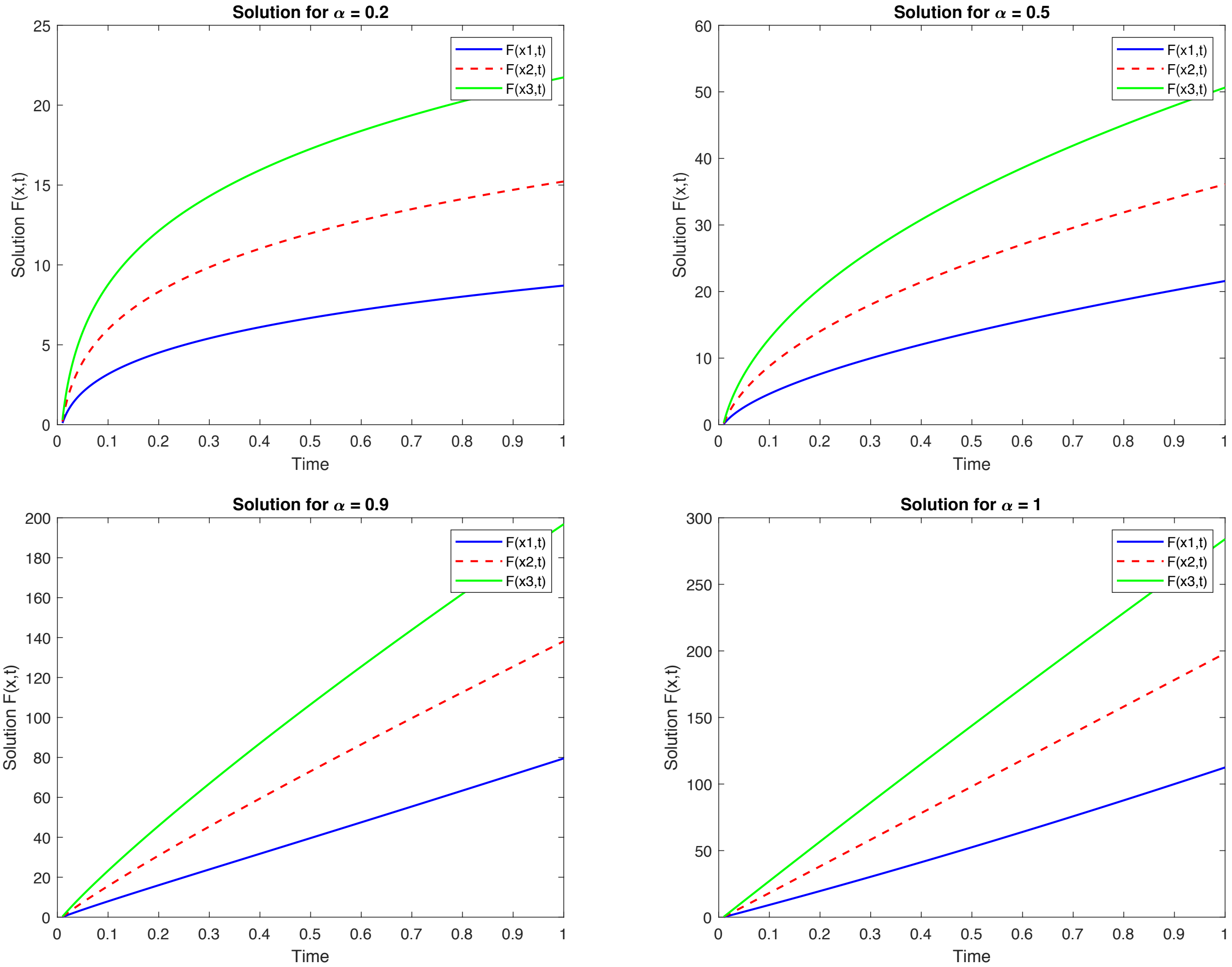

Figure 2: Solution of fractional order diffusion equation on complete graph with 3 vertices and same weight.

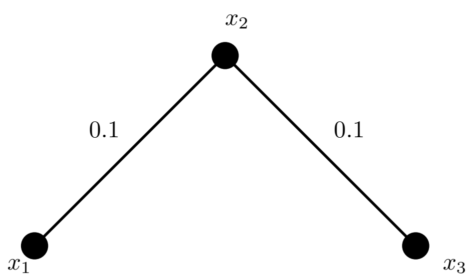

Figure 3:

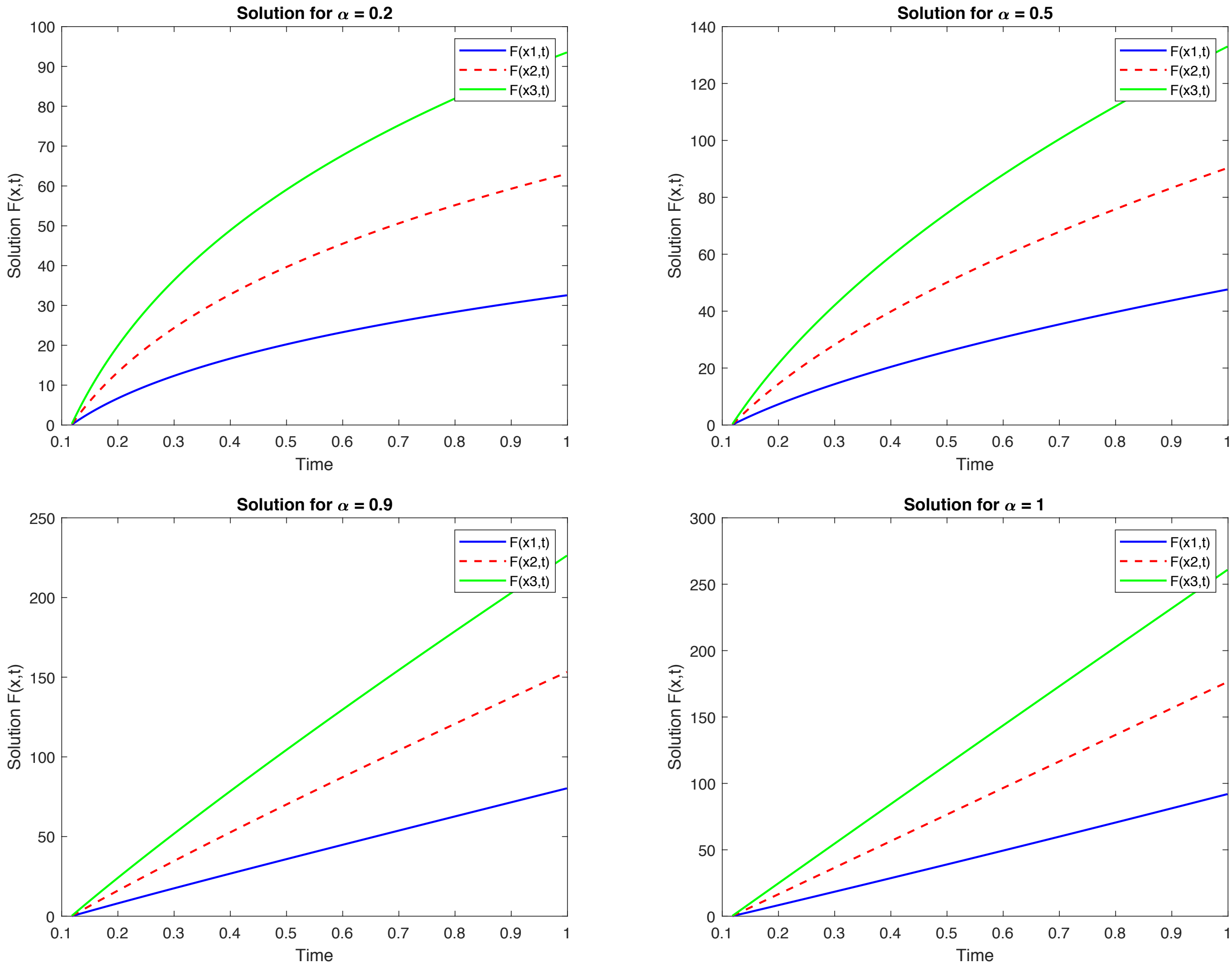

Figure 4: Solution of fractional order diffusion equation on connected graph with 3 vertices and same weight.

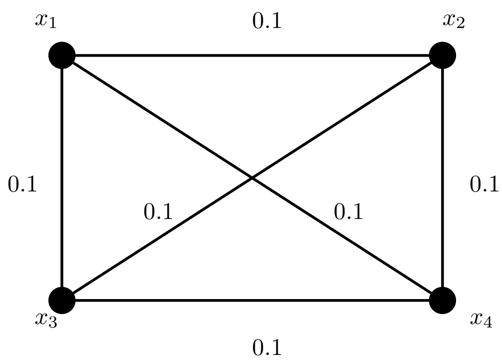

Figure 5:

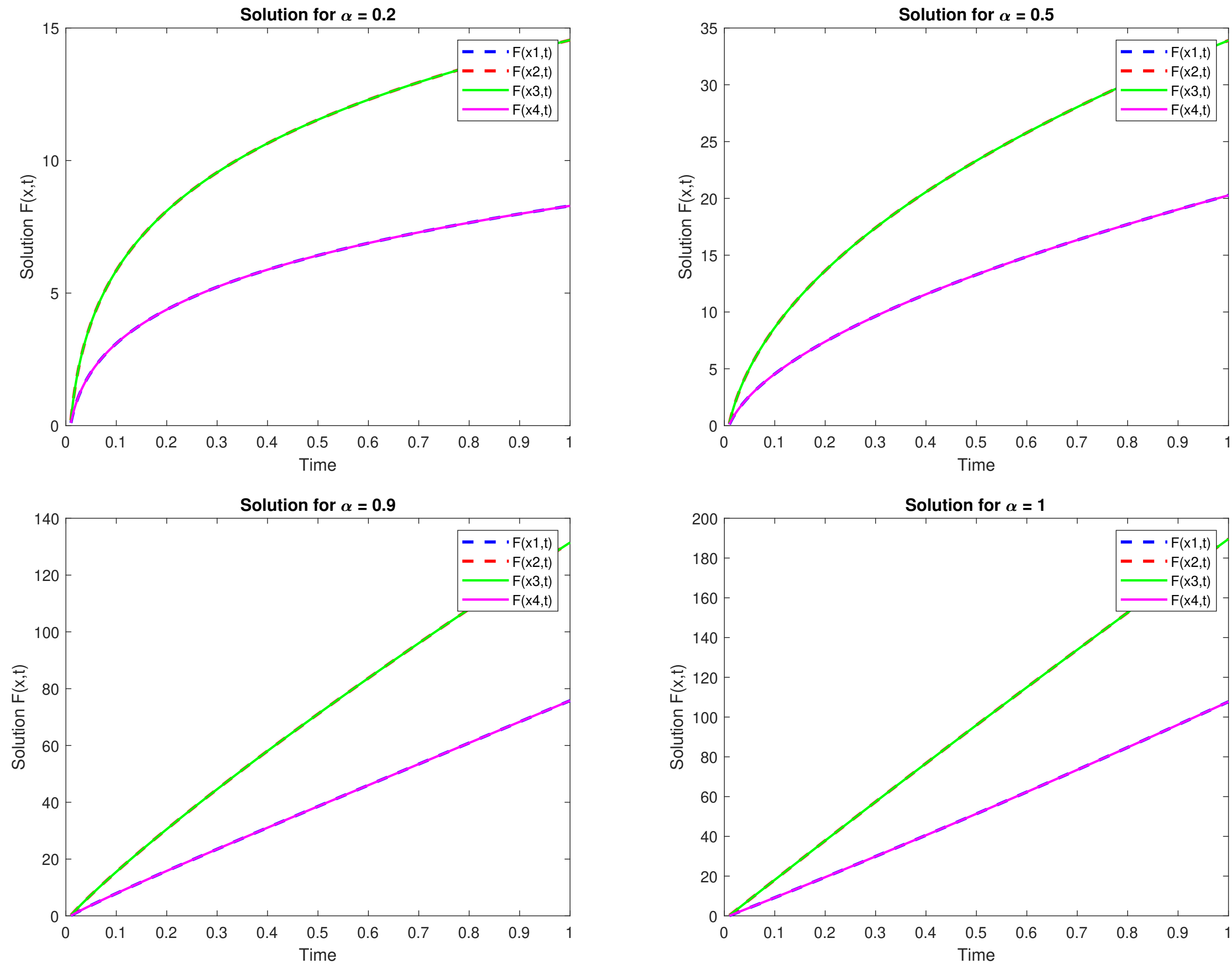

Figure 6: Solution of fractional order diffusion equation on complete graph with 4 vertices and same weight.

Example 4.3. Let G be a graph with weight ω as Figure 5.
From the graph, we can write down the ω-Laplacian matrix:

$$\triangle_\omega = \begin{bmatrix} -0.3 & 0.1 & 0.1 & 0.1 \\ 0.1 & -0.3 & 0.1 & 0.1 \\ 0.1 & 0.1 & -0.3 & 0.1 \\ 0.1 & 0.1 & 0.1 & -0.3 \end{bmatrix}$$

which $-\triangle_\omega$ has eigenvalues $\lambda_1 = 0, \lambda_2 = 0.4, \lambda_3 = 0.4$ and $\lambda_4 = 0.4$, with corresponding eigenvectors:

$$\Phi_1(x) = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \Phi_2(x) = \begin{bmatrix} -0.2113 \\ 0.7887 \\ -0.5774 \\ 0 \end{bmatrix}, \Phi_3(x) = \begin{bmatrix} -0.2887 \\ -0.2887 \\ -0.2887 \\ 0.8660 \end{bmatrix} \text{ and } \Phi_4(x) = \begin{bmatrix} 0.7887 \\ -0.2113 \\ -0.5774 \\ 0 \end{bmatrix}.$$

Then the solution when the intial values at $t_0 = 0.01$ given by $F(x_1, t_0) = 0.1, F(x_2, t_0) = 0.2, F(x_3, t_0) = 0.2$ and $F(x_4, t_0) = 0.1$ are shown in Figure 6. Figure 6 shows the solution at each vertex in graph when fractional order derivative varies from $\alpha = 0.2$, 0.5, 0.9 and 1. It can be seen that the value on each vertex increase. When fractional order derivative is closed to 1, the trend becomes linear and the magnitude of the solution increases faster. Moreover, there is a symmetry in the solution on vertices x_1 and x_4, and the solution on vertices x_2 and x_3. It can be seen that the solution on vetices x_1 and x_4 is less than the solution on vertices x_2 and x_3.

Example 4.4. Let G be a graph with weight ω as Figure 7.
From the graph, we can write down the ω-Laplacian matrix:

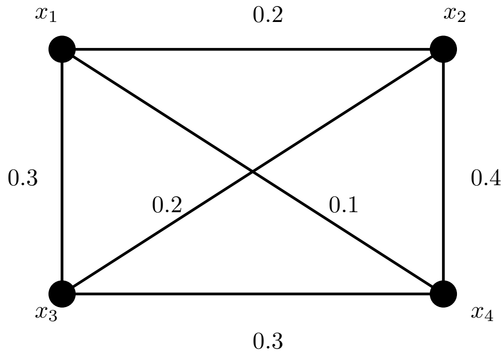

Figure 7:

$$\triangle_\omega = \begin{bmatrix} -0.6 & 0.2 & 0.3 & 0.1 \\ 0.2 & -0.8 & 0.2 & 0.4 \\ 0.3 & 0.2 & -0.8 & 0.3 \\ 0.1 & 0.4 & 0.3 & -0.8 \end{bmatrix}$$

which $-\triangle_\omega$ has the eigenvalues $\lambda_1 = 0, \lambda_2 = 1.7222, \lambda_3 = 1.0289$ and $\lambda_4 = 1.2489$, with corresponding eigenvectors:

$$\Phi_1(x) = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \Phi_2(x) = \begin{bmatrix} -0.7751 \\ 0.3648 \\ -0.0966 \\ 0.5068 \end{bmatrix}, \Phi_3(x) = \begin{bmatrix} 0.3033 \\ 0.5501 \\ -0.7739 \\ -0.0795 \end{bmatrix} \text{ and } \Phi_4(x) = \begin{bmatrix} 0.2393 \\ -0.5606 \\ -0.3764 \\ 0.6977 \end{bmatrix}.$$

Then solution when the intial values at $t_0 = 0.01$ given by $F(x_1, t_0) = 0.1, F(x_2, t_0) = 0.1, F(x_3, t_0) = 0.1$ and $F(x_4, t_0) = 0.2$ are shown in Figure 8. Figure 8 shows the solution at each vertex in graph when fractional order derivative varies from $\alpha = 0.2$, 0.5, 0.9 and 1. It can be seen that the solution on each vertex increase. When fractional order derivative is closed to 1, the trend becomes linear and the magnitude of the solution increases faster. However, the solution on each vertex is different due to the rate of diffusion term resulted from unequal weight.

Next, we consider non-homogeneous fractional diffusion equations on graphs in (3.3).

Example 4.5. Consider the graph G is in Example 1 and non-homogeneous fractional diffusion equations (3.3) with source term at x_1, x_2 and x_3 given by $H(x_1, t) = 0.8$, $H(x_2, t) = 0.3$ and $H(x_3, t) = 0$ respectively. We obtain the solutions as shown in Figure 9. Figure 9 shows the solution at each vertex in graph when fractional order derivative varies from $\alpha = 0.2$, 0.5, 0.9 and 1. It can be seen that the solution on each vertex increase. When fractional order derivative is closed to 1, the trend becomes linear and the magnitude of the solution increases faster. However, the solutions on each vertex which has source term increase faster.

5 Conclusions

In conclusion, we generalize diffusion equation on graphs to fractional order diffusion equations on graphs (2.9) using Riemann-Lioville's derivative and Caputo's derivative. It can be seen from numerical simulation that the solution of fractional order diffusion equations on graphs exhibit blow up behavior. In particular, the solution increase over the time. When order of fractional derivative is closed to 1, the solution becomes more linear. The effect of non-homogenous source term accelerates the magnitude of solution to blow up faster.

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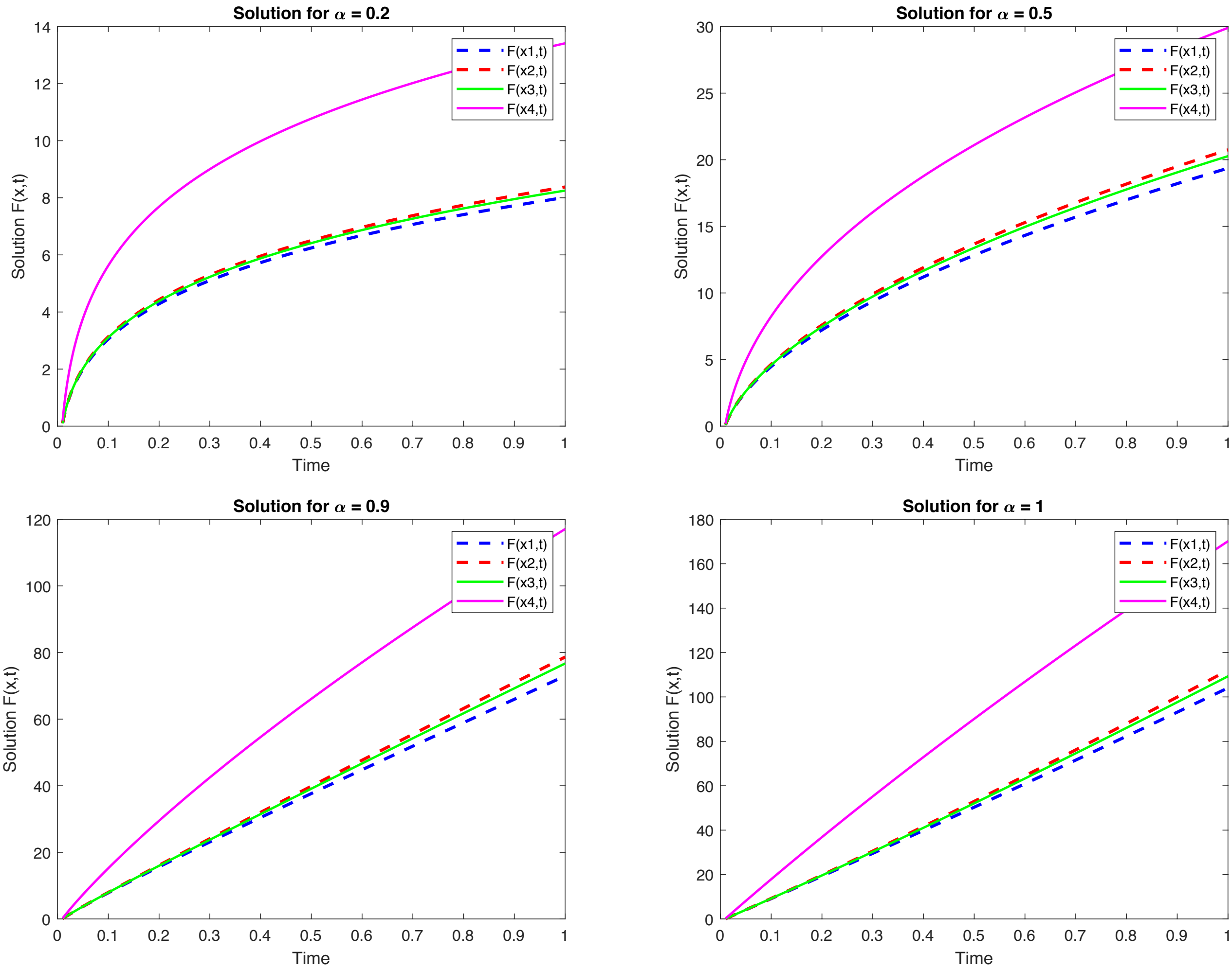

Figure 8: Solution of fractional order diffusion equation on complete graph with 4 vertices but unequal weight.

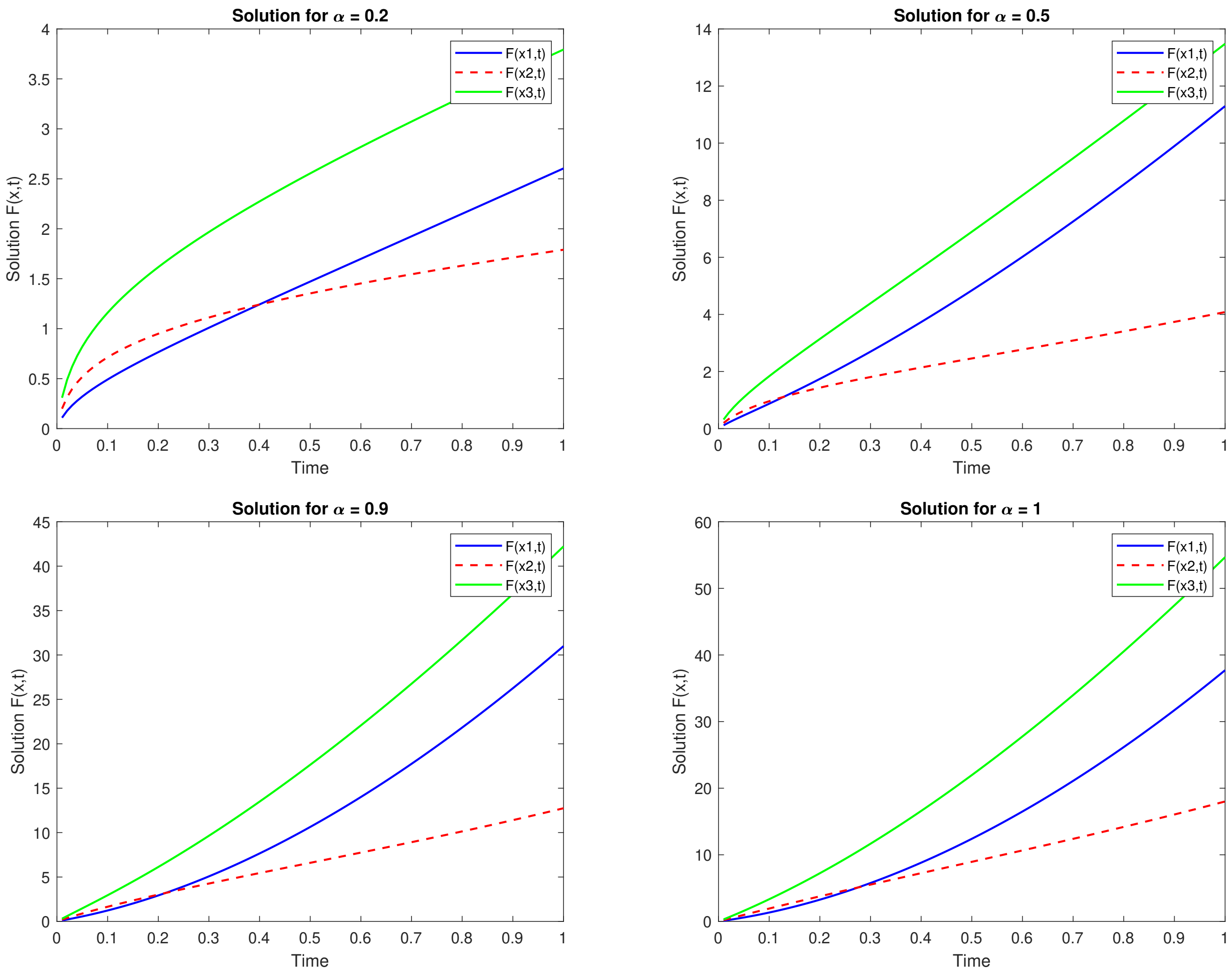

Figure 9: Solution of fractional order diffusion equation on complete graph with 3 vertices and same weight, but add source term at x_1 and x_2

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